A VARIATIONAL PROBLEM ARISES FROM MUD CRACKING

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Main Problem

We consider the following minimizing problem:

Minimize

$$\mathbf{T}(E) = P(E, \Omega) + \mathbf{G}(E)$$

among all sets $E \subset \Omega$ of finite perimeter with a fixed volume.

Here

- Ω is any open subset of $\mathbb{R}^n$ with $n \geq 2$.
- $P(E, \Omega)$ denotes the perimeter of E,
- $\mathbf{G}$ is a lower semicontinuous functional on the sets of finite perimeter [a] in Ω with the property that [b]

$$\mathbf{G}(E) \leq \mathbf{G}(F) + C|E \Delta F|^{\beta}$$

  for any sets E, F in Ω of finite perimeter with $|E| = |F|$, for some constant $C > 0$ and a number $\beta > 1 - \frac{1}{n}$.()

[a] (44)

[b] See notations in 43

The special case: $\mathbf{G}(E) \equiv 0$

That is, *minimize*

$$\mathbf{T}(E) = P(E, \Omega)$$

among all sets $E \subset \Omega$ *of finite perimeter with a fixed volume.*

This problem is often encountered in the field of capillarity theory. Liquid drops, resting on or hanging from a given surface, are some typical examples.

The regularity of the corresponding minimizers has been studied extensively in:

E. Gonzalez, U. Massari & I. Tamanini, On the regularity of boundaries of sets minimizing perimeter with a volume constraint. Indiana University Mathematics Journal, Vol. 32, No.1 (1983), 25-37.

Another example

Minimize

$$\mathbf{T}(E) = P(E,\Omega) + \int_E H(x)\,dx$$

among all sets $E \subset \Omega$ of finite perimeter with a fixed volume. Here $H \in L^p(\Omega)$, for some $p > n$, is a given function.

Without a volume constraint, this is the problem of finding sets with prescribed mean curvature H, and has been studied for instance in

U. Massari. Frontiere orientate di curvatura media assegnata in L^p. Rend. Sem. Mat. Univ. Padova 53 (1975), 37-52.

In our case, we impose an additional volume constraint on it. So,

$$G(E) = \int_E H(x)\,dx.$$

From Hölder inequality, we see that $\beta = 1 - \frac{1}{p} > 1 - \frac{1}{n}$ here.

Main motivation: mud cracking

Mud cracking represents a very typical physical phenomenon.
After losing a certain amount of moisture, a material such as a piece of mud will begin to crack.
People are interested in why, how and where the material cracks.

My approach

Let Ω represent a piece of mud. After losing a certain amount of moisture, say $\sigma|\Omega|$ for some $\sigma \in (0,1)$, the volume of the mud decreases, and thus a crack E of volume $\sigma|\Omega|$ must come out to replace the losing volume.

The selection of cracking is not totally random, but the actual physics of it might be too complicated to handle.

Instead, we may assume that it minimizes the total work of transporting the old mud Ω to the new mud $\Omega - E$, with multiplicity $\frac{1}{1-\sigma}$, under a volume preserving map.

Justification

Let us think about a mud of the shape of a disk.

To replace the volume of losing moisture, it can

- either shrink evenly to a smaller disk
- or dig some space out by cracking inside it.

Which way is better? As we know, the mud will possibly choose the later way. This is because the corresponding transport costs of two ways are different. The mud just chooses a cheaper way to reduce the total work.

A reasonable way to represent the total work is given by the Wasserstein distances W_p between Radon measures of equal total mass.
As a result, one would like to minimize

$$W_p^q\left(\mathcal{L}^n\lfloor\Omega, \frac{1}{1-\sigma}\mathcal{L}^n\lfloor(\Omega - E)\right) + P\left(E, \Omega\right)$$

among all sets E of finite perimeter in Ω with volume $|E| = \sigma\,|\Omega|$ for some $\sigma \in (0,1)$. Here,$q = min(1, 1/p)$, and the perimeter $P(E,\Omega)$ of E is used to represent the cracking energy for breaking the mud.
Using the properties of Wasserstein distances, it is easy to see that

$$\begin{aligned} W_p\left(\mathcal{L}^n\lfloor\Omega, \frac{1}{1-\sigma}\mathcal{L}^n\lfloor(\Omega - E)\right) &= W_p\left(\frac{1}{1-\sigma}\mathcal{L}^n\lfloor E, \frac{\sigma}{1-\sigma}\mathcal{L}^n\lfloor\Omega\right) \\ &= \lambda W_p\left(\mathcal{L}^n\lfloor E, \sigma\mathcal{L}^n\lfloor\Omega\right) \end{aligned}$$

for some constant $\lambda > 0$.

Thus, the problem becomes to minimize

$$P(E,\Omega) + \lambda W_p^q(\mathcal{L}^n \lfloor E, \sigma \mathcal{L}^n \lfloor \Omega)$$

among all sets E in Ω of finite perimeter and with a volume constraint $|E| = \sigma |\Omega|$. In this case,

$$\mathbf{G}(E) = \lambda W_p^q(\mathcal{L}^n \lfloor E, \sigma \mathcal{L}^n \lfloor \Omega).$$

It is easy to see that $\beta = 1$ here and thus $\beta > 1 - \frac{1}{n}$.

More general form

Minimize the quasi perimeter

$$\mathbf{T}(E) = P(E,\Omega) + \mathbf{G}(E)$$

among all sets in

$$\mathcal{F}_\sigma = \{E \subset \Omega : P(E,\Omega) < +\infty, |E| = \sigma|\Omega|\}.$$

for some $0 < \sigma < 1$.

Here, G satisfies the property that

$$\mathbf{G}(E) \leq \mathbf{G}(F) + C|E\Delta F|^\beta$$

for any sets E, F in $\mathcal{F}_\sigma$ and $\beta > 1 - \frac{1}{n}$.

- Existence: follows from the compactness of sets of finite perimeter.
- Main question: the regularity of these minimizers.
- Further properties and numerical simulation will be considered later.

Main Theorem

Theorem 1. *Suppose E is a minimizer of the quasi perimeter $\mathbf{T}$ in $\mathcal{F}_\sigma$. Then E is a quasi minimizer of perimeter (without the volume constraint) in the sense that*

$$P(E,\Omega) \leq P(F,\Omega) + C|E\Delta F|^{\min(1,\beta)} \tag{0.1}$$

for all subsets F of Ω with $E\Delta F$ contained in a small open ball.

$C^{1,\alpha}$Regularity

As a corollary, we get the following $C^{1,\alpha}$ regularity:

Corollary 1. *Suppose E is a minimizer of the quasi perimeter $\mathbf{T}$ in $\mathcal{F}_\sigma$. Then the reduced boundary $\partial^* E$ is an $(n-1)$ dimensional $C^{1,\min(1/2,\beta/2)}$ hypersurface in Ω, and moreover $\dim((\partial E - \partial^* E) \cap \Omega) \leq n-8$.*

Strategy for regularity results

The special case $\mathbf{G}(E) \equiv 0$ was studied by Gonzalez, Massari & Tamanini. We adapt a similar approach to deal with more general cases.

- Step 1 (key step): show that these $\mathbf{T}$ minimizers indeed have both interior points and exterior points;
- Step 2: show that they are quasi minimizers of perimeter without the volume constraint;
- Step 3: using the known results about quasi minimizers of perimeter to get the desired regularity of these $\mathbf{T}$ minimizers.

Exterior points of sets of finite perimeter

Let E be a set of finite perimeter in Ω. A point $p \in \Omega$ is said to be an *exterior point* of E (or *interior point of* E) if

$$|E \cap B\left(p,r\right)| = 0 \text{ (or } |E \cap B\left(p,r\right)| = |B(p,r)|\text{, respectively)}$$

for some open ball neighborhood $B\left(p,r\right)$ of p.

- If $|E| = 0$, then every point is an exterior point of E.
- If $|E| = |\Omega|$, then every point is an interior point of E.
- In general, if $0 < |E| < |\Omega|$, E may not necessarily have an exterior or interior point in Ω.

The existence of exterior points and interior points itself becomes an interesting problem to study.

Existence of exterior points

To study the existence of exterior points, we consider the following function. For any $r \geq 0$, let

$$f(r) = \inf_{x \in \Omega} |E \cap Q(x,r)|,$$

where $Q(x,r)$ denotes the n-dimensional open cube in $\mathbb{R}^n$ centered at x and with edge length r. Note that $f(0) = 0$ and $f(r)$ is an increasing function of r. Also, E has exterior points in Ω if and only if $f(r) \equiv 0$ in a small neighborhood of 0.

For any point p in Ω with metric density 0, one may say directly that

$$|E \cap B(p,r)| = o(r^n),$$

and thus we may conclude that $f(r) = o(r^n)$ as r approaches 0. This is true even if E is not of finite perimeter. When E is indeed a set of finite perimeter, we can get a better result by saying that $f(r) = o\left(r^{n+1}\right)$, which is demonstrated by the following theorem.

Theorem 2. *Suppose E is a set of finite perimeter in Ω with $|E| < |\Omega|$. Then, there exists an $\eta > 0$ such that for any $r \in [0, \eta)$,*

$$0 \leq f(r) \leq C_1 r^{n+1+\frac{1}{n-1}}$$

for some constant $C_1 \geq 0$, depending on E.

Proof. Let $p \in \Omega \backslash E$ be any point with metric density 0 . That is,

$$\lim_{r \to 0} \frac{|E \cap B(p,r)|}{\alpha_n r^n} = 0.$$

Thus, there exists a $\eta_1 > 0$ such that

$$|Q(p, \eta_1) \cap E| \leq \frac{(\eta_1)^n}{4}.$$

Now, for any $r \leq \eta_1$, one can subdivide $Q = Q(p, \eta_1)$ into totally $\left[\frac{\eta_1}{r}\right]^n$ disjoint smaller cubes $\{Q_j\}$ with edge length r, where $[x]$ denotes the integer part of x.

Let

$$\mathbf{A} = \left\{ Q_j : |E \cap Q_j| > \frac{1}{2} r^n \right\}$$

and

$$\mathbf{B} = \left\{ Q_j : \left| E \cap Q_j \right| \leq \frac{1}{2} r^n \right\}.$$

Then, the total number $|\mathbf{A}| + |\mathbf{B}| = \left[\frac{\eta_1}{r}\right]^n$ and

$$\frac{1}{2} |\mathbf{A}| \, r^n \leq \sum_{Q_j \in \mathbf{A}} \left| E \cap Q_j \right| \leq |E \cap Q| \leq \frac{1}{4} (\eta_1)^n,$$

where $|\mathbf{A}|$ denotes the total number of elements in set $\mathbf{A}$. Thus, $|\mathbf{A}| \leq \frac{1}{2} \left(\frac{\eta_1}{r}\right)^n$ and

$$|\mathbf{B}| = \left[\frac{\eta_1}{r}\right]^n - |\mathbf{A}| \geq \left[\frac{\eta_1}{r}\right]^n - \frac{1}{2} \left(\frac{\eta_1}{r}\right)^n \geq \frac{1}{4} \left(\frac{\eta_1}{r}\right)^n$$

if $r < \left(1 - \sqrt[n]{0.75}\right) \eta_1$.

By the relative isoperimetric inequality,[1] for any $Q_i \in \mathbf{B}$,

$$\left| E \cap Q_j \right|^{1-1/n} \leq \gamma (n) \, P \left(E, Q_i\right)$$

[1] 0.19

for some constant $\gamma(n)$. Therefore,

$$\begin{aligned} |\mathbf{B}|\, f(r)^{1-1/n} &= \sum_{Q_i \in \mathbf{B}} f(r)^{1-1/n} \\ &\leq \sum_{Q_i \in \mathbf{B}} \left|E \cap Q_j\right|^{1-1/n} \\ &\leq \sum_{Q_i \in \mathbf{B}} \gamma(n)\, P(E, Q_i) \\ &\leq \gamma(n)\, P(E, Q) < +\infty \end{aligned}$$

because E has finite perimeter in Q. Thus,

$$f(r) \leq \left(\frac{\gamma(n)\, P(E, Q)}{|\mathbf{B}|}\right)^{\frac{n}{n-1}} \leq \left(\frac{4\gamma(n)\, P(E, Q)}{(\eta_1)^n}\right)^{\frac{n}{n-1}} r^{\frac{n^2}{n-1}}$$

whenever $r \leq \eta = \left(1 - \sqrt[n]{0.75}\right)\eta_1$. $\square$

Remark 1. *This theorem enables one to find a good cube inside which the total volume of E is very tiny.*

Proposition 1. *Suppose E is a set in Ω of finite perimeter with $|E| < |\Omega|$, and τ is any positive real number. Let Q be any n-dimensional open cube in Ω with edge length r satisfying*

$$r < \left(\frac{\tau}{2^{n+2}}\right)^{n-1} \frac{1}{(2C_1)^{\alpha}} \tag{0.2}$$

and

$$|E \cap Q| \leq 2C_1 r^{\frac{n^2}{n-1}}. \tag{0.3}$$

Then there exists an $s \in \left[\frac{r}{2}, r\right]$ such that

$$|E \cap Q(p, s)| \leq 2C_1 s^{\frac{n^2}{n-1}}$$

and

$$\mathcal{H}^{n-1}(E \cap \partial Q(p, s)) \leq \tau \, |E \cap Q(p, s)|^{\alpha},$$

where $\alpha = 1 - \frac{1}{n}$ and $Q(p, s)$ is the cube having the same center p of Q and with edge length s.

Proof. Let p be the center of Q and $Q(p,s)$ be the cube centered at p and with edge length s. We consider the function $g:[0,r]\to[0,+\infty)$ defined by

$$g(s)=\left|E\cap Q(p,s)\right|,$$

for each $s\in[0,r]$. Then, $g(0)=0$ and

$$g'(s)=\mathcal{H}^{n-1}\left(E\cap\partial Q(p,s)\right)$$

for almost all s.

From (0.2) and (0.3), we have

$$\begin{aligned}\int_0^r\left(g(s)^{1/n}\right)'ds &= g(r)^{1/n}-g(0)^{1/n}\\ &\le (2C_1)^{1/n}r^{\frac{n}{n-1}}\le\frac{\tau}{n}r.\end{aligned}$$

Therefore, there exists an $s\in[0,r]$ such that

$$\left(g(s)^{1/n}\right)'\le\frac{\tau}{n}.$$

That is,

$$g'(s)\le\tau g(s)^{\alpha}.$$

Let

$$s_0 = \max\left\{s \in [0, r] : g'(s) \le \tau g(s)^\alpha\right\} \le r.$$

We claim that

$$g(s_0) \le 2C_1 (s_0)^{\frac{n^2}{n-1}}.$$

In fact, if $s_0 = r$, then it follows from our assumption (0.3).

If $s_0 < r$, then for any $s \in (s_0, r)$, we have

$$g'(s) > \tau g(s)^\alpha,$$

which yields

$$\left(g(s)^{1/n}\right)' > \frac{\tau}{n}.$$

Integrating it from s_0 to r yields

$$g(r)^{1/n} - g(s_0)^{1/n} \ge \frac{\tau}{n}(r - s_0). \tag{0.4}$$

Therefore,

$$\begin{aligned} g\left(s_0\right)^{1/n} &\leq g\left(r\right)^{1/n} - \frac{\tau}{n}\left(r - s_0\right) \\ &\leq \left(2C_1\right)^{1/n} r^{\frac{n}{n-1}} - \frac{\tau}{n}\left(r - s_0\right) \\ &\leq \left(2C_1\right)^{1/n} \left(s_0\right)^{\frac{n}{n-1}}. \end{aligned}$$

The last inequality follows from the fact that the function

$$h\left(x\right) = \left(2C_1\right)^{1/n} x^{\frac{n}{n-1}} - \frac{\tau}{n}x$$

is decreasing on $[0, r]$ because $h'\left(x\right) = \left(2C_1\right)^{1/n} \frac{n}{n-1} x^{\frac{1}{n-1}} - \frac{\tau}{n} \leq 0.$

Moreover, since $g\left(s_0\right) \geq 0$, by (0.4) and (0.2), we have

$$\begin{aligned} r - s_0 &\leq \frac{n}{\tau} g\left(r\right)^{1/n} \\ &\leq \frac{n}{\tau}\left(2C_1\right)^{1/n} r^{\frac{n}{n-1}} \leq \frac{r}{2}. \end{aligned}$$

Therefore, we have

$$s_0 \geq r/2.$$

$\square$

Let τ be a small positive number and

$$A_\tau(E)$$

be the family of all n-dimensional open cubes in Ω satisfying

$$|E \cap Q| \leq 2C_1 r^{\frac{n^2}{n-1}}, \mathcal{H}^{n-1}(E \cap \partial Q) \leq \tau |E \cap Q|^\alpha, \tag{0.5}$$

where r is the edge length of Q satisfying (0.2).

Question: How to control the perimeter $P(E, Q)$?

The isoperimetric inequality says that

$$P(E, Q) + \mathcal{H}^{n-1}(E \cap \partial Q) \geq C_n |E \cap Q|^\alpha,$$

where $C_n = n|B(0,1)|^{1/n}$.

Question: For any set E of finite perimeter, is there a cube Q such that

$$P(E, Q) \geq (C_n + 2\tau) |E \cap Q|^\alpha?$$

Proposition 2. *Let E be a set of finite perimeter with $|E| < |\Omega|$, and Q be any cube in $\mathrm{A}_\tau(E)$. Then either*

$$P(E, Q) \geq (C_n + 2\tau)\,|E \cap Q|^\alpha$$

or there exists a smaller cube $\tilde{Q} \subset Q$ such that $\tilde{Q} \in \mathrm{A}_\tau(E)$ and the edge length of $\tilde{Q}$ satisfying

$$\tilde{r} \in (r/8, 3r/4)\,.$$

Proof. Without losing generality, we may assume that Q is centered at the origin O. Let

$$Z = \left\{ s \in \left(-\frac{r}{2}, \frac{r}{2}\right) : \sum_{i=1}^{n} \mathcal{H}^{n-1}\left(E \cap \{x = (x_1, \cdots, x_n) \in Q : x_i = s\}\right) \geq \frac{\tau}{2}\,|E \cap Q|^\alpha \right\}.$$

Then, since

$$
\begin{aligned}
n\,|E\cap Q| &\geq \sum_{i=1}^{n}\int_{-r/2}^{r/2}\mathcal{H}^{n-1}\left(E\cap\{x\in Q : x_i = s\}\right)ds\\
&\geq \int_{Z}\sum_{i=1}^{n}\mathcal{H}^{n-1}\left(E\cap\{x\in Q : x_i = s\}\right)ds\\
&\geq \frac{\tau}{2}\,|E\cap Q|^{\alpha}\,\mathcal{H}^{1}\left(Z\right),
\end{aligned}
$$

by applying (0.2) and (0.3), we have

$$
\begin{aligned}
\mathcal{H}^{1}\left(Z\right) &\leq \frac{2n}{\tau}\,|E\cap Q|^{\frac{1}{n}}\\
&< \frac{2n}{\tau}\left(2C_1\right)^{\frac{1}{n}}r^{\frac{n}{n-1}} < \frac{r}{2}.
\end{aligned}
$$

Therefore, there exists $s_0 \in \left(\frac{-r}{4},\frac{r}{4}\right)$ such that

$$
\sum_{i=1}^{n}\mathcal{H}^{n-1}\left(E\cap\{x=(x_1,\cdots,x_n)\in Q : x_i = s_0\}\right) < \frac{\tau}{2}\,|E\cap Q|^{\alpha}. \tag{0.6}
$$

Using the hyperplanes $\{x : x_i = s_0\}$, we decompose Q into the union of 2^n smaller n-dimensional rectangles $\{Q_1, Q_2, \cdots, Q_{2^n}\}$. Each of these Q_i's is located in one corner of Q, and two of these rectangles are in fact n dimensional cubes and with edge lengths $\frac{r}{2} \pm s_0$. Since $s_0 \in \left(-\frac{r}{4}, \frac{r}{4}\right)$, we have

$$\frac{r}{4} < \frac{r}{2} \pm s_0 < \frac{3r}{4}. \tag{0.7}$$

Now, if

$$\max_i |E \cap Q_i| \leq (1-\delta)\,|E \cap Q|\,,$$

then we can rearrange $\{Q_i\}$ into two groups

$$V_1 = \cup\left\{\bar{Q}_{i_1}, \bar{Q}_{i_2}, \cdots, \bar{Q}_{i_k}\right\} \text{ and } V_2 = Q \backslash V_1$$

such that

$$\delta\,|E \cap Q| \leq |E \cap V_i| \leq (1-\delta)\,|E \cap Q|$$

for each $i = 1, 2$. A well known inequality says

$$(2 - 2^\alpha)\,(\min(a,b))^\alpha \leq a^\alpha + b^\alpha - (a+b)^\alpha\,, \text{ for any } a, b \geq 0.$$

Therefore,

$$
\begin{aligned}
&|E\cap V_1|^{\alpha}+|E\cap V_2|^{\alpha}-|E\cap Q|^{\alpha}\\
\geq\ &\left(2-2^{\alpha}\right)\min\left\{|E\cap V_1|^{\alpha},|E\cap V_2|^{\alpha}\right\}\\
\geq\ &\left(2-2^{\alpha}\right)\delta^{\alpha}\,|E\cap Q|^{\alpha}\geq\frac{4\tau}{C_n}\,|E\cap Q|^{\alpha}.
\end{aligned}
$$

Hence, by the isoperimetric inequality, (0.6), and (0.5),

$$
\begin{aligned}
P(E,Q)\ \geq\ & P(E,V_1)+P(E,V_2)\\
\geq\ & C_n\,|E\cap V_1|^{\alpha}+C_n\,|E\cap V_2|^{\alpha}-\mathcal{H}^{n-1}(E\cap\partial V_1)-\mathcal{H}^{n-1}(E\cap\partial V_2)\\
\geq\ & C_n\,|E\cap V_1|^{\alpha}+C_n\,|E\cap V_2|^{\alpha}-2\frac{\tau}{2}\,|E\cap Q|^{\alpha}-\mathcal{H}^{n-1}(E\cap\partial Q)\\
\geq\ & C_n\,|E\cap Q|^{\alpha}+4\tau\,|E\cap Q|^{\alpha}-\tau\,|E\cap Q|^{\alpha}-\tau\,|E\cap Q|^{\alpha}\\
=\ & (C_n+2\tau)\,|E\cap Q|^{\alpha}.
\end{aligned}
$$

This gives one the first case.

If

$$
\max_i|E\cap Q_i|>(1-\delta)\,|E\cap Q|,
$$

then at least one of the two cubes in Q_i's, say Q_1, satisfies

$$\begin{aligned} |E \cap Q_1| &< \delta\, |E \cap Q| \\ &\leq \delta\,(2C_1)\, r^{\frac{n^2}{n-1}} \\ &\leq \delta\,(2C_1)\,(4r_1)^{\frac{n^2}{n-1}} \leq 2C_1\,(r_1)^{\frac{n^2}{n-1}}, \end{aligned}$$

where $r_1 = \frac{r}{2} \pm s_0$ is the edge length of Q_1. By proposition 1, there exists a smaller cube $\tilde{Q} \subset Q_1$ such that $\tilde{Q} \in A_\tau(E)$ with edge length $\tilde{r} \in \left(\frac{r_1}{2}, r_1\right) \subset \left(\frac{r}{8}, \frac{3r}{4}\right)$, due to (0.7). This completes the second part. $\square$

Classification

Corollary 2. *Let $E \subset \Omega$ be a set of finite perimeter with $|E| < |\Omega|$. Then one of the following two cases must be true:*

1. either for any $\lambda > 0$, there exists a cube $Q \in A_\tau(E)$ with edge length $r < \lambda$ and
$$P(E, Q) \geq (C_n + 2\tau)\, |E \cap Q|^\alpha ;$$
2. or there exists a sequence of cubes $\{Q_i\} \subset A_\tau(E)$ such that
$$Q_{i+1} \subset Q_i$$
and their edge lengths satisfy
$$\frac{1}{8} r_i \leq r_{i+1} \leq \frac{3}{4} r_i$$
for each i.

Remark 2. *In the second case of corollary 2, we may associate a family of open cubes to it as follows. Let* Q_0 *be any given cube in* Ω. *By picking the first cube* Q_1 *inside* Q_0, *we get a sequence of cubes* $\{Q_i\}_{i=1}^{\infty}$ *as in the second case of corollary 2, and all these smaller cubes are contained in* Q_0. *By rescaling and translation, each cube* Q_i *is the image of* $[-1,1]^n$ *under some affine map* f_i *for each* $i = 0, 1, \cdots$. *Using these affine maps, we define a continuous map* $F : [-1,1]^n \times (0, +\infty) \to \mathbb{R}^n$ *by setting*

$$F(x,s) = \begin{cases} \frac{s}{r_1}\left(f_0(x) - f_0(0)\right) + f_0(0), & \text{if } s > r_0 \\ \frac{1}{r_i - r_{i+1}}\left((s - r_{i+1}) f_i(x) + (r_i - s) f_{i+1}(x)\right), & \text{if } s \in [r_{i+1}, r_i] \end{cases}$$

for some i. *Note that, for each* $s > 0$,*the image*

$$F_s = F(Q_0, s)$$

is also a cube with edge length s. *Also,* $F_{r_i} = Q_i$ *for each* $i = 0, 1, 2, \cdots$, *and*

$$F_s \subseteq F_t$$

whenever $s \le t$. *Moreover, if* $s \in [r_{i+1}, r_i]$ *for some* $i = 1, 2, \cdots$, *we have*

$$Q_{i+1} \subseteq F_s \subseteq Q_i \subset Q_0$$

and

$$\begin{aligned} |E \cap F_s| &\leq |E \cap Q_i| \\ &\leq 2C_1 (r_i)^{\frac{n^2}{n-1}} \\ &\leq 2C_1 (8s)^{\frac{n^2}{n-1}}. \end{aligned}$$

Therefore, $|E \cap F_s|$ is continuous in s and

$$|E \cap F_s| \leq C_E s^{\frac{n^2}{n-1}}$$

for any $s \in (0, r_1)$, where $C_E = 2C_1 8^{\frac{n^2}{n-1}}$. Similarly, for any open ball B in Ω, we may pick the first cube Q_1 inside B, and then construct a family of cubes $\{F_s\}$ as above. If we set K_s to be the largest open ball of diameter s inscribed in the cube F_s, then we get a family of open balls $\{K_s\}$ and $|K_s \cap E|$ is also continuous in s with

$$|E \cap K_s| \leq C_E s^{\frac{n^2}{n-1}}.$$

Lemma 1. *Let E be any* $\mathbf{T}$ *minimizer in* $\mathcal{F}_\sigma$. *Then the second case of corollary will happen.*

Proof. It is trivial if E has exterior points. Therefore, we may assume that E has no exterior points in Ω. Under this assumption, we will prove the result by showing that the first case in corollary 2 will not happen here.

Assume that there exists a cube Q in $A_\tau(E)$ such that

$$P(E, Q) \geq (C_n + 2\tau) |E \cap Q|^\alpha \tag{0.8}$$

and its edge length

$$r < \lambda = \left(\frac{\tau}{2^\beta C (2C_1)^{\beta-\alpha}} \right)^{\frac{n-1}{n^2(\beta-\alpha)}}. \tag{0.9}$$

Then, we consider another set

$$\tilde{E} = (E \backslash Q) \cup B,$$

where B is the ball having the same center as Q and with

$$|B| = |E \cap Q| \leq 2C_1 r^{\frac{n^2}{n-1}} < \frac{1}{2} r^n.$$

Note that B is strictly contained in Q. Since Q is in $A_\tau(E)$ and E has no exterior points, we have

$$0 < |E \cap Q| \leq 2C_1 r^{\frac{n^2}{n-1}}.$$

Therefore, by (0.9),

$$\begin{aligned} C\left|E \Delta \tilde{E}\right|^\beta &\leq C\left(2|E \cap Q|\right)^\beta \\ &= 2^\beta C|E \cap Q|^{\beta-\alpha}|E \cap Q|^\alpha \\ &\leq 2^\beta C\left(2C_1 r^{\frac{n^2}{n-1}}\right)^{\beta-\alpha}|E \cap Q|^\alpha \\ &< \tau|E \cap Q|^\alpha. \end{aligned}$$

Now, $|E| = \left|\tilde{E}\right|$ and

$$\begin{aligned}
\mathbf{T}\left(\tilde{E}\right) &= P\left(\tilde{E},\Omega\right) + \mathbf{G}\left(\tilde{E}\right) \\
&= P\left(E,\Omega\right) - P\left(E,Q\right) + \mathcal{H}^{n-1}\left(E\cap\partial Q\right) + C_n\left|E\cap Q\right|^{\alpha} + \mathbf{G}\left(\tilde{E}\right) \\
&\leq P\left(E,\Omega\right) - \tau\left|E\cap Q\right|^{\alpha} + \mathbf{G}\left(\tilde{E}\right), \text{ by } (0.8) \\
&< P\left(E,\Omega\right) - C\left|E\Delta\tilde{E}\right|^{\beta} + \mathbf{G}\left(\tilde{E}\right) \\
&\leq P\left(E,\Omega\right) + \mathbf{G}\left(E\right) = \mathbf{T}\left(E\right).
\end{aligned}$$

This is a contradiction with the minimality of E. Therefore, by the corollary 2, only the second case of the corollary will happen here. □

Suppose E is a $\mathbf{T}$ minimizer in $\mathcal{F}_\sigma$. We may consider another operator

$$\tilde{\mathbf{T}}(F) = P(F, \Omega) + \tilde{\mathbf{G}}(F)$$

for any $F \in \mathcal{F}_{1-\sigma}$, where $\tilde{\mathbf{G}}(F) = \mathbf{G}(F^c)$.

Lemma 2. *E is a $\mathbf{T}$ minimizer in $\mathcal{F}_\sigma$ if and only if E^c is a $\tilde{\mathbf{T}}$ minimizer in $\mathcal{F}_{1-\sigma}$.*

Proof. This is because

$$\begin{aligned}
& \mathbf{T}(E) \leq \mathbf{T}(F) \\
\Longleftrightarrow\ & \mathbf{G}(E) + P(E, \Omega) \leq \mathbf{G}(F) + P(F, \Omega) \\
\Longleftrightarrow\ & \tilde{\mathbf{G}}(E^c) + P(E^c, \Omega) \leq \tilde{\mathbf{G}}(F^c) + P(F^c, \Omega) \\
\Longleftrightarrow\ & \tilde{\mathbf{T}}(E^c) \leq \tilde{\mathbf{T}}(F^c),
\end{aligned}$$

for any $F \in \mathcal{F}_\sigma$. $\square$

Theorem 3. *Let E be any $\mathbf{T}$ minimizer in $\mathcal{F}_\sigma$. Then E has both interior points and exterior points in Ω.*

Proof. Assume E has no exterior points. Since $0 < |E| = \sigma|\Omega| < |\Omega|$, there exists at least one open cube Q in Ω such that $|E \cap Q| > 0$, and also an open ball B in Ω such that $|E^c \cap B| > 0$. We may also require that Q and B are disjoint. Now, by lemma 1, lemma 2 and the remark 2, there exist a family of n-dimensional open cubes $\{F_s\}$ for E and a family of n-dimensional open balls $\{K_s\}$ for E^c such that

1. for each $s > 0$, both the edge length of F_s and the diameter of K_s equal to s;
2. let s_0 be the edge length of Q and t_0 be the diameter of B, then $F_{s_0} = Q$ and $K_{t_0} = B$;
3. whenever $s < t$, we have $F_s \subseteq F_t$ and $K_s \subseteq K_t$;
4. there exists a decreasing sequence of positive numbers $\{s_i\}_{i=0}^{\infty}$ with limit 0 such that $F_{s_i} \in A_T(E)$ for each $i = 1, 2, \cdots$;
5. both $|F_s \cap E|$ and $|K_s \cap E^c|$ are nondecreasing continuous functions of $s \in$

$(0, +\infty)$.

6. moreover, for any $s \le s_1$, we have

$$0 < |F_s \cap E| \le C_E s^{\frac{n^2}{n-1}} \tag{0.10}$$

for some constant $C_E > 0$. Also, there exists a positive number $t_1 \le t_0$ such that for any $t \le t_1$, we have

$$0 < |K_t \cap E^c| \le C_{E^c} t^{\frac{n^2}{n-1}} \tag{0.11}$$

for some constant $C_{E^c} > 0$.

Now, we pick a positive number $\epsilon_o \le \min\{s_1, t_1\}$ small enough so that

$$2n\left(C_{E^c}\right)^{1/n} \epsilon_0^{\frac{1}{n-1}} \le \tau \tag{0.12}$$

and

$$\left|F_{\epsilon_0} \cap E\right| < |B \cap E^c|\,.$$

For any $s \in (0, \epsilon_0)$, since

$$|F_s \cap E| \le \left|F_{\epsilon_0} \cap E\right| < |B \cap E^c| = \left|K_{t_0} \cap E^c\right|,$$

by the mean value theorem, there exists at least one $t \le t_0$ such that

$$\left| K_t \cap E^c \right| = \left| F_s \cap E \right| . \tag{0.13}$$

Since E has no exterior points, we have $|F_s \cap E| > 0$. By the fact that $\lim_{s \to 0} |K_s \cap E^c| = 0$, the set of all t satisfying (0.13) must have a minimum in $(0, t_0)$, and we denote this minimum by $g(s)$. Thus, $g(s) \in (0, t_0)$ and

$$\left| F_s \cap E \right| = \left| K_{g(s)} \cap E^c \right| . \tag{0.14}$$

Note that since $F_s \subseteq Q$ and $K_{g(s)} \subseteq K_{t_0} = B$, we know that F_s and $K_{g(s)}$ are still disjoint. Now, we fix an $s \in (0, \epsilon_0)$ small enough so that $F_s \in A_\tau(E)$,

$$\frac{2n}{\epsilon_0} (C_E)^{1/n} s^{\frac{n}{n-1}} \le \tau \tag{0.15}$$

and

$$C \left(C_E s^{\frac{n^2}{n-1}} \right)^{\beta - 1 + \frac{1}{n}} < \tau . \tag{0.16}$$

For this particular s, we consider the set

$$\tilde{E} = (E - F_s) \cup K_{g(s)} .$$

Then, by (0.14), we have $|\tilde{E}| = |E| = \sigma|\Omega|$ and

$$P\left(\tilde{E}, \Omega\right) = P(E, \Omega) - P(E, F_s) + \mathcal{H}^{n-1}(E \cap \partial F_s) - P\left(E^c, K_{g(s)}\right) + \mathcal{H}^{n-1}\left(E^c \cap \partial K_{g(s)}\right). \tag{0.17}$$

By the isoperimetric inequality (),

$$P(E, F_s) + \mathcal{H}^{n-1}(E \cap \partial F_s) \geq C_n |E \cap F_s|^\alpha .$$

Also, by (**??**),

$$
\begin{aligned}
&\mathcal{H}^{n-1}\left(E^c \cap \partial K_{g(s)}\right) - P\left(E^c, K_{g(s)}\right) \\
\le\ & \frac{2n}{g(s)}\left|E^c \cap K_{g(s)}\right|, \text{ by } (0.14) \\
\le\ & \begin{cases} \frac{2n}{g(s)}\left|E\cap F_s\right|^{\alpha}\left(C_{E^c}\right)^{1/n} g(s)^{\frac{n}{n-1}}, & \text{if } g(s) \le \epsilon_0, \text{ by } (0.11) \\ \frac{2n}{g(s)}\left|E\cap F_s\right|^{\alpha}\left(C_{E}\right)^{1/n} s^{\frac{n}{n-1}}, & \text{if } g(s) > \epsilon_0, \text{ by } (0.10) \end{cases} \\
\le\ & \begin{cases} \left|E\cap F_s\right|^{\alpha} 2n\left(C_{E^c}\right)^{1/n} \epsilon_0^{\frac{1}{n-1}}, & \text{if } g(s) \le \epsilon_0 \\ \left|E\cap F_s\right|^{\alpha}\frac{2n}{\epsilon_0}\left(C_{E}\right)^{1/n} s^{\frac{n}{n-1}}, & \text{if } g(s) > \epsilon_0, \end{cases} \\
\le\ & \tau\left|E\cap F_s\right|^{\alpha}, \text{ by } (0.15) \text{ and } (0.12).
\end{aligned}
$$

Therefore, by (0.17), the fact $F_s \in A_\tau(E)$, and (**??**), we have

$$
\begin{aligned}
P\left(\tilde{E},\Omega\right) &\le P(E,\Omega) - C_n\left|E\cap F_s\right|^{\alpha} + 2\mathcal{H}^{n-1}\left(E\cap\partial F_s\right) + \tau\left|E\cap F_s\right|^{\alpha} \\
&\le P(E,\Omega) - C_n\left|E\cap F_s\right|^{\alpha} + 3\tau\left|E\cap F_s\right|^{\alpha} \\
&\le P(E,\Omega) - \tau\left|E\cap F_s\right|^{\alpha},
\end{aligned}
$$

due to (**??**). Hence,

$$\begin{aligned}\mathbf{T}\left(\tilde{E}\right) &= P\left(\tilde{E},\Omega\right)+\mathbf{G}\left(\tilde{E}\right)\\ &\leq P\left(E,\Omega\right)-\tau\left|E\cap F_s\right|^{\alpha}+\mathbf{G}\left(E\right)+C\left|E\cap F_s\right|^{\beta}\\ &= \mathbf{T}\left(E\right)+\left|E\cap F_s\right|^{\alpha}\left(C\left|E\cap F_s\right|^{\beta-1+\frac{1}{n}}-\tau\right)\\ &< \mathbf{T}\left(E\right).\end{aligned}$$

The last inequality follows from (0.10) and (0.16). This contradicts with the $\mathbf{T}$ minimality of E. Therefore, E must have an exterior points. Since, by lemma 2, E^c is a $\tilde{\mathbf{T}}$ minimizer, we see E^c have also some exterior points. Therefore, E has both interior points and exterior points.

$\square$

Proof of the Main Theorem

Since E is an T minimizer, E has both interior points and exterior points in Ω. Thus, there exist two open balls B_1 and B_2 in Ω of same radius $2R$ such that

$$|E \cap B_1| = 0 \text{ and } |E^c \cap B_2| = 0.$$

Let F be any set of finite perimeter with $E \Delta F$ contained in some open ball B_ρ of radius $\rho < R$.
Suppose that $\left|E \cap B_\rho\right| \geq \left|F \cap B_\rho\right|$. We can move a ball B in $B_1 \backslash B_\rho$ of radius R, while remaining strictly in $\Omega - \overline{B}(\rho)$, until it reaches a new position B_3 such that

$$|B_3 \cap E^c| = \left|E \cap B_\rho\right| - \left|F \cap B_\rho\right|.$$

Let $F_\rho = F \cup B_3$. Then $\left|F_\rho\right| = |E|$. Since E is a $\mathbf{T}$ minimizer, we get

$$\mathbf{T}(E) \leq \mathbf{T}\left(F_\rho\right).$$

That is,

$$\mathbf{G}(E)+P\left(E,\Omega\right)\leq\mathbf{G}(F_\rho)+P\left(F_\rho,\Omega\right).$$

Therefore,

$$\begin{aligned}P\left(E,\Omega\right) &\leq \mathbf{G}(F_\rho)-\mathbf{G}(E)+P\left(F_\rho,\Omega\right)\\ &\leq C|E\Delta F_\rho|^{\beta}+P\left(F,\Omega\right)-P\left(B_3\cap E^c,B_3\right)+\mathcal{H}^{n-1}\left(\partial B_3\cap E^c\right)\\ &\leq P\left(F,\Omega\right)+C|E\Delta F_\rho|^{\beta}+\frac{n}{R}\left|B_3\cap E^c\right|\\ &\leq P\left(F,\Omega\right)+c|E\Delta F|^{\min(1,\beta)},\end{aligned}$$

where $c=C+\frac{n}{R}$.
We arrive at the same relation

$$P\left(E,\Omega\right)\leq P\left(F,\Omega\right)+c\left|E\Delta F\right|^{\min(1,\beta)}$$

if we suppose $\left|E\cap B_\rho\right|\leq\left|F\cap B_\rho\right|$.

Notations

We mention here only the basic notations and definitions about perimeters.

- We assume that Ω is an open (bounded) subset of $\mathbb{R}^n$ with $n \geq 2$.
- If $E \subseteq \Omega$, $|E|$ is the Lebesgue measure of E,
- $\chi_E(x)$ is the characteristic function of E.
- $\mathcal{H}^s(\cdot)$ denotes the s dimensional Hausdorff measure.
- $E \Delta F$ is the symmetric difference $(E \backslash F) \cup (F \backslash E)$.
- Finally, E^c is the complement of E in Ω.

Sets of finite perimeter

Recall that a function $f \in L^1(\Omega)$ is of bounded variation in Ω if

$$\| Df \| (\Omega) = \sup \left\{ \int_\Omega f div \phi dx : \phi \in C_0^1(\Omega, \mathbb{R}^n), |\phi|(x) \leq 1 \right\} < \infty. \quad \textbf{(0.18)}$$

A set $E \subset \Omega$ is said to be of finite perimeter in Ω if its characteristic function χ_E is of bounded variation in Ω. We will use the notation $P(E, \Omega)$ for the perimeter so that

$$P(E, \Omega) = \| D\chi_E \| (\Omega).$$

For $\partial E \cap \Omega$ sufficiently smooth, $P(E, \Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$.
Moreover, for each n-dimensional open cube $Q \subset \Omega$, a relative isoperimetric inequality says that

$$\gamma(n) P(E, Q) \geq \min \left\{ |E \cap Q|^\alpha, |E^c \cap Q|^\alpha \right\} \quad \textbf{(0.19)}$$

for some constant $\gamma(n) > 0$.

Reduced boundary