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The Geodesic Problem in Quasimetric Spaces

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Abstract In this article, we study the geodesic problem in a generalized metric space, in which the distance function satisfies a relaxed triangle inequality $d(x,y) \leq \sigma(d(x,z)+d(z,y))$ for some constant $\sigma \geq 1$, rather than the usual triangle inequality. Such a space is called a quasimetric space. We show that many well-known results in metric spaces (e.g. Ascoli-Arzelà theorem) still hold in quasimetric spaces. Moreover, we explore conditions under which a quasimetric will induce an intrinsic metric. As an example, we introduce a family of quasimetrics on the space of atomic probability measures. The associated intrinsic metrics induced by these quasimetrics coincide with the d_α metric studied early in the study of branching structures arisen in ramified optimal transportation. An optimal transport path between two atomic probability measures typically has a "tree shaped" branching structure. Here, we show that these optimal transport paths turn out to be geodesics in these intrinsic metric spaces.

Keywords Optimal transport path · Quasimetric · Geodesic distance · Branching structure

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1 Introduction

This article aims at studying some classical analysis problems in semimetric spaces, in which the distance is not required to satisfy the triangle inequity. During the au-

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thor's recent study of optimal transport path between probability measures, he observes that there exists a family of very interesting semimetrics on the space of atomic probability measures. These semimetrics satisfy a relaxed triangle inequality $d(x, y) \leq \sigma(d(x, z) + d(z, y))$ for some constant $\sigma \geq 1$, rather than the usual triangle inequality. Such semimetric spaces are called quasimetric spaces[1] in [6]. Moreover, these family of quasimetrics indeed induce a family of intrinsic metrics on the space of atomic probability measures. Furthermore, optimal transport paths studied early in [10–14] etc turn out to be exactly geodesics in these induced metric spaces. This observation motivates us to study the geodesic problem in quasimetric spaces in this article. Other closely related works on ramified optimal transportation may be found in [3–5, 7] etc.

This article is organized as follows. In Sect. 2, we first introduce the concept as well as some basic properties of quasimetric spaces, then we extend some well-known results (e.g. Ascoli-Arzelà theorem) about continuous functions in metric spaces to continuous functions in quasimetric spaces. After that, in Sect. 3, we consider the geodesic problem in quasimetric spaces. We show that every continuous quasimetric will induce an intrinsic pseudometric on the space. In case that the quasimetric is nice enough (e.g. either "ideal" or "perfect" in the sense of Definition 2.5 or Definition 3.14), then the quasimetric will indeed induce an intrinsic metric. In the end, we spend the last section in discussing our motivation example: optimal transport paths between atomic probability measures. We first introduce a family of quasimetrics on the space of atomic probability measures. Each of these quasimetric is both ideal and perfect, and thus it induces an intrinsic metric on the space of atomic probability measures. We showed that the d_α-metrics introduced in [10] is simply the intrinsic metrics induced by these quasimetrics. Furthermore, each geodesic in these length spaces corresponds to an optimal transport path studied in [10].

2 Continuous Maps in Quasimetric Spaces

2.1 Quasimetric Spaces

Definition 2.1 Let X be any nonempty set. A function $J : X \times X \to \mathbb{R}$ is called a quasimetric if for any $x, y, z \in X$, we have

(1) (non-negativity) $J(x, y) \geq 0$;
(2) (identity of indiscernibles) $J(x, y) = 0$ if and only if $x = y$
(3) (symmetry) $J(x, y) = J(y, x)$;
(4) (relaxed triangle inequality) $J(x, y) \leq \sigma[J(x, z) + J(z, y)]$ for some constant $\sigma \geq 1$.

When J is a quasimetric on X, the pair (X, J) is called *a quasimetric space*. Let $\sigma(J)$ denote the smallest number σ satisfying condition (2.1).

[1] When this article was submitted, the author used the term "nearmetric" as in [8] instead of "quasimetric". Later, Professor Nigel Kalton kindly let the author know the term "quasimetric" used in the book [6]. Thus, in the final version of the article, we replaced the previous term "nearmetric" with this more suitable term "quasimetric".

Every metric space is clearly a quasimetric space with $\sigma = 1$.

Example 2.2 Suppose d is a metric on a nonempty set X. Then, for any $\beta > 1$, $\lambda \geq 0$, $\mu > 0$, $J(x, y) = \lambda d(x, y) + \mu d(x, y)^\beta$ is typically not a metric on X. However, J defines a quasimetric on X with $\sigma(J) \leq 2^{\beta-1}$. Indeed,

$$\begin{aligned} J(x, y) &= \lambda d(x, y) + \mu d(x, y)^\beta \\ &\leq \lambda \left[d(x, z) + d(y, z)\right] + \mu \left[d(x, z) + d(y, z)\right]^\beta \\ &\leq \lambda \left[d(x, z) + d(y, z)\right] + 2^{\beta-1}\mu \left[d(x, z)^\beta + d(y, z)^\beta\right] \\ &\leq 2^{\beta-1}\left[J(x, z) + J(z, y)\right]. \end{aligned}$$

In Section 4 we will provide a family of interesting quasimetrics on the space of atomic probability measures.

More generally, suppose J is a distance function on X satisfying conditions (2.1), (2.1), (2.1) in Definition 2.1. For each n, let $\sigma_n(J)$ be the smallest number $\sigma_n \geq 1$ satisfying

$$J(x_1, x_{n+1}) \leq \sigma_n \sum_{i=1}^{n} J(x_i, x_{i+1}), \tag{2.1}$$

for any $x_1, \ldots, x_{n+1} \in X$. In particular, $\sigma_1(J) = 1$ and $\sigma_2(J) = \sigma(J)$.

Lemma 2.3 *Suppose (X, J) is a quasimetric space. Then, for each n,*

$$\sigma_n(J) \leq \sigma(J)^{n-1}.$$

Proof We show this using the mathematical induction. It is trivial when $n = 1$ or 2. Then, from condition (2.1), we see that for any n and any points $\{x_1, x_2, \ldots, x_n\}$ in X, we have

$$\begin{aligned} J(x_1, x_n) &\leq \sigma(J)\left(J(x_1, x_{n-1}) + J(x_{n-1}, x_n)\right) \\ &\leq \sigma(J)\left(\sigma(J)^{n-2}\sum_{i=1}^{n-2} J(x_i, x_{i+1}) + J(x_{n-1}, x_n)\right) \\ &\leq \sigma(J)^{n-1}\sum_{i=1}^{n-1} J(x_i, x_{i+1}) \quad \text{since } \sigma(J) \geq 1. \end{aligned}$$

Therefore, $\sigma_n(J) \leq \sigma(J)^{n-1}$ for all n. □

Proposition 2.4 *Suppose (X, J) is a quasimetric space. Then, for each n and m in $\mathbb{N}$,*

$$\sigma_{nm}(J) \leq \sigma_n(J)\,\sigma_m(J).$$

Proof Note that, for any $\{x_1, x_2, \dots, x_{mn+1}\}$ in X, from (2.1), we have

$$\begin{aligned}
&J(x_1, x_{mn+1}) \\
&\quad \le \sigma_n(J)\left(J(x_1, x_{m+1}) + J(x_{m+1}, x_{2m+1}) + \cdots + J\left(x_{(n-1)m+1}, x_{nm+1}\right)\right) \\
&\quad \le \sigma_n(J)\left(\sigma_m(J)\sum_{i=1}^{m} J(x_i, x_{i+1}) + \cdots + \sigma_m(J)\sum_{i=(n-1)m+1}^{nm} J(x_i, x_{i+1})\right) \\
&\quad = \sigma_n(J)\,\sigma_m(J)\sum_{i=1}^{nm} J(x_i, x_{i+1}).
\end{aligned}$$

Therefore,

$$\sigma_{nm}(J) \le \sigma_n(J)\,\sigma_m(J).$$

□

Clearly, $\sigma_n(J)$ is nondecreasing as n increases. Thus, we define

$$\sigma_\infty(J) := \lim_n \sigma_n(J) \tag{2.2}$$

for any quasimetric J on X.

Definition 2.5 Suppose J is a quasimetric on X. If $\sigma_\infty(J) < \infty$, then J is called an ideal quasimetric on X.

Note that J is an ideal quasimetric if and only if for some $\sigma \ge 1$,

$$J(x, y) \le \sigma \sum_{i=1}^{n} J(x_i, x_{i+1}), \tag{2.3}$$

for any finitely many points $x_1, \dots, x_{n+1} \in X$ with $x_1 = x$, $x_{n+1} = y$. The smallest σ satisfying (2.3) is just $\sigma_\infty(J)$.

A sequence $\{x_n\}$ is *convergent* to x in a quasimetric space (X, J) if $J(x_n, x) \to 0$, and we denote it by $x_n \xrightarrow{J} x$. A sequence $\{x_n\}$ is *Cauchy in* (X, J) if for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $J(x_n, x_m) \le \epsilon$ for all $n, m \ge N$. Since $J(x_n, x_m) \le \sigma(J)(J(x_n, x) + J(x, x_m))$, it follows that every convergent sequence in (X, J) is a Cauchy sequence. If every Cauchy sequence in (X, J) is convergent, then we say J is a *complete* quasimetric on X. A quasimetric J on X always gives a topology on X where a subset A is closed if it contains every point $a \in X$ for which there is some sequence $a_i \in A$ with $\lim_{i\to\infty} J(a_i, a) = 0$.

Definition 2.6 A quasimetric J on X is *continuous* if for any convergent sequences $x_n \xrightarrow{J} x$, $y_n \xrightarrow{J} y$, we have

$$J(x_n, y_n) \to J(x, y), \quad \text{as } n \to \infty. \tag{2.4}$$

If for any convergent sequences $x_n \xrightarrow{J} x$, $y_n \xrightarrow{J} y$, we have

$$J(x, y) \leq \liminf_n J(x_n, y_n), \tag{2.5}$$

then we say J is lower semicontinuous.

For instance, suppose J satisfies conditions (2.1), (2.1), (2.1) in Definition 2.1, and also the following condition

$$|J(x, y) - J(z, w)| \leq \sigma (J(x, z) + J(w, y)) \tag{2.6}$$

for any $x, y, z, w \in X$ and some $\sigma \geq 1$. By setting $z = w$, we get $J(x, y) \leq \sigma[J(x, z) + J(z, y)]$, and hence J is a quasimetric on X. Also, since for each n,

$$|J(x_n, y_n) - J(x, y)| \leq \sigma (J(x, x_n) + J(y, y_n)),$$

J is automatically satisfying the continuous condition (2.4) in this case. When J is indeed a metric on X, then (2.6) trivially holds.

2.2 Continuous Maps in Quasimetric Spaces

In this section, we extend some well-known results (see for instance in [9] or [1]) about continuous maps in metric spaces to continuous maps in quasimetric spaces.

Suppose (X, J) is a quasimetric space, and K is a compact metric space with a metric d_K. A map $f : K \to (X, J)$ is *continuous* if $J(f(x_n), f(x)) \to 0$ in X whenever $d_K(x_n, x) \to 0$ in K as $n \to \infty$. A map $f : K \to (X, J)$ is *uniformly continuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $J(f(x), f(y)) \leq \epsilon$ whenever $x, y \in K$ with $d_K(x, y) \leq \delta$. A map $f : K \to (X, J)$ is *Lipschitz* if there exists a constant $C \geq 0$ such that

$$J(f(x), f(y)) \leq C d_K(x, y)$$

for any $x, y \in K$. Let $C(K, (X, J))$ be the family of all continuous maps from K to (X, J), and $Lip(K, (X, J))$ be the family of all Lipschitz maps from K to (X, J).

Proposition 2.7 *Suppose J is a continuous quasimetric on X. Then, every continuous map $f : K \to (X, J)$ is uniformly continuous.*

Proof Suppose $f : K \to (X, J)$ is continuous. If f is not uniformly continuous, then there exists an $\epsilon > 0$, and two sequences $\{x_n\}$, $\{y_n\}$ in K such that $d(x_n, y_n) \leq \frac{1}{n}$, but $J(f(x_n), f(y_n)) \geq \epsilon$. By the compactness of K and taking subsequence if necessary, we may assume that both $\{x_n\}$ and $\{y_n\}$ converge to the same point $x^* \in K$. So, by the continuity of J in (2.4) and the continuity of f at x^*, we have

$$0 = J\left(f\left(x^*\right), f\left(x^*\right)\right) = \lim_{n\to\infty} J(f(x_n), f(y_n)) \geq \epsilon.$$

A contradiction. Thus, f must be uniformly continuous. $\square$

For any maps $f, h : K \to (X, J)$, let

$$J_\infty (f, h) := \sup_{x \in K} J (f (x), h (x)). \tag{2.7}$$

If $J_\infty(f_n, f) \to 0$, then we say that f_n is *uniformly convergent* to f.

Proposition 2.8 *Suppose J is a quasimetric on X. Then, J_∞ is a quasimetric on $C(K, (X, J))$.*

Proof For any $f, h \in C(K, (X, J))$, by definition (2.7), we have $J_\infty(f, h) \geq 0$ and $J_\infty(f, h) = J_\infty(h, f)$. Also, $J_\infty(f, h) = 0$ if and only if $f(x) = h(x)$ for all $x \in K$. Moreover, for any $g \in C(K, (X, J))$,

$$\begin{aligned} J_\infty (f, h) &= \sup_{x \in K} J (f (x), h (x)) \\ &\leq \sup_{x \in K} \sigma (J) \left[J (f (x), g (x)) + J (g (x), h (x))\right] \\ &\leq \sigma (J) \left[\sup_{x \in K} J (f (x), g (x)) + \sup_{x \in K} J (g (x), h (x))\right] \\ &= \sigma (J) \left[J_\infty (f, g) + J_\infty (g, h)\right]. \end{aligned}$$

Therefore, $(C(K, (X, J)), J_\infty)$ is also a quasimetric space. $\square$

Proposition 2.9 *Suppose $\{f_n : K \to (X, J)\}$ is a sequence of continuous maps. If $J_\infty(f_n, f) \to 0$, then f is also continuous.*

Proof Since $J_\infty(f_n, f) \to 0$, for any $\epsilon > 0$, there exists an n such that

$$\sup_{x \in K} J (f_n (x), f (x)) \leq \epsilon/3. \tag{2.8}$$

For any $x \in K$, since f_n is continuous at x, there exists a $\delta = \delta(x) > 0$ such that $J(f_n(x), f_n(y)) \leq \epsilon/3$ whenever $y \in K$ with $d_K(x, y) \leq \delta$. Therefore, by lemma 2.3 and (2.8), we have

$$\begin{aligned} J (f (x), f (y)) &\leq \sigma (J)^2 \left[J (f (x), f_n (x)) + J (f_n (x), f_n (y)) \right. \\ &\quad \left. + J (f_n (y), f (y))\right] \\ &\leq \epsilon \sigma (J)^2 \end{aligned}$$

and thus f is continuous at every $x \in K$. $\square$

Theorem 2.10 *Suppose (X, J) is a complete quasimetric space and J is lower semicontinuous. Then, the space $(C(K, (X, J)), J_\infty)$ is also a complete quasimetric space.*

Proof Let $\{f_n\}$ be any Cauchy sequence in $C(K,(X,J))$ with respect to J_∞. That is, for any $\epsilon > 0$, there exists an N such that whenever $m, n \geq N$, we have $J_\infty(f_n, f_m) \leq \epsilon$. So, for each $x \in K$, $\{f_n(x)\}$ is Cauchy in X. Since X is complete, $\{f_n(x)\}$ converges to some $f(x) \in X$ with respect to J. Now,

$$\begin{aligned} J_\infty(f_n, f) &= \sup_{x\in K} J(f_n(x), f(x)) \\ &\leq \sup_{x\in K} \lim_{m\to\infty} J(f_n(x), f_m(x)), \quad \text{because } J \text{ is lower semicontinuous} \\ &\leq \limsup_{m\to\infty} \left[\sup_{x\in K} J(f_n(x), f_m(x))\right] \leq \epsilon \end{aligned}$$

So, $J_\infty(f_n, f) \to 0$. By proposition 2.9, f is continuous. Hence, by proposition 2.8, J_∞ is a complete quasimetric on $C(K,(X,J))$. □

Definition 2.11 A subset $\mathcal{F}$ of $C(K,(X,J))$ is equicontinuous if for every $x \in K$ and $\epsilon > 0$, there is a $\delta = \delta(x,\epsilon) > 0$, such that whenever $y \in K$ with $d_K(x,y) \leq \delta$, we have $J(f(x), f(y)) \leq \epsilon$ for all $f \in \mathcal{F}$.

Now, we have the following Ascoli-Arzelà theorem in quasimetric spaces:

Theorem 2.12 *Suppose (X, J) is a complete quasimetric space and J is lower semicontinuous. A subset $\mathcal{F}$ of $(C(K,(X,J)), J_\infty)$ is precompact if and only if it is bounded and equicontinuous.*

Proof Suppose $\mathcal{F}$ is a precompact (i.e. every sequence has a convergent subsequence) subset of $C(K,(X,J))$. Then, for each fixed $\epsilon > 0$, there exists a finite subset $\{f_1,\ldots,f_k\}$ of $\mathcal{F}$ such that

$$\mathcal{F} \subset \bigcup_{i=1}^{k} B_{\epsilon/3}(f_i), \tag{2.9}$$

where the notation $B_\epsilon(g) = \{h \in C(K,(X,J)) | J_\infty(g,h) < \epsilon\}$. Otherwise, for any finite subset $\{f_1,\ldots,f_k\}$, there exists an $f_{k+1} \notin \bigcup_{i=1}^{k} B_{\epsilon/3}(f_i)$, and thus we get a sequence $\{f_k\}$ in $\mathcal{F}$. Since $J_\infty(f_m, f_n) \geq \epsilon/3$ for any $m \neq n$, we know $\{f_n\}$ does not contain any Cauchy subsequence, which contradicts to $\mathcal{F}$ being precompact. Therefore, (2.9) must be true, which also implies that $\mathcal{F}$ is bounded.

Now, for any $x \in K$ and each f_i in (2.9), there exists a $\delta_i > 0$ such that whenever $y \in K$ with $d_K(x,y) < \delta_i$, we have $J(f_i(x), f_i(y)) \leq \frac{\epsilon}{3}$. For every $f \in \mathcal{F}$, by (2.9), there is an $1 \leq i \leq k$ such that $J_\infty(f, f_i) \leq \frac{\epsilon}{3}$. We conclude that for any $y \in K$ with $d_K(x,y) < \delta = \min\{\delta_1,\ldots,\delta_k\}$, we have

$$\begin{aligned} J(f(x), f(y)) &\leq \sigma(J)^2 \left[J(f(x), f_i(x)) + J(f_i(x), f_i(y)) + J(f_i(y), f(y))\right] \\ &\leq \epsilon\sigma(J)^2. \end{aligned}$$

Therefore, $\mathcal{F}$ is equicontinuous at every $x \in K$.

On the other hand, suppose $\mathcal{F}$ is equicontinuous and bounded. Then, for any sequence $\{f_n\}$ in $\mathcal{F}$, by using the diagonal process and taking subsequence if necessary, we may assume $\{f_n\}$ is convergent to f on a countable dense subset S in K. We now prove that $\{f_n\}$ is Cauchy in $C(K, (X, J))$ with respect to J_∞. Indeed, for any $\epsilon > 0$, since $\mathcal{F}$ is equicontinuous and K is compact, there exists a finite many points $\{r_1, \ldots, r_k\}$ in S such that for any $x \in K$, there is a r_i, such that

$$J\left(f_n\left(x\right), f_n\left(r_i\right)\right) \leq \frac{\epsilon}{3}$$

for all n. Now, whenever m, n are large enough, for all $x \in K$,

$$\begin{aligned} &J\left(f_n\left(x\right), f_m\left(x\right)\right) \\ &\quad\leq \sigma\left(J\right)^2\left[J\left(f_n\left(x\right), f_n\left(r_i\right)\right) + J\left(f_n\left(r_i\right), f_m\left(r_i\right)\right) + J\left(f_m\left(r_i\right), f_m\left(x\right)\right)\right] \\ &\quad\leq \sigma\left(J\right)^2 \epsilon. \end{aligned}$$

Therefore, $\{f_n\}$ is a Cauchy sequence in $C(K, (X, J))$. By the completeness of $C(K, (X, J))$ stated in theorem 2.10, the sequence $\{f_n\}$ is convergent with respect to J_∞. Thus, $\mathcal{F}$ is precompact. □

Corollary 2.13 *Suppose (X, J) is a complete quasimetric space and J is lower semicontinuous. A subset $\mathcal{F}$ of $C(K, (X, J))$ is sequentially compact with respect to J_∞ if and only if it is closed, bounded and equicontinuous.*

3 Intrinsic Metrics Induced by Quasimetrics

This section is devoted to study the geodesic problem in a quasimetric space (X, J). Let $[a, b]$ be a bounded closed interval.

Definition 3.1 Let N be a natural number. A curve $f \in C([a, b], (X, J))$ is called an N-piecewise metric Lipschitz curve in (X, J) if there exists a partition

$$P_f = \{a = a_0 < a_1 < \cdots < a_N = b\}$$

of $[a, b]$ such that for each $i = 0, 1, \ldots, N-1$,

(1) J is a metric on the subset $f([a_i, a_{i+1}])$ of X and
(2) the restriction of f on $[a_i, a_{i+1}]$ is Lipschitz.

Here, requiring J to be a metric on $f([a_i, a_{i+1}])$ is the same as asking it to satisfy the triangle inequality: $J(f(t_1), f(t_2)) \leq J(f(t_1), f(t_2)) + J(f(t_2), f(t_3))$ for any $t_1, t_2, t_3 \in [a_i, a_{i+1}]$. Let

$$\mathcal{P}_N\left([a, b], (X, J)\right)$$

be the family of all $N-$piecewise metric Lipschitz curves in (X, J), and $\mathcal{P}([a, b], (X, J))$ be the union of $\mathcal{P}_N([a, b], (X, J))$ over all N's.

3.1 Length of Rectifiable Curves

Recall that when (X, d) is a metric space, and $f : [a, b] \to (X, d)$ is a (continuous) curve. Then, one may define its length as

$$L(f) = \sup_P V_P(f) \in [0, +\infty],$$

where the supremum is over all partitions P of $[a, b]$, and $V_P(f)$ is the variation of f over the partition $P = \{a = t_0 < t_1 < \cdots < t_N = b\}$ given by

$$V_P(f) = \sum_{i=1}^{N} d(f(t_{i-1}), f(t_i)).$$

In case f is Lipschitz, an equivalent formula for the length of f is

$$L(f) = \int_a^b |\dot{f}(t)|_d \, dt,$$

where $|\dot{f}(t)|_d$ is the metric derivative of f at $f(t)$ defined by

$$|\dot{f}(t)|_d := \lim_{s \to t} \frac{d(f(s), f(t))}{|s - t|},$$

provided the limit exists. When f is Lipschitz, $|\dot{f}(t)|_d$ exists almost everywhere, and is bounded and measurable in t.

Now, suppose (X, J) is a quasimetric space, and $f \in \mathcal{P}_N([a, b], (X, J))$. Then on each interval $[a_i, a_{i+1}]$, $f : [a_i, a_{i+1}] \to (X, J)$ is a Lipschitz curve in the metric space $(f([a_i, a_{i+1}]), J)$, and thus the length of the restriction of f on $[a_i, a_{i+1}]$ is well defined. As a result, we may define the length of f to be

$$L(f) := \sum_{i=0}^{N-1} L\left(f \llcorner_{[a_i, a_{i+1}]}\right).$$

In other words, we have

Definition 3.2 For any $f \in \mathcal{P}_N([a, b], (X, J))$, the length of f is defined as

$$L_J(f) := \int_a^b |\dot{f}(t)|_J \, dt,$$

where the metric derivative

$$|\dot{f}(t)|_J := \lim_{s \to t} \frac{J(f(s), f(t))}{|s - t|}$$

provided the limit exists. We may simply write $L_J(f)$ as $L(f)$ if J is obvious.

Lemma 3.3 *Suppose J is a continuous quasimetric on X, $C > 0$ is a constant, and $P = \{a = a_0 < a_1 < \cdots < a_N = b\}$ is a partition of the interval $[a, b]$. Then, for any $x, y \in X$, the family*

$$\mathcal{F} = \left\{ \begin{array}{l} f \in C([a,b],(X,J)) : f(a) = x,\ f(b) = y, \text{ and } J \text{ is a metric on} \\ f([a_i, a_{i+1}]) \text{ and } Lip(f\lfloor_{[a_i,a_{i+1}]}) \leq C, \text{ for each } i = 0, \ldots, N-1 \end{array} \right\}$$

is a bounded, closed and equicontinuous subset of $C([a,b],(X,J))$. Moreover, if f_n is uniformly convergent to f in J_∞, then,

$$L(f) \leq \liminf_n L(f_n).$$

Proof For any $g \in \mathcal{F}$ and any $t \in [a, b]$, we have $t \in [a_j, a_{j+1}]$ for some $j \leq N-1$ and

$$\begin{aligned} J(g(t), x) &= J(g(t), g(a)) \\ &= \sigma(J)^j \left(\sum_{i=0}^{j-1} J(g(a_i), g(a_{i+1})) + J\left(g\left(a_j\right), g(t)\right) \right) \\ &\leq \sigma(J)^j C|t-a| \leq C\sigma(J)^{N-1}|b-a| \end{aligned}$$

Therefore, $\mathcal{F}$ is bounded.

Suppose $\{f_n\}$ is any convergent sequence in $\mathcal{F}$ with respect to J_∞ with $f \in C([a,b],(X,J))$ being the limit. Then, for each fixed i, and any $t_1, t_2, t_3 \in [a_i, a_{i+1}]$, we have

$$J(f_n(t_1), f_n(t_2)) \leq J(f_n(t_1), f_n(t_3)) + J(f_n(t_3), f_n(t_2))$$

and

$$J(f_n(t_1), f_n(t_2)) \leq C|t_1 - t_2|.$$

Let $n \to \infty$, we have J is a metric on $f([a_i, a_{i+1}])$ and $Lip(f\lfloor[a_i, a_{i+1}]) \leq C$. Therefore, $f \in \mathcal{F}$. This shows that $\mathcal{F}$ is closed and also equicontinuous. Moreover, for any partition Q of $[a_i, a_{i+1}]$, the variation

$$V_Q\left(f\lfloor\left[a_i, a_{i+1}\right]\right) = \lim_n V_Q\left((f_n)\lfloor\left[a_i, a_{i+1}\right]\right) \leq \liminf_n L\left((f_n)\lfloor\left[a_i, a_{i+1}\right]\right).$$

So,

$$L\left(f\lfloor\left[a_i, a_{i+1}\right]\right) = \sup_Q V_Q\left(f\lfloor\left[a_i, a_{i+1}\right]\right) \leq \liminf_n L\left(f_n\lfloor\left[a_i, a_{i+1}\right]\right).$$

Hence,

$$L(f) = \sum_{i=0}^{N-1} L\left(f\lfloor_{[a_i,a_{i+1}]}\right) \leq \sum_{i=0}^{N-1} \liminf_n L\left(f_n\lfloor_{[a_i,a_{i+1}]}\right) = \liminf_n L(f_n).$$

□

Proposition 3.4 *Suppose (X, J) is a quasimetric space, and $f \in \mathcal{P}_N([a,b],(X,J))$. If $L(f)=0$, then f is a constant map.*

Proof $L(f)=0$ implies that $L(f\lfloor_{[a_i,a_{i+1}]})=0$ for each i. Thus, f is a constant on $[a_i, a_{i+1}]$ for each i. Since f is continuous, f is a constant on $[a,b]$. □

Since any Lipschitz curve in a metric space has an arc parametrization, by applying arc parametrizations piecewisely, we also have

Proposition 3.5 (*Reparametrization*) *For any $f \in \mathcal{P}_N([a,b],(X,J))$ and $L = L(f)$, there exists a homeomorphism $\phi : [0,L] \to [a,b]$ so that $\gamma = f \circ \phi \in \mathcal{P}_N([0,L],(X,J))$ has $|\dot{\gamma}(t)|_J = 1$ almost everywhere in $[0,L]$.*

3.2 The Geodesic Problem

Let N be a fixed natural number. For any $x, y \in X$, we consider the geodesic problem

$$\min\{L(f)\} \tag{3.1}$$

among all f in the family

$$Path_N(x,y) = \{f \in \mathcal{P}_N([0,1],(X,J)) \quad \text{with } f(0)=x;\ f(1)=y\}.$$

Note that, by a linear change of variable, one may replace $[0,1]$ in $Path_N(x,y)$ by any closed interval $[a,b]$ without changing the infimum value in the geodesic problem (3.1).

Definition 3.6 Suppose J is a quasimetric on X. For any $x, y \in X$, and $N \in \mathbb{N}$, define

$$D_J^{(N)}(x,y) = \inf\{L_J(f) : f \in Path_N(x,y)\}$$

whenever $Path_N(x,y)$ is not empty, and set $D_J^{(N)}(x,y)=\infty$ when $Path_N(x,y)$ is empty. Since $D_J^{(N)}(x,y)$ is a decreasing function of N, we define

$$D_J(x,y) = \lim_{N\to\infty} D_J^{(N)}(x,y).$$

Theorem 3.7 *Suppose J is a continuous complete quasimetric on a nonempty set X. For any $N \in \mathbb{N}$, and $x, y \in X$, the geodesic problem (3.1) admits a solution $f \in Path_N(x,y)$ provided that $Path_N(x,y)$ is not empty. So, $L(f) = D_J^{(N)}(x,y)$.*

Proof Suppose $Path_N(x,y)$ is not empty. Let $L=\inf\{L(f) : f \in Path_N(x,y)\}$. Note that for each $f \in Path_N(x,y)$, we have

$$J(x,y) \le \sigma(J)^{N-1} \sum_{i=0}^{N-1} J(f(a_i), f(a_{i+1}))$$

$$\leq \sigma (J)^{N-1} \sum_{i=0}^{N-1} L\left(f\lfloor_{[a_i,a_{i+1}]}\right) = \sigma (J)^{N-1} L(f).$$

This implies that if $L = 0$, then we have $J(x, y) = 0$. Therefore, $x = y$ and the constant $f(t) \equiv x$ is the desired solution.

So, without losing generality, we may assume that $L > 0$. Let $\{f_n\}$ be a length minimizing sequence in $Path_N(x, y)$ with $L(f_n) \to L$. Let

$$P_{f_n} = \left\{0 = a_0^{(n)} < a_1^{(n)} < \cdots < a_N^{(n)} = 1\right\}$$

be the partition of [0, 1], associated with f_n. By reparametrization if necessary, we may assume that each f_n is Lipschitz with $Lip(f_n) \leq 1.5L$ on $[a_i^{(n)}, a_{i+1}^{(n)}]$ for each $i = 0, \ldots, N-1$. Then, by choosing a subsequence if necessary, we may assume that each sequence $\{a_i^{(n)}\}$ is convergent to some point a_i as $n \to \infty$ for each $i = 0, 1, \ldots, N$. Using a linear change of variable, we may assume that for each i, $a_i^{(n)} = a_i$ and $Lip(f_n) \leq 2L$ on $[a_i, a_{i+1}]$. Now, $\{f_n\}$ is a sequence in the family

$$\mathcal{F} = \left\{ \begin{array}{l} f \in C([0,1], (X, J)) : f(0) = x, f(1) = y, \text{ and } J \text{ is a metric on} \\ f([a_i, a_{i+1}]) \text{ and } Lip(f\lfloor_{[a_i,a_{i+1}]}) \leq 2L, \text{ for each } i = 0, \ldots, N-1 \end{array} \right\}.$$

By Lemma 3.3, $\mathcal{F}$ is a bounded, closed and equicontinuous subset of $C([0, 1], (X, J))$. By the Ascoli-Arzelà theorem shown in corollary 2.13, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ in $\mathcal{F}$ is uniformly convergent to some $f \in \mathcal{F}$ with respect to J_∞. By the lower semicontinuity of L in the family $\mathcal{F}$, we have $L(f) \leq \liminf_k L(f_{n_k}) = L$. Therefore, f is a length minimizer in $Path_N(x, y)$. □

Note that each $D_J^{(N)}$ is a semimetric[2] on X in the sense that $D_J^{(N)}(x, y) \geq 0$, $D_J^{(N)}(x, y) = 0$ if and only if $x = y$, and $D_J^{(N)}(x, y) = D_J^{(N)}(y, x)$. In general, $D_J^{(N)}$ may fail to satisfy the triangle inequality. Nevertheless, we have

$$D_J^{(n+m)}(x, y) \leq D_J^{(n)}(x, z) + D_J^{(m)}(z, y)$$

for any m, n and $x, y, z \in X$. As a result, by letting $N \to \infty$, we have

Proposition 3.8 *Suppose J is a quasimetric on X, then D_J is a peudometric*[3] *on X.*

Since D_J is a pseudometric, D_J is a metric on X if and only if

$$D_J(x, y) > 0 \quad \text{whenever } x \neq y.$$

[2] A function $d : X \times X \to [0, +\infty)$ is a *semimetric* on X if d satisfies conditions (2.1), (2.1), (2.1) in Definition 2.1. So, a semimetric d is not required to satisfy the triangle inequality.

[3] A function $d : X \times X \to [0, +\infty)$ is a *pseudometric* on X if d satisfies conditions (2.1), (2.1) in Definition 2.1, and the triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$. But $d(x, y) = 0$ does not necessarily imply $x = y$.

When D_J becomes a metric on X. This metric is called the *intrinsic metric*, or *geodesic distance*, on X induced by the quasimetric J.

3.3 Examples of Metrics Induced by Quasimetrics

Now, we are interested in cases that D_J is indeed a metric on X.

3.3.1 Ideal Quasimetrics

Let J be any semimetric on X. For any $x, y \in X$, we set

$$d_J(x, y)$$

to be the infimum of

$$\sum_{i=1}^{n-1} J\left(x_i, x_{i+1}\right)$$

over all finitely many points $x_1, \ldots, x_n \in X$ with $x_1 = x$ and $x_n = y$.

This d_J defines a pseudometric on X, but not necessarily a metric on X.

Example 3.9 For instance, let $X = [0, 1]$ and $J(x, y) = |x - y|^p$ for some $p > 1$ defines a quasimetric on X. Then, for each n,

$$\begin{aligned} d_J\left(0, 1\right) &\leq \sum_{i=0}^{n-1} J\left(\frac{i}{n}, \frac{i+1}{n}\right) \\ &= \sum_{i=0}^{n-1} \left(\frac{1}{n}\right)^p = \frac{1}{n^{p-1}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus, $d_J(0, 1) = 0$, but $0 \neq 1$. Hence d_J is not a metric on X. Also, note that in this example, $Path_N(x, y)$ is empty whenever $x \neq y$. Thus, $D_J(x, y) = \infty$ whenever $x \neq y$.

As in the case of D_J, d_J is a metric on X if and only if

$$d_J\left(x, y\right) > 0 \quad \text{whenever } x \neq y.$$

Note also that

$$d_J\left(x, y\right) \leq D_J^{(N)}\left(x, y\right)$$

for each N, and thus,

$$d_J\left(x, y\right) \leq D_J\left(x, y\right).$$

Therefore, $d_J(x, y) > 0$ will automatically imply $D_J(x, y) > 0$. As a result, we have

Proposition 3.10 *Suppose J is a quasimetric on X. If d_J is a metric on X and $D_J(x, y) < \infty$ for every $x, y \in X$, then D_J also defines a metric on X.*

Remark 3.11 When J is indeed a metric on X, then both d_J and D_J are metrics. In this case, d_J is just the metric J itself, while D_J is the intrinsic metric induced by J.

In general, by means of definition, we have

$$d_J(x, y) \leq J(x, y) \leq \sigma_\infty(J)\, d_J(x, y),$$

where $\sigma_\infty(J)$ is defined as in (2.2).

Now, suppose J is an ideal quasimetric, then $\sigma_\infty(J) < \infty$ and J satisfies the condition

$$J(x_1, x_n) \leq \sigma_\infty(J) \sum_{i=1}^{n-1} J(x_i, x_{i+1})$$

for any finitely many points $\{x_1, x_2, \ldots, x_n\} \subset X$. Clearly, we have the following proposition:

Proposition 3.12 *Suppose (X, J) is an ideal quasimetric space. Then for any N and any $f \in \mathcal{P}_N([a, b], (X, J))$, we have*

$$J(f(a), f(b)) \leq \sigma_\infty(J)\, L(f).$$

Lemma 3.13 *Suppose J is an ideal quasimetric on X. Then, d_J is a metric on X. Moreover, if $D_J(x, y) < \infty$ for every $x, y \in X$, then D_J also defines a metric on X.*

Proof This is simply because when $x \neq y$, $d_J(x, y) \geq \frac{1}{\sigma_\infty(J)} J(x, y) > 0$. □

3.3.2 Perfect Quasimetrics

Here is another kind of quasimetric J which also induces a metric D_J.

Definition 3.14 A quasimetric J on X is a perfect near metric if for any $x, y \in X$, the value $D_J^{(N)}(x, y)$ becomes a real valued constant $D_J(x, y)$ when N is large enough.

Since for each N, $D_J^{(N)}(x, y) = 0$ if and only if $x = y$, we have the following theorem.

Proposition 3.15 *On a perfect quasimetric space (X, J), D_J defines a metric on X.*

When J is indeed a metric on X, then for each N, the metric $D_J^{(N)}$ agrees with the intrinsic metric induced by J. Thus, every metric space is automatically a perfect quasimetric space. In Sect. 4, we will discuss a family of very important perfect quasimetric spaces, which are not metric spaces.

Theorem 3.16 *Suppose (X, J) is a perfect quasimetric space, and the geodesic problem (3.1) has solution for N large enough. Then, (X, D_J) is a length space in the* *sense that for every $x, y \in X$, there exists a curve $f : [0, L] \to (X, D_J)$ such that $f(0) = x$, $f(L) = y$ and*

$$D_J(f(t), f(s)) = |t - s|$$

for every $t, s \in [0, L]$ where $L = D_J(x, y)$.

Proof For every $x, y \in X$, since (X, J) is a perfect quasimetric space, we have $D_J^{(N)}(x, y) = D_J(x, y) < \infty$ whenever N is large enough. Now, for each large enough N, there exists a curve $f : [0, L] \to (X, J)$ such that f is the length minimizer in $Path_N(x, y)$ with $L(f) = D_J^{(N)}(x, y) = D_J(x, y)$. Without losing generality, we may assume f has its arc parametrization. Now for any $0 \leq s < t \leq L$, we have

$$D_J(f(s), f(t)) \leq L\left(f\lfloor_{[s,t]}\right) = \int_s^t \left|\dot{f}\right|_J dt = t - s.$$

Similarly, $D_J(f(0), f(s)) \leq s$ and $D_J(f(t), f(L)) \leq L - t$. Thus, we have

$$\begin{aligned} L = D_J(x, y) &\leq D_J(f(0), f(s)) + D_J(f(s), f(t)) + D_J(f(t), f(L)) \\ &\leq s + (t - s) + (L - t) = L. \end{aligned}$$

Therefore, all inequalities becomes equalities at every step and for any $t, s \in [0, L]$, we have $D_J(f(t), f(s)) = |t - s|$. □

Corollary 3.17 *Suppose J is a complete, continuous, perfect quasimetric on X. Then, (X, D_J) is a length space.*

The curve f in Theorem 3.16 is called a *geodesic* from x to y in the perfect quasimetric space (X, J).

4 Optimal Transport Paths as Geodesics

We now begin to introduce a family of both ideal and perfect quasimetrics on the space of atomic probability measures.

4.1 A Family of Quasimetrics on the Space of Atomic Probability Measures

Let (Y, d) be any metric space. For any $y \in Y$, let δ_y be the Dirac measure centered at y. An atomic probability measure in Y is in the form of

$$\sum_{i=1}^{m} a_i \delta_{y_i}$$

with distinct points $y_i \in Y$, and $a_i > 0$ with $\sum_{i=1}^{m} a_i = 1$.

Given two atomic probability measures

$$\mathbf{a}=\sum_{i=1}^{m} a_i \delta_{x_i} \quad \text{and} \quad \mathbf{b}=\sum_{j=1}^{n} b_j \delta_{y_j} \tag{4.1}$$

in Y, a *transport plan* from $\mathbf{a}$ to $\mathbf{b}$ is an atomic probability measure

$$\gamma=\sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_{ij}\delta_{(x_i,y_j)} \tag{4.2}$$

in the product space $Y \times Y$ such that

$$\sum_{i=1}^{m} \gamma_{ij}=b_j \quad \text{and} \quad \sum_{j=1}^{n} \gamma_{ij}=a_i \tag{4.3}$$

for each i and j. Let $Plan(\mathbf{a},\mathbf{b})$ be the space of all transport plans from $\mathbf{a}$ to $\mathbf{b}$.

For any $\alpha<1$, we now introduce the functional H_α on transport plans. For any atomic probability measure γ in $Y \times Y$ of the form (4.2), we define

$$H_\alpha(\gamma):=\sum_{i=1}^{m}\sum_{j=1}^{n} \left(\gamma_{ij}\right)^{\alpha} d\left(x_i,y_j\right),$$

where d is the given metric on Y.

Using H_α, we may define

Definition 4.1 For any two atomic probability measures $\mathbf{a}, \mathbf{b}$ on Y, and $\alpha<1$, define

$$J_\alpha(\mathbf{a},\mathbf{b}):=\min\{H_\alpha(\gamma): \gamma \in Plan(\mathbf{a},\mathbf{b})\}.$$

For any given natural number $N \in \mathbb{N}$, let $\mathcal{A}_N(Y)$ be the space of all atomic probability measures

$$\sum_{i=1}^{m} a_i \delta_{x_i}$$

on Y with $m \le N$, and $\mathcal{A}(Y)=\bigcup_N \mathcal{A}_N(Y)$ be the space of all atomic probability measures on Y.

Proposition 4.2 *J_α defines a quasimetric on $\mathcal{A}_N(Y)$ with $\sigma(J_\alpha)\le N^{1-\alpha}$.*

Proof For any $\mathbf{a},\mathbf{b} \in \mathcal{A}_N(Y)$ in the form of (4.1), clearly $J_\alpha(\mathbf{a},\mathbf{b}) \ge 0$ and $J_\alpha(\mathbf{a},\mathbf{b})=J_\alpha(\mathbf{b},\mathbf{a})$.

If $J_\alpha(\mathbf{a},\mathbf{b})=0$, then there exists a $\gamma \in Plan(\mathbf{a},\mathbf{b})$ such that $H_\alpha(\gamma)=0$. Thus, $d(x_i,y_j)=0$ whenever $\gamma_{ij}\neq 0$. Since $\{y_j\}$'s are distinct, at most one of γ_{ij} can be nonzero for each i. On the other hand, by (4.3), at least one of γ_{ij} must be nonzero for each i. Therefore, for each i, there is a unique $j=\sigma(i)$ such that $x_i=y_j$ and $\gamma_{ij}=a_i=b_j$. This shows that $\mathbf{a}=\mathbf{b}$.

Now, we prove that J satisfies the relaxed triangle inequality as in condition (2.1) in Definition 2.1. Indeed, for any

$$\mathbf{a}=\sum_{i=1}^{m}a_i\delta_{x_i},\quad \mathbf{b}=\sum_{j=1}^{n}b_j\delta_{y_j}\quad\text{and}\quad \mathbf{c}=\sum_{k=1}^{h}c_k\delta_{z_k}$$

in $\mathcal{A}_N(Y)$, and any

$$u_{\mathbf{a}}^{\mathbf{c}}=\sum_{i=1}^{m}\sum_{k=1}^{h}u_{ik}\delta_{(x_i,z_k)}\in Path\,(\mathbf{a},\mathbf{c}),\qquad \tau_{\mathbf{c}}^{\mathbf{b}}=\sum_{j=1}^{n}\sum_{k=1}^{h}\tau_{kj}\delta_{(z_k,y_j)}\in Path\,(\mathbf{c},\mathbf{b}),$$

we denote

$$\gamma_{ij}=\sum_{k=1}^{h}\frac{u_{ik}\tau_{kj}}{c_k}$$

for each i, j. Note that

$$\sum_{i=1}^{m}\gamma_{ij}=\sum_{i=1}^{m}\left(\sum_{k=1}^{h}\frac{u_{ik}\tau_{kj}}{c_k}\right)=\sum_{k=1}^{h}\left(\sum_{i=1}^{m}\frac{u_{ik}\tau_{kj}}{c_k}\right)=\sum_{k=1}^{h}\tau_{kj}=b_j$$

and similarly $\sum_j\gamma_{ij}=a_i$. Therefore, we find a transport plan

$$\gamma=\sum_{i=1}^{m}\sum_{j=1}^{n}\gamma_{ij}\delta_{(x_i,y_j)}\in Plan\,(\mathbf{a},\mathbf{b}).$$

We now want to show

$$H_\alpha(\gamma)\le N\left(H_\alpha\left(u_{\mathbf{a}}^{\mathbf{c}}\right)+H_\alpha\left(\tau_{\mathbf{c}}^{\mathbf{b}}\right)\right).$$

Indeed,

$$\begin{aligned}
H_\alpha(\gamma)&=\sum_{i=1}^{m}\sum_{j=1}^{n}(\gamma_{ij})^\alpha d\left(x_i,y_j\right)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=1}^{h}\frac{u_{ik}\tau_{kj}}{c_k}\right)^\alpha d\left(x_i,y_j\right)\\
&\le\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{h}\left(\frac{u_{ik}\tau_{kj}}{c_k}\right)^\alpha\left(d\left(x_i,z_k\right)+d\left(z_k,y_j\right)\right),\quad\text{because }\alpha<1\\
&=\sum_{i=1}^{m}\sum_{k=1}^{h}\left(\sum_{j=1}^{n}\left(\frac{u_{ik}\tau_{kj}}{c_k}\right)^\alpha\right)d\left(x_i,z_k\right)\\
&\quad+\sum_{j=1}^{n}\sum_{k=1}^{h}\left(\sum_{i=1}^{m}\left(\frac{u_{ik}\tau_{kj}}{c_k}\right)^\alpha\right)d\left(z_k,y_j\right)
\end{aligned}$$

$$\leq N^{1-\alpha}\left(\sum_{i=1}^{m}\sum_{k=1}^{h}(u_{ik})^{\alpha}\, d\,(x_i, z_k)+\sum_{j=1}^{n}\sum_{k=1}^{h}\left(\tau_{kj}\right)^{\alpha} d\left(z_k, y_j\right)\right)$$
$$= N^{1-\alpha}\left(H_\alpha\left(u_{\mathbf{a}}^{\mathbf{c}}\right)+H_\alpha\left(\tau_{\mathbf{c}}^{\mathbf{b}}\right)\right),$$

where the 2nd inequality follows from the inequality $\sum_{i=1}^{N}(t_i)^{\alpha} \leq N^{1-\alpha}(\sum_{i=1}^{N} t_i)^{\alpha}$. Therefore, by taking infimum, we have

$$J_\alpha\,(\mathbf{a},\mathbf{b}) \leq N^{1-\alpha}\,(J_\alpha\,(\mathbf{a},\mathbf{c})+J_\alpha\,(\mathbf{c},\mathbf{b}))\,. \qquad \square$$

Proposition 4.3 *Suppose* (Y, d) *is a complete metric space. Then,* J_α *is a complete quasimetric on* $\mathcal{A}_N(Y)$.

Proof Let $\{\mathbf{a}_n\}$ be any Cauchy sequence in $\mathcal{A}_N(Y)$. Then, for any $\epsilon > 0$, there exists a natural number $\tilde{N}$, such that

$$J_\alpha\,(\mathbf{a}_n, \mathbf{a}_m) \leq \epsilon$$

whenever $n, m \geq \tilde{N}$. Note that each atomic probability measure $\mathbf{a}_n$ may be expressed as

$$\mathbf{a}_n = \sum_{i=1}^{N} a_i^{(n)} \delta_{x_i^{(n)}}$$

for some $a_i^{(n)} \geq 0$, $\sum_{i=1}^{N} a_i^{(n)} = 1$ and $x_i^{(n)} \in Y$.

Now, let $\gamma^{(n,m)}$ be an H_α minimizer in $Plan(\mathbf{a}_n, \mathbf{a}_m)$ with

$$J_\alpha\,(\mathbf{a}_n, \mathbf{a}_m) = H_\alpha\left(\gamma^{(n,m)}\right).$$

This transport plan $\gamma^{(n,m)}$ is expressed as

$$\gamma^{(n,m)} = \sum_{i,j=1}^{N} \gamma_{ij}^{(n,m)} \delta_{\left(x_i^{(n)}, x_j^{(m)}\right)}$$

for some $\gamma_{ij}^{(n,m)} \geq 0$ with $\sum_{i=1}^{N} \gamma_{ij}^{(n,m)} = a_j^{(m)}$ and $\sum_{j=1}^{N} \gamma_{ij}^{(n,m)} = a_i^{(n)}$ for all $i, j = 1, 2, \ldots, N$.

By picking a subsequence if necessary, without losing generality, we may use the diagonal argument and assume that for all $i, j = 1, 2, \ldots, N$ and all $n \geq \tilde{N}$

$$\gamma_{ij}^{(n,m)} \to \gamma_{ij}^{(n)}$$

as $m \to \infty$. Then, for each i, j and each $n \geq \tilde{N}$, we have

$$\sum_{i=1}^{N} \gamma_{ij}^{(n)} = \lim_{m\to\infty} \sum_{i=1}^{N} \gamma_{ij}^{(n,m)} = \lim_{m\to\infty} a_j^{(m)} \quad \text{and} \quad \sum_{j=1}^{N} \gamma_{ij}^{(n)} = a_i^{(n)}. \tag{4.4}$$

Let

$$a_j = \lim_{m\to\infty} a_j^{(m)}$$

for each j. If $a_j > 0$, then by (4.4), there exists an i such that $\gamma_{ij}^{(n)} > 0$. So

$$d\left(x_i^{(n)}, x_j^{(m)}\right) \le \frac{H_\alpha(\gamma^{(n,m)})}{[\gamma_{ij}^{(n,m)}]^\alpha} = \frac{J_\alpha(\mathbf{a}_n, \mathbf{a}_m)}{[\gamma_{ij}^{(n,m)}]^\alpha}$$

which implies that

$$\left\{x_j^{(m)}\right\}_{m=1}^{\infty}$$

is a Cauchy sequence in the complete metric space (Y, d). Thus, $x_j^{(m)} \to x_j$ as $m \to \infty$ for some $x_j \in Y$.

Let

$$\mathbf{a} = \sum_{a_j>0} a_j \delta_{x_j} \in \mathcal{A}_N(Y)$$

and for each $n \ge \tilde{N}$, let

$$\gamma^{(n)} = \sum_{ij} \gamma_{ij}^{(n)} \delta_{\left(x_i^{(n)}, x_j\right)}.$$

Then, $\gamma^{(n)} \in Plan(\mathbf{a}_n, \mathbf{a})$ and

$$\begin{aligned} J_\alpha(\mathbf{a}_n, \mathbf{a}) &\le H_\alpha\left(\gamma^{(n)}\right) \\ &= \sum_{ij} \left[\gamma_{ij}^{(n)}\right]^\alpha d\left(x_i^{(n)}, x_j\right) \\ &= \lim_{m\to\infty} \sum_{ij} \left[\gamma_{ij}^{(n,m)}\right]^\alpha d\left(x_i^{(n)}, x_j^{(m)}\right) \\ &= \lim_{m\to\infty} J_\alpha(\mathbf{a}_n, \mathbf{a}_m) \le \epsilon. \end{aligned}$$

Therefore, $\{\mathbf{a}_n\}$ is (subsequentially) convergent to $\mathbf{a}$ in $(A_N(Y), J_\alpha)$. This shows that $\mathcal{A}_N(Y)$ is complete with respect to the quasimetric J_α. □

Note that, in general, J_α may fail to be a metric on $\mathcal{A}_N(Y)$ as demonstrated in the following example.

Example 4.4 For any $\alpha < 1$, let y be a positive real number. Then, we consider three atomic measures in $Y = \mathbb{R}^2$:

$$\mathbf{a} = \frac{1}{2}\delta_{(-1,y+1)} + \frac{1}{2}\delta_{(1,y+1)}, \quad \mathbf{b} = \delta_{(0,0)} \quad \text{and} \quad \mathbf{c} = \delta_{(0,y)}.$$

Then,

$$J_\alpha(\mathbf{a},\mathbf{c})+J_\alpha(\mathbf{c},\mathbf{b})-J_\alpha(\mathbf{a},\mathbf{b})$$
$$=2\left(\frac{1}{2}\right)^\alpha\sqrt{2}+y-2\left(\frac{1}{2}\right)^\alpha\sqrt{1+(y+1)^2}<0$$

whenever y is large enough. Thus, J_α does not satisfy the triangle inequality.

4.2 Optimal Transport Paths between Atomic Probability Measures

Now, we want to show that the quasimetric J_α is both ideal and perfect. To achieve these results, we first recall some concepts about optimal transport paths between probability measures as studied in [10].

Let $\mathbf{a}$ and $\mathbf{b}$ be two fixed atomic probability measures in the form of (4.1).

Definition 4.5 A *transport path* from $\mathbf{a}$ to $\mathbf{b}$ is a weighted directed graph G consists of a vertex set $V(G)$, a directed edge set $E(G)$ and a weight function

$$w: E(G)\to(0,+\infty)$$

such that $\{x_1, x_2,\dots,x_k\}\cup\{y_1,y_2,\dots,y_l\}\subset V(G)$ and for any vertex $v\in V(G)$,

$$\sum_{\substack{e\in E(G)\\ e^-=v}} w(e)=\sum_{\substack{e\in E(G)\\ e^+=v}} w(e)+\begin{cases} a_i, & \text{if } v=x_i \text{ for some } i=1,\dots,k,\\ -b_j, & \text{if } v=y_j \text{ for some } j=1,\dots,l,\\ 0, & \text{otherwise}\end{cases} \tag{4.5}$$

where e^- and e^+denotes the starting and ending endpoints of each edge $e\in E(G)$.

Remark 4.6 The balance equation (4.5) simply means that the total mass flows into v equals to the total mass flows out of v. When G is viewed as a polyhedral chain or current, (4.5) can be simply expressed as

$$\partial G=\mathbf{b}-\mathbf{a}.$$

Also, when G is viewed as a vector valued measure, the balance equation is simply

$$\mathrm{div}(G)=\mathbf{a}-\mathbf{b}$$

in the sense of distributions.

Let $Path(\mathbf{a},\mathbf{b})$ be the space of all transport paths from $\mathbf{a}$ to $\mathbf{b}$.

Definition 4.7 For any $\alpha\le 1$, and any $G\in Path(\mathbf{a},\mathbf{b})$, define

$$\mathbf{M}_\alpha(G):=\sum_{e\in E(G)} w(e)^\alpha\, length(e).$$

Remark 4.8 In [10], the parameter α was restricted in $[0, 1]$. Later, the author observed that $\alpha < 0$ is also very interesting, and related to studying the dimension of fractals. So, negative α is also allowed here.

We first recite two lemmas that were proved in [10, Proposition 2.1] and [10, Definition 7.1 and Lemma 7.1] respectively.

Lemma 4.9 *For any transport path $G \in Path(\mathbf{a}, \mathbf{b})$, there exists another transport path $\tilde{G} \in Path(\mathbf{a}, \mathbf{b})$ such that*

$$\mathbf{M}_\alpha \left(\tilde{G}\right) \leq \mathbf{M}_\alpha (G),$$

the set of vertices $V(\tilde{G}) \subset V(G)$ and $\tilde{G}$ contains no cycles.

Here, a weighted directed graph $G = \{V(G), E(G), W : E(G) \to (0, 1]\}$ contains a *cycle* if for some $k \geq 3$, there exists a list of distinct vertices $\{v_1, v_2, \ldots, v_k\}$ in $V(G)$ such that for each $i = 1, \ldots, k$, either the segment $[v_i, v_{i+1}]$ or $[v_{i+1}, v_i]$ is a directed edge in $E(G)$, with the agreement that $v_{k+1} = v_1$. When a directed graph G contains no cycles, it becomes a directed tree.

Lemma 4.10 *For any transport path $G \in Path(\mathbf{a}, \mathbf{b})$ containing no cycles, there exists*

(1) *an $m \times n$ real matrix*

$$u = \left(u_{ij}\right) \quad \textit{with}$$

$$u_{ij} \geq 0, \quad \sum_{i=1}^{m} u_{ij} = b_j, \quad \sum_{j=1}^{n} u_{ij} = a_i \quad \textit{for each } i, j \textit{ and } \sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij} = 1,$$

(2) *and an $m \times n$ matrix*

$$g = \left(g_{ij}\right)$$

with each g_{ij} is either 0 *or an oriented polyhedral curve g_{ij} from x_i to y_j,*

such that

$$G = \sum_{i,j} u_{ij} g_{ij}$$

as real coefficients polyhedral chains.

By means of Lemma 4.9, it is easy to see that for each $\alpha \leq 1$, there exists an optimal transport path in $Path(\mathbf{a}, \mathbf{b})$ which minimizes the cost functional $\mathbf{M}_\alpha$.

For the sake of visualization we provide some numerical simulations (see the forthcoming paper [15]) for different values of α.

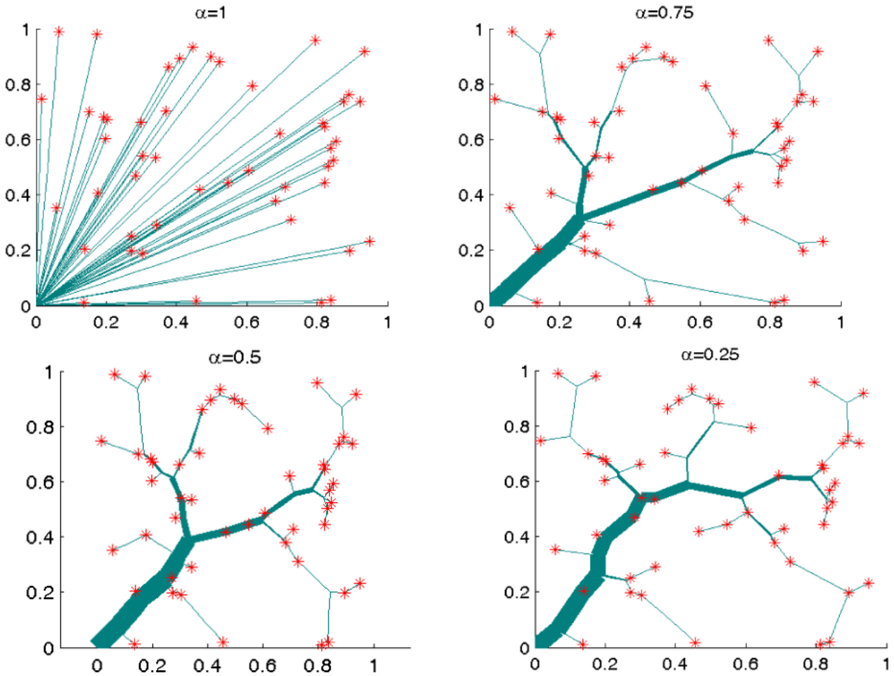

Fig. 1 A family of optimal transport paths from 50 random points to a single point

Example 4.11 Let $\{x_i\}$ be 50 random points in the square $[0, 1] \times [0, 1]$. Then, $\{x_i\}$ determines an atomic probability measure

$$\mathbf{a} = \sum_{i=1}^{50} \frac{1}{50} \delta_{x_i}.$$

Let $\mathbf{b} = \delta_O$ where $O = (0, 0)$ is the origin. Then an optimal transport path from $\mathbf{a}$ to $\mathbf{b}$ looks like Fig. 1 with $\alpha = 1, 0.75, 0.5$ and 0.25 respectively.

Example 4.12 Let $\{x_i\}$ be 100 random points in the rectangle $[-2.5, 2.5] \times [0, 1]$. Then, $\{x_i\}$ determines an atomic probability measure

$$\mathbf{a} = \sum_{i=1}^{100} \frac{1}{100} \delta_{x_i}.$$

Let $\mathbf{b} = \delta_O$ where $O = (0, 0)$ is the origin, and let $\alpha = 0.85$. Then an optimal transport path from $\mathbf{a}$ to $\mathbf{b}$ looks like Fig. 2.

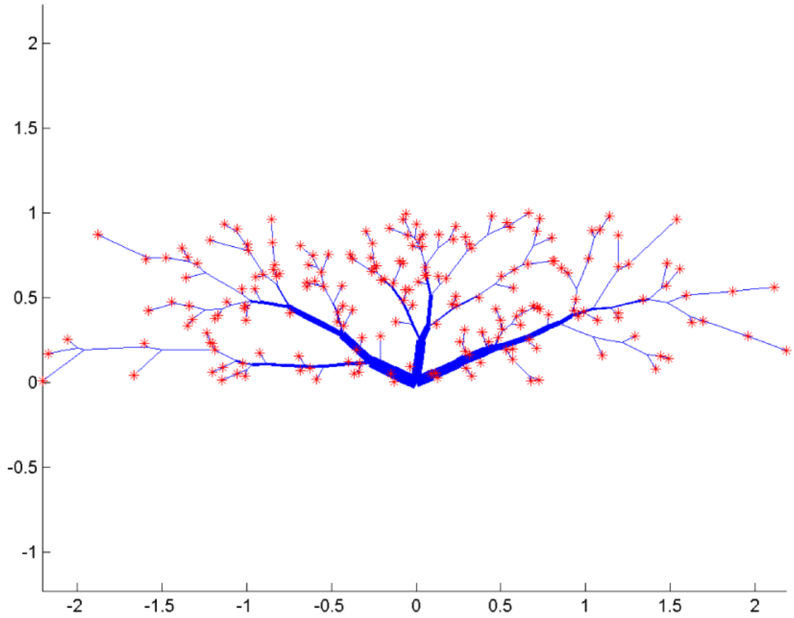

Fig. 2 An example of an optimal transport path from 100 random points to a single point with $\alpha = 0.85$

4.3 Relation between Optimal Transport Paths and Quasimetrics J_α

We now start to investigate the relationship between optimal transport path and the quasimetric J_α on $\mathcal{A}_N(Y)$. We first observe that any transport plan $\gamma \in Plan(\mathbf{a}, \mathbf{b})$ in the form of (4.2) determines a transport path $G_\gamma \in Path(\mathbf{a}, \mathbf{b})$. Indeed, we consider the weighted directed graph G_γ with

$$V\left(G_\gamma\right) = \{x_1, \ldots, x_m, y_1, \ldots, y_n\},$$
$$E\left(G_\gamma\right) = \left\{\text{a pair } [x_i, y_j] \text{ if } \gamma_{ij} \neq 0\right\},$$

and setting the weight $W([x_i, y_j]) = \gamma_{ij}$ for each i, j with $\gamma_{ij} \neq 0$. Moreover,

$$\mathbf{M}_\alpha\left(G_\gamma\right) = \sum_{e \in E(G_\gamma)} w(e)^\alpha \, length(e) = \sum_{i,j} \left(\gamma_{ij}\right)^\alpha d\left(x_i, y_j\right) = H_\alpha(\gamma).$$

Proposition 4.13 *For any $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)} \in \mathcal{A}(Y)$, there exists a transport path $G \in Path(\mathbf{a}^{(1)}, \mathbf{a}^{(k)})$ such that*

$$\mathbf{M}_\alpha(G) \leq \sum_{i=1}^{k-1} J_\alpha\left(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)}\right)$$

and G contains no cycles.

Proof Let γ_i be an optimal transport path from $\mathbf{a}^{(i)}$ to $\mathbf{a}^{(i+1)}$, for each $i = 1, 2, \ldots, k-1$. Each γ_i determines a transport path $G_{\gamma_i} \in Path(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)})$ as above. Then, viewed as real coefficients polyhedral chains,

$$G = \sum_{i=1}^{k-1} G_{\gamma_i}$$

is a transport path from $\mathbf{a}^{(1)}$ to $\mathbf{a}^{(k)}$. Moreover, we have

$$\mathbf{M}_\alpha(G) \leq \sum_{i=1}^{k-1} \mathbf{M}_\alpha\left(G_{\gamma_i}\right) = \sum_{i=1}^{k-1} H_\alpha(\gamma_i) = \sum_{i=1}^{k-1} J_\alpha\left(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)}\right).$$

By Lemma 4.9, there exists a transport path $\tilde{G}$ from $\mathbf{a}^{(1)}$ to $\mathbf{a}^{(k)}$ such that $\tilde{G}$ contains no cycles, $V(\tilde{G}) \subset V(G)$, and

$$\mathbf{M}_\alpha\left(\tilde{G}\right) \leq \mathbf{M}_\alpha(G) \leq \sum_{i=1}^{k-1} J_\alpha\left(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)}\right).$$

□

Theorem 4.14 *J_α is an ideal quasimetric on $\mathcal{A}_N(Y)$ with $\sigma_\infty(J_\alpha) \leq N^{2(1-\alpha)}$.*

Proof For any $k \in \mathbb{N}$ and any points $\{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)}\} \subset \mathcal{A}_N(Y)$, by proposition 4.13, there exists a transport path $G \in Path(\mathbf{a}^{(1)}, \mathbf{a}^{(k)})$ such that

$$\mathbf{M}_\alpha(G) \leq \sum_{i=1}^{k-1} J_\alpha\left(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)}\right)$$

and G contains no cycles. Moreover, by Lemma 4.10, there exists a matrix (u_{ij}) of real numbers and a metric (g_{ij}) of polyhedral curves such that

$$G = \sum_{ij} u_{ij} g_{ij}$$

as real coefficients polyhedral chains. Let

$$\gamma = \sum_{ij} u_{ij} \delta_{(x_i, y_j)}$$

be any transport plan in $Plan(\mathbf{a}^{(1)}, \mathbf{a}^{(k)})$. Then,

$$H_\alpha(\gamma) = \sum_{ij} \left(u_{ij}\right)^\alpha d\left(x_i, y_j\right) \leq \sum_{ij} \left(u_{ij}\right)^\alpha \mathit{length}\left(g_{ij}\right)$$

$$= \sum_{e \in E(G)} \left(\sum_{g_{ij} \text{ contains } e} \left(u_{ij}\right)^\alpha \right) \mathit{length}(e)$$

$$\leq \sum_e \left(N^{2(1-\alpha)} \left(\sum_{g_{ij} \text{ contains } e} u_{ij} \right)^\alpha \right) length\,(e)$$
$$= N^{2(1-\alpha)} \sum_{e \in E(G)} (w\,(e))^\alpha \, length\,(e)$$
$$= N^{2(1-\alpha)} \mathbf{M}_\alpha\,(G) \leq N^{2(1-\alpha)} \sum_{i=1}^{k-1} J_\alpha \left(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)} \right).$$

Therefore,

$$J_\alpha \left(\mathbf{a}^{(1)}, \mathbf{a}^{(k)} \right) \leq N^{2(1-\alpha)} \sum_{i=1}^{k-1} J_\alpha \left(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)} \right)$$

and thus J_α is an ideal quasimetric on $\mathcal{A}_N(Y)$ with $\sigma_\infty(J_\alpha) \leq N^{2(1-\alpha)}$. □

Suppose (Y, d) is a geodesic metric space. That is, for any $x, y \in Y$, there exists a Lipschitz curve $\Gamma_{x,y} : [0, 1] \to (Y, d)$ with $\Gamma_{x,y}(0) = x$, $\Gamma_{x,y}(1) = y$ and length $L(\Gamma_{x,y}) = d(x, y)$.

Lemma 4.15 *Suppose (Y, d) is a geodesic metric space. Let $G \in Path(\mathbf{a}, \mathbf{b})$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{A}_N(Y)$. If each edge of G is a geodesic curve between its endpoints in the metric space Y, then there exists a piecewise metric Lipschitz curve $g \in \mathcal{P}_{N_G}([0, 1], (\mathcal{A}_N(Y), J_\alpha))$ such that*

$$L_{J_\alpha}\,(g) = \mathbf{M}_\alpha\,(G)\,,$$

where N_G is total number of edges in the graph G.

Proof We may prove it using the mathematical induction on N_G. When $N_G = 1$, G itself is a geodesic in Y. Then, it is clearly true in this case. Now, assume $N_G > 1$. Pick an edge e of G with its starting endpoint e^- being a vertex in $\mathbf{a}$. Let

$$\tilde{\mathbf{a}} = \mathbf{a} + w\,(e)\,(\delta_{e^+} - \delta_{e^-})\,,$$

where e^+ is the targeting endpoint of the directed edge e, and $w(e)$ is the associated weight on e. Removing edge e from G, we get another transport path $\tilde{G} \in Path(\tilde{\mathbf{a}}, b)$. Then, $N_{\tilde{G}} = N_G - 1 \geq 1$. By the principle of the mathematical induction, we may assume that $\tilde{G}$ corresponds to a piecewise metric Lipschitz curve $\tilde{g} \in \mathcal{P}_{N_{\tilde{G}}}([0, 1], (\mathcal{A}_N(Y), J_\alpha))$ such that

$$L_{J_\alpha}\,(\tilde{g}) = \mathbf{M}_\alpha \left(\tilde{G} \right).$$

Now, let

$$g\,(t) = \begin{cases} \tilde{g}(\frac{t}{\lambda}), & 0 \leq t \leq \lambda, \\ \Gamma_e(\frac{t-\lambda}{1-\lambda}), & \lambda \leq t \leq 1, \end{cases}$$

where $\lambda = \frac{N_G - 1}{N_G}$, and Γ_e is the associated geodesic in Y from e^- to e^+. Then, $g \in \mathcal{P}_{N_G}([0,1], (\mathcal{A}_N(Y), J_\alpha))$ and

$$L_{J_\alpha}(g) = L_{J_\alpha}(\tilde{g}) + L_{J_\alpha}(\Gamma_e) = \mathbf{M}_\alpha\left(\tilde{G}\right) + w(e)^\alpha \, length(e) = \mathbf{M}_\alpha(G). \qquad \square$$

Remark 4.16 From this lemma, we see that for any transport path $G \in Path(\mathbf{a}, \mathbf{b})$ in a geodesic metric space (Y, d), we have a simple formula for the transport cost:

$$\mathbf{M}_\alpha(G) = \int_0^1 |\dot{g}(t)|_{J_\alpha} \, dt.$$

On the other hand, in [2], the authors studied another kind of ramified transportation in which the cost of a path is given by

$$\int_0^1 |\dot{g}(t)|_W \, J(g(t)) \, dt$$

where W is the Wasserstein distance on probability measures, and J is some function on the space of atomic probability measures. It is interesting to see this difference between these two different approaches.

Theorem 4.17 *Suppose (Y, d) is a geodesic metric space. Then, J_α is a perfect quasimetric on $\mathcal{A}_N(Y)$, and thus it induces a metric D_{J_α} on $\mathcal{A}_N(Y)$.*

Proof Suppose $\mathbf{a}, \mathbf{b}$ are two points in $\mathcal{A}_N(Y)$. For any $f \in \mathcal{P}_k([0,1], (\mathcal{A}_N(Y), J_\alpha))$ with $f(0) = \mathbf{a}$ and $f(1) = \mathbf{b}$, there exists a partition $P = \{0 = a_0 < \cdots < a_k = 1\}$ of $[0,1]$ such that J_α is a metric on $f([a_i, a_{i+1}])$ and $f\lfloor_{[a_i, a_{i+1}]}$ is Lipschitz for each $i = 0, 1, \ldots, k-1$. Let $x_i = f(a_i)$ for each i, by Proposition 4.13, there exists a transport path G from $f(0) = \mathbf{a}$ to $f(1) = \mathbf{b}$ such that

$$\mathbf{M}_\alpha(G) \leq \sum J_\alpha(x_i, x_{i+1}) \leq \sum_i L\left(f\lfloor_{[a_i, a_{i+1}]}\right) = L(f)$$

and G contains no cycles.When (Y, d) is a geodesic metric space, each edge of G is realized by a geodesic curve between its endpoints. By Lemma 4.15, G determines a curve $g \in \mathcal{P}_{N_G}([0,1], (\mathcal{A}_N(Y), J_\alpha))$ with $L(g) = \mathbf{M}_\alpha(G) \leq L(f)$. Since $\mathbf{a}, \mathbf{b} \in \mathcal{A}_N(Y)$ and $G \in Path(\mathbf{a}, \mathbf{b})$, the total number of vertices of G with degree one is no more than $2N$. Since G contains no cycles, the total number N_G of edges of G is no more than $4N - 3$. Thus, $g \in \mathcal{P}_{4N-3}([0,1], (\mathcal{A}_N(Y), J_\alpha))$. Hence, for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}_N(Y)$,

$$D_{J_\alpha}^{(k)}(\mathbf{a}, \mathbf{b}) = D_{J_\alpha}^{(4N-3)}(\mathbf{a}, \mathbf{b})$$

for any $k \geq 4N - 3$. This shows that J_α is a perfect quasimetric on $\mathcal{A}_N(Y)$. $\square$

Corollary 4.18 *Suppose (Y, d) is a geodesic metric space. Then, for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}_N(Y)$ and $\alpha \leq 1$, we have*

$$D_{J_\alpha}(\mathbf{a}, \mathbf{b}) = \min\{\mathbf{M}_\alpha(G) : G \in Path(\mathbf{a}, \mathbf{b})\}.$$

Proof Let G be any optimal transport path from $\mathbf{a}$ to $\mathbf{b}$. From the proof of the above theorem, we see $D_{J_\alpha}(\mathbf{a},\mathbf{b}) \le \mathbf{M}_\alpha(G) \le L(f)$ for any $f \in \mathcal{P}_k([0,1], (\mathcal{A}_N(Y), J_\alpha))$ with $k \ge 4N-3$. Hence, $D_{J_\alpha}(\mathbf{a},\mathbf{b}) = \mathbf{M}_\alpha(G)$. □

Corollary 4.19 *Suppose (Y,d) is a geodesic metric space. Then, $(\mathcal{A}_N(Y), D_{J_\alpha})$ is a length space.*

Proof By Corollary 4.18, each optimal transport path G determines a solution g to the geodesic problem (3.1). Then, by Theorem 3.16, $(\mathcal{A}_N(Y), D_{J_\alpha})$ becomes a length space. □

Since $\mathcal{A}_1(Y) \subset \mathcal{A}_2(Y) \subset \cdots \subset \mathcal{A}_N(Y) \subset \cdots$, and $(\mathcal{A}_N(Y), D_{J_\alpha})$ is a length space for each N, we have

Proposition 4.20 *Suppose (Y,d) is a geodesic metric space. Then, D_{J_α} is a metric on the space $\mathcal{A}(Y)$ of all atomic probability measures on Y. Moreover, $(\mathcal{A}(Y), D_{J_\alpha})$ is a length space.*

We now give some conclusive remarks.

Remark 4.21 In [10], we defined $d_\alpha(\mathbf{a},\mathbf{b}) := \min\{\mathbf{M}_\alpha(G) : G \in Path(\mathbf{a},\mathbf{b})\}$ for $0 \le \alpha < 1$ and showed that d_α defines a metric on the space of (atomic) probability measures. Moreover, we showed $(\mathcal{A}(Y), d_\alpha)$ is a length space. Now, from Corollary 4.18, we see that $d_\alpha = D_{J_\alpha}$. That is, the metric d_α is just the intrinsic metric on $\mathcal{A}(Y)$ induced by the quasimetric J_α. Proposition 4.20 simply gives another proof of $(\mathcal{A}(Y), d_\alpha)$ being a length space. Furthermore, an optimal transport path studied in [10] is simply a geodesic in the length space $(\mathcal{A}(Y), D_{J_\alpha})$.

Remark 4.22 Suppose (Y,d) is a geodesic metric space, and $\mathcal{P}_\alpha(Y)$ is the completion of the metric space $(\mathcal{A}(Y), D_{J_\alpha})$. Then, $(\mathcal{P}_\alpha(Y), D_{J_\alpha})$ is also a length space. A geodesic in the length space $(\mathcal{P}_\alpha(Y), D_{J_\alpha})$ is also called an optimal transport path between its endpoints.

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