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Interior regularity of optimal transport paths

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Abstract. In a previous paper [12], we considered problems for which the cost of transporting one probability measure to another is given by a transport path rather than a transport map. In this model overlapping transport is frequently more economical. In the present article we study the interior regularity properties of such optimal transport paths. We prove that an optimal transport path of finite cost is rectifiable and simply a finite union of line segments near each interior point of the path.

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1. Introduction

In a previous paper [12], the author considered a phenomena of mass transport problems, for which the actual transport cost is given by the actual transport path, rather than by transport maps as in Monge's problems [7, 10, 8, 3, 6, 2]. In this model, an optimal transport path in [12] between two probability measures will often overlap in some cost efficient way. Such a phenomenon is very common in the nature such as trees, railways, circulatory system and so on. This article is a continuation of the paper [12]. Namely, we will consider here the regularity properties of these optimal transport paths.

There are mainly three steps to get the regularity results. The first step is about the rectifiability of transport paths of finite cost. We achieve this by viewing a transport path as a real flat 1-chain. We showed that a real flat k-chain with finite mass and finite $\mathbf{M}^\alpha$ mass is rectifiable. The idea of the proof is partly from Brian White's result [11] on rectifiability of flat chains with coefficients in general groups. Thus, an optimal transport path with finite $\mathbf{M}^\alpha$ cost (which automatically implies finite mass) is rectifiable . The second step is achieving tangent cone properties of optimal transport paths. Some readers may find that proofs here are analogous to the proofs of tangent cone properties of classical integer multiplicity minimizing currents as in the book [9]. In the third part, by some comparisons, we achieve some local finiteness property at every interior point on the support of the path away from the boundary. Namely, in a neighborhood of such points, the path is given simply by a cone consisting of a finite union of segments. An interesting aspect of this proof is that we do not assume the usual positive lower bound condition on the density

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of flat chains as in the study of stationary varifolds in [1]. In fact one should not expect a global positive lower bound on the density to exist here.

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In this paper, we will use the following notations as in [12]:

- X : a compact convex subset of a Euclidean space $\mathbb{R}^m$.
- α : a positive number in $[0,1)$.
- μ^+ : a probability Radon measure on X as the initial measure.
- μ^- : a probability Radon measure on X as the target measure.
- $Path\,(\mu^+,\mu^-)$: the space of all transport path from μ^+ to μ^-.
- $\mathcal{M}_\Lambda\,(X)$: the space of Radon measure μ on X with total mass $\mu\,(X)=\Lambda$.
- δ_p: the Dirac measure at the point p.

2. Rectifiability of real flat chains with finite $\mathbf{M}^\alpha$ mass

Let P_k denote the group of polyhedral chains of dimension k in $\mathbb{R}^m$ with real coefficients. Given any $\alpha\in[0,1]$, and any polyhedral chain $P=\sum g_i\sigma_i\in P_k$ with each $g_i\in\mathbb{R}$, we define the $\mathbf{M}^\alpha$ mass of P to be

$$\mathbf{M}^\alpha\,(P):=\sum|g_i|^\alpha\,\mathcal{H}^k\,(\sigma_i)\,, \tag{2.1}$$

where the σ_i are nonoverlapping oriented k-dimensional convex cells. Let

$$W^\alpha\,(P)=\inf_{R\in P_{k+1}}\{\mathbf{M}^\alpha\,(R)+\mathbf{M}^\alpha\,(P-\partial R)\}\,,$$

where ∂ is the usual boundary operation on P_k. The *Whitney flat $\alpha-$distance* between $P_1,P_2\in P_k$ is $W^\alpha\,(P_1-P_2)$. Let $\mathcal{F}_k$ be the W^1-completion of P_k. The elements of $\mathcal{F}_k$ are called (real)*flat $k-$chains* in [4]. For any real flat chain $T\in\mathcal{F}_k$, we define its $\mathbf{M}^\alpha$ mass to be

$$\mathbf{M}^\alpha\,(T):=\inf_{\{P_i\}\overset{W^1}{\to}T}\ \liminf_{i\to\infty}\mathbf{M}^\alpha\,(P_i)\,. \tag{2.2}$$

Remark 2.1. An element T of $\mathcal{F}_k$ with finite $\mathbf{M}^\alpha$ mass is not necessarily a flat chain with coefficients in the group $\left(\mathbb{R}^1,|x|^\alpha\right)$ as in [11] or [5], because T is the limit of polyhedral chains under $\mathbf{W}^1$ flat distance , rather than under the $\mathbf{W}^\alpha$ flat distance.

Remark 2.2. Suppose $k=0$ or 1 and $\{P_i\}$ is a sequence of polyhedral k-chains with uniformly bounded mass and boundary mass. Then, $\{P_i\}\to T$ under flat 1-distance W^1 is equivalent to $\{P_i\}\to T$ weakly as signed measures or vector measures. (See [9, 31.2] and [11].)

2.1. Measures of finite $\mathbf{M}^\alpha$ *mass*

Since every signed measure on X can be approximated weakly by atomic measures, every signed measure is a real flat 0-chain. However, for any $\alpha < 1$, not every signed measure has finite $\mathbf{M}^\alpha$ mass. In fact, as shown in [11] by White, we have the following proposition:

Proposition 2.3. *Suppose* $0 \leq \alpha < 1$. *For any signed measure* μ *on* X *of finite* $\mathbf{M}^\alpha$ *mass, there exists two sequences* $\{x_i\} \subseteq X$ *and* $\{a_i\} \subseteq (-\infty, +\infty)$, *such that*

$$\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i} \text{ with } \sum_{i=1}^{\infty} |a_i|^\alpha = \mathbf{M}^\alpha(\mu) < +\infty. \tag{2.3}$$

Proof. We may assume μ is a Radon measure, for every signed measure is the difference of two Radon measures. We will apply induction on the dimension n of $\mathbb{R}^n \supset X$. The case $n = 0$ is trivial. We assume it is true whenever $X \subset \mathbb{R}^k$ for some integer k. Now, suppose $X \subset \mathbb{R}^{k+1}$. Let

$$\nu = \mu - \mu \llcorner \{x : \mu(\{x\}) > 0\}.$$

For every hyperplane H of $\mathbb{R}^{k+1}$, by induction, $\nu \llcorner H = 0$.

Assume $\nu \neq 0$ so that $\nu(X) > 0$.

Let $L : \mathbb{R}^{k+1} \to \mathbb{R}$ be the projection map to its first coordinate. We consider a map

$$\gamma : [a, b] \to (\mathbb{R}, ||\cdot||_\alpha)$$

defined by

$$\gamma(t) = \nu(\{x = \{x_1, \cdots, x_{k+1}\} \in X : x_1 \leq t\}),$$

where $[a, b]$ is an interval whose interior contains $L(spt(\nu))$ and $||\cdot||_\alpha$ is the metric given by $||x||_\alpha = |x|^\alpha$ for any $x \in \mathbb{R}$. Here $|x|$ is the absolute value of x.

Now, for any $s < t$,

$$\begin{aligned} ||\gamma(s) - \gamma(t)||_\alpha &= (\nu(\{x = \{x_1, \cdots, x_{k+1}\} \in X : s < x_1 \leq t\}))^\alpha \\ &\leq \mathbf{M}^\alpha(\nu \llcorner \{x = \{x_1, \cdots, x_{k+1}\} \in X : s < x_1 \leq t\}). \end{aligned}$$

This implies that

$$\lim_{t \to s+} ||\gamma(t) - \gamma(s)||_\alpha = 0,$$

and

$$\lim_{s \to t-} ||\gamma(s) - \gamma(t)||_\alpha = (\nu(\{x = \{x_1, \cdots, x_{k+1}\} \in X : x_1 = t\}))^\alpha = 0,$$

since $\{x_1 = t\}$ is a hyperplane of $\mathbb{R}^{k+1}$. Thus, $\gamma : [a, b] \to (\mathbb{R}, ||\cdot||_\alpha)$ is an increasing continuous function with $\gamma(a) = 0$ and $\gamma(b) = \nu(X) > 0$.

Now, we calculate the length of the curve γ under the metric $||\cdot||_\alpha$. On one hand, for any partition $a = t_0 \leq \cdots \leq t_m = b$, we have

$$\begin{aligned}\sum ||\gamma(t_i) - \gamma(t_{i-1})||_\alpha &= \sum \left(\nu\left(\{x = \{x_1, \cdots, x_{k+1}\} \in X : t_{i-1} < x_1 \leq t_i\}\right)\right)^\alpha \\ &\leq \sum \mathbf{M}^\alpha\left(\nu \llcorner \{x = \{x_1, \cdots, x_{k+1}\} \in X : t_{i-1} < x_1 \leq t_i\}\right) \\ &\leq \mathbf{M}^\alpha\left(\nu \llcorner \{x = \{x_1, \cdots, x_{k+1}\} \in X : a < x_1 \leq b\}\right) \\ &= \mathbf{M}^\alpha(\nu) < +\infty.\end{aligned}$$

This implies that γ has finite length.

However, on the other hand, $\gamma([a, b]) = [0, \nu(X)] \subset (\mathbb{R}^n, ||...||_\alpha)$ has length infinity unless $\nu(X) = 0$. A contradiction! Therefore, $\nu = 0$ and

$$\mu = \mu \llcorner \{x : \mu(\{x\}) > 0\} \ .$$

Hence μ must be of the form

$$\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i}$$

for some $a_i \in \mathbb{R}$ and $x_i \in X$. □

Remark 2.4. Proposition 2.3 is not true for $\alpha = 1$, since every Radon measure μ on X has finite $\mathbf{M}^1$ mass.

Any signed measure of the form (2.3) is called an *infinite atomic measure*. Let

$$\mathcal{A}_\Lambda(X)$$

be the space of all infinite atomic measures in X with total mass Λ. Some properties of infinite atomic measures will be studied in Section 4.3.

2.2. Rectifiability of real flat chains

As in [9], a subset $M \subset \mathbb{R}^m$ is called (countably) $k-$*rectifiable* if $M = \bigcup_{i=0}^{\infty} M_i$, where $\mathcal{H}^k(M_0) = 0$ under the $k-$Hausdorff measure $\mathcal{H}^k$ and each M_i, for $i = 1, 2, \cdots$, is a subset of a $k-$dimensional C^1 submanifold in $\mathbb{R}^m$.

It is well known that a real flat chain with finite mass may not be rectifiable. However, it satisfies the following rectifiable slicing theorem ([4, 11]):

Proposition 2.5. *Let A be a k-flat chain of finite mass in $\mathbb{R}^m$. Then the following two conditions are equivalent:*

1. *A is rectifiable.*
2. *For almost every $(m-k)$-plane P parallel to a coordinate plane, the slice $A \cap P$ is a rectifiable 0-chain.*

Remark 2.6. In [11], the above theorem actually holds for flat chains with coefficients in more general normed groups. We only need the simplest case here, namely the group is the real line with Euclidean metric.

The main result of this section is the following rectifiability theorem:

Theorem 2.7. *Given any* $0 \leq \alpha < 1$. *Any real flat* k*-chain* A *of finite* $\mathbf{M}^1$ *mass and finite* $\mathbf{M}^\alpha$ *mass is rectifiable.*

Proof. Let $\Pi : \mathbb{R}^m \to L$ be an orthogonal projection onto an affine k-plane L. Then, we have

$$\int_L \mathbf{M}^\alpha \left(A \cap \Pi^{-1}(x) \right) dx \leq \mathbf{M}^\alpha (A) < +\infty.$$

This is obvious when A is a polyhedral chain, and by Fatou's lemma, we know it is also true when A is a real flat chain. Therefore,

$$\mathbf{M}^\alpha \left(A \cap \Pi^{-1}(x) \right) < +\infty$$

for a.e. $x \in L$. By Proposition (2.3), $A \cap \Pi^{-1}(x)$ is an infinite atomic measure for a.e. $x \in L$. On the other hand, A is a flat chain of finite $\mathbf{M}^1$ mass. By Proposition 2.5, A is rectifiable. □

3. Tangent cone properties of rectifiable $\mathbf{M}^\alpha$ mass minimizers

The goal of this section is to achieve the tangent cone properties of any one dimensional $\mathbf{M}^\alpha$ mass minimizer. This result will be used in the next section for optimal transport paths. We first recall some terminology about rectifiable currents as in [4] or [9].

Let $\Omega \subset \mathbb{R}^m$ be an open subset and $\mathcal{D}^k(\Omega)$ be the set of all C^∞ differential k-forms in Ω with compact support with the usual Fréchet topology [4].

A *k-dimensional current T in Ω* is a continuous linear functional on $\mathcal{D}^k(\Omega)$. Let $\mathcal{D}_k(\Omega)$ denote the set of all k-dimensional currents in Ω. Motivated by Stokes' theorem, the *boundary* of a current $T \in \mathcal{D}_k(\Omega)$ with $k \geq 1$ is the current $\partial T \in \mathcal{D}_{k-1}(\Omega)$ defined by

$$\partial T(\psi) := T(d\psi)$$

for any $\psi \in \mathcal{D}^{k-1}(\Omega)$. When $k = 0$, one defines $\partial T := 0$.

A *rectifiable current* T is a current coming from an oriented rectifiable set with multiplicities. More precisely, $T \in \mathcal{D}_k(\Omega)$ is a *rectifiable current* if it can be expressed as

$$T(\omega) = \int_M < \omega(x), \xi(x) > \theta(x)\, d\mathcal{H}^k(x), \ \forall \omega \in \mathcal{D}^k(\Omega)$$

where

- M is an $\mathcal{H}^k$ measurable and $k-$rectifiable subset of Ω,
- θ is an $\mathcal{H}^k \lfloor M$ integrable positive function,
- $\xi : M \to \Lambda_k(\mathbb{R}^m)$ is a $\mathcal{H}^k$ measurable unit tangent k vector field on M.

The rectifiable current T described as above is often denoted by

$$T = \underset{=}{\tau}(M, \theta, \xi).$$

Let $\mathcal{R}_k(\Omega)$ be the space of all rectifiable k-currents in Ω.

From now on, we always assume $\alpha \in [0, 1)$. For any $T = \underset{=}{\tau}(M, \theta, \xi) \in \mathcal{R}_k(\Omega)$, its $\mathbf{M}^\alpha$ mass is given by

$$\mathbf{M}^\alpha \left(\underset{=}{\tau}(M, \theta, \xi) \right) = \int_M \theta^\alpha d\mathcal{H}^k(x) < +\infty.$$

A rectifiable k-current $T = \underset{=}{\tau}(M, \theta, \xi) \in \mathcal{R}_k(\Omega)$ is *an M^α mass minimizer* if

$$\mathbf{M}^\alpha(T) \le \mathbf{M}^\alpha(S)$$

for any rectifiable current $S \in \mathcal{R}_k(\Omega)$ with $\partial S = \partial T$.

The upshot here is to show that there exists a tangent cone of any $\mathbf{M}^\alpha$ mass minimizer $T = \underset{=}{\tau}(M, \theta, \xi) \in \mathcal{R}_1(\Omega)$ at any fixed point p on $spt(T) \setminus spt(\partial T)$. We will apply this result to an optimal transport path in the next section.

3.1. First Variation

Suppose a rectifiable 1-current $T = \underset{=}{\tau}(M, \theta, \xi) \in \mathcal{R}_1(\Omega)$ is an $\mathbf{M}^\alpha$ mass minimizing with a fixed point p is $spt(T) \setminus spt(\partial T)$.

For any open ball $U \subset \Omega \backslash spt(\partial T)$, suppose $\{\phi_t\}_{-\epsilon<t<\epsilon}$ is a 1-parameter family of diffeomorphisms of U satisfying

$$\phi_0 = id_U, \exists \text{ compact } K \subset U \text{ such that } \phi_t | U \sim K = id_{U \backsim K} \; \forall t \in (-\epsilon, \epsilon);$$
$$(x, t) \to \phi_t(x) \text{ is a smooth map } U \times (-\epsilon, \epsilon) \to U.$$

Then,

$$\mathbf{M}^\alpha(\phi_{t\#}(T)) = \int_M |D(\phi_t \lfloor M)| \theta^\alpha d\mathcal{H}^1.$$

We can compute the first variation and have

$$\frac{d}{dt}\mathbf{M}^\alpha(\phi_{t\#}(T))|_{t=0} = \int_M D_\xi Y \cdot \xi \theta^\alpha d\mathcal{H}^1,$$

where the vector field

$$Y(x) = \frac{\partial}{\partial t}\phi(t, x)|_{t=0},$$

$D_\xi Y$ is the partial derivative of Y in the direction ξ.

Thus, since T is an $\mathbf{M}^\alpha$ mass minimizer, we have

$$\int_M D_\xi Y \cdot \xi \theta^\alpha d\mathcal{H}^1 = 0. \tag{3.1}$$

3.2. Monotonicity formula

Let ψ be a smooth function such that

$$\psi(t) = 1 \text{ for } t \leq 1/2, \psi(t) = 0 \text{ for } t \geq 1 \text{ and } \psi'(t) \leq 0 \text{ for all } t.$$

For any fixed $\rho > 0$ small enough, choose $Y(x) = \psi(r/\rho)(x-p)$ with $r = |x-p|$, and $U = B_\rho(p)$ in (3.1), then

$$\int_{B_\rho(p)} D_\xi\left(\psi(r/\rho)(x-p)\right) \cdot \xi \theta^\alpha d\mathcal{H}^1 = 0.$$

Since

$$\begin{aligned} & D_\xi\left(\psi(r/\rho)(x-p)\right) \cdot \xi \\ &= \psi(r/\rho) + r\frac{\partial}{\partial r}\left(\psi(r/\rho)\right)\left(\frac{x-p}{r}\cdot\xi\right)^2 \\ &= \psi(r/\rho) - \rho\frac{\partial}{\partial \rho}\left(\psi(r/\rho)\right)\left(\frac{x-p}{r}\cdot\xi\right)^2. \end{aligned}$$

Therefore,

$$\int_{B_\rho(p)} \left[\psi(r/\rho) - \rho\frac{\partial}{\partial \rho}\left(\psi(r/\rho)\right)\left(\frac{x-p}{r}\cdot\xi\right)^2\right]\theta^\alpha d\mathcal{H}^1 = 0.$$

$$\frac{d}{d\rho}\int_{B_\rho(p)} \frac{\psi(r/\rho)}{\rho}\theta^\alpha d\mathcal{H}^1 = \frac{1}{\rho}\frac{d}{d\rho}\int_{B_\rho(p)} \psi(r/\rho)\left[1 - \left(\frac{x-p}{r}\cdot\xi\right)^2\right]\theta^\alpha d\mathcal{H}^1.$$

Now, letting ψ increase to the characteristic function of the interval $(-\infty, 1)$, we obtain the following *monotonicity formula*, in the sense of distribution,

$$\begin{aligned} & \frac{d}{d\rho}\left(\frac{\int_{B_\rho(p)} \theta^\alpha(x)\, d\mathcal{H}^1(x)}{\rho}\right) \\ &= \frac{d}{d\rho}\int_{B_\rho(p)} \frac{1}{|x-p|}\left(1 - \frac{x-p}{|x-p|}\cdot\xi\right)\theta^\alpha(x)\, d\mathcal{H}^1(x). \end{aligned} \tag{3.2}$$

Corollary 3.1. $\frac{\int_{B_\rho(p)} \theta^\alpha(x) d\mathcal{H}^1(x)}{\rho}$ *is a nondecreasing function of* ρ *for* $0 < \rho < dist(p, spt(\partial T))$.

Proof. This is because $\int_{B_\rho(p)} \frac{1}{|x-p|}\left(1 - \frac{x-p}{|x-p|}\cdot\xi\right)\theta^\alpha(x)\, d\mathcal{H}^1(x)$ is nondecreasing in ρ . □

Corollary 3.2. *If* $\frac{\int_{B_\rho(p)} \theta^\alpha(x) d\mathcal{H}^1(x)}{\rho}$ *is constant in* ρ*, then*

$$\xi(x) = \frac{x-p}{|x-p|}.$$

Proof. Since $\frac{\int_{B_\rho(p)} \theta^\alpha(x) d\mathcal{H}^1(x)}{\rho}$ is constant, by the monotonicity formula (3.2),

$$\frac{d}{d\rho} \int_{B_\rho(p)} \frac{1}{|x-p|} \left(1 - \frac{x-p}{|x-p|} \cdot \xi\right) \theta^\alpha(x) \, d\mathcal{H}^1(x) = 0.$$

Thus,

$$\int_{B_\rho(p)} \frac{1}{|x-p|} \left(1 - \frac{x-p}{|x-p|} \cdot \xi\right) \theta^\alpha(x) \, d\mathcal{H}^1(x)$$

is constant and equals to $\int_{B_0(p)} \frac{1}{|x-p|} \left(1 - \frac{x-p}{|x-p|} \cdot \xi\right) \theta^\alpha(x) \, d\mathcal{H}^1(x) = 0$. Since $1 - \frac{x-p}{|x-p|} \cdot \xi$ is nonnegative in $B_\rho(p)$, we have

$$1 - \frac{x-p}{|x-p|} \cdot \xi = 0, \text{ i.e. } \xi(x) = \frac{x-p}{|x-p|},$$

for any $x \in M$. □

3.3. Tangent cone

For any $\lambda > 0$, consider a map

$$\begin{aligned} \eta_{p,\lambda} \ &: \ \mathbb{R}^m \to \mathbb{R}^m \\ y &\to \frac{y-p}{\lambda}. \end{aligned}$$

For each λ, and the rectifiable $1-$current $T = \underline{\underline{\tau}}(M, \theta, \xi) \in \mathcal{R}_1(\Omega)$, let

$$T_\lambda = \eta_{p,\lambda\#} T \lfloor_{\bar{B}_{\rho\lambda}(p)}.$$

Note that $T_\lambda = \underline{\underline{\tau}}(\eta_{p,\lambda\#} M \cap B_\rho, \theta \circ \eta_{p,\lambda^{-1}}, \xi_\circ \eta_{p,\lambda^{-1}})$ is also rectifiable. Since T is an $\mathbf{M}^\alpha$ mass minimizer, so is T_λ for each λ.

Proposition 3.3. *Suppose* $\mathbf{M}(T) + \mathbf{M}^\alpha(T) < +\infty$. *There exists a sequence* $\{\lambda_j\}$ *approaching zero such that* $\{T_{\lambda_j}\}$ *converges to some locally rectifiable* $1-$*current* C *in both* W^α *flat metric and* W^1 *flat metric. Moreover* C *is a cone as well as an* $\mathbf{M}^\alpha$ *mass minimizer.*

Proof. Since $\mathbf{M}(T) + \mathbf{M}^\alpha(T) < +\infty$, there exists a sequence of positive numbers $\{\lambda_j\}$ converges to zero such that the slicing $\{T_{\lambda_j}\}$ of T by the sphere $\{x : |x-p| = \lambda_j\}$ satisfies the condition:

$$\sup \left\{\mathbf{M}^\alpha\left(\partial T_{\lambda_j}\right), \mathbf{M}^\alpha\left(T_{\lambda_j}\right), \mathbf{M}\left(\partial T_{\lambda_j}\right), \mathbf{M}\left(T_{\lambda_j}\right)\right\} < +\infty.$$

Thus, by the compactness theorem of flat chains, $\{T_{\lambda_j}\}$ is subsequently convergent to some flat current C in both W^α flat metric and W^1 flat metric. C is a rectifiable current $\underline{\underline{\tau}}(W, \theta_C, \xi_C)$ because it has finite $\mathbf{M}^\alpha$ mass and finite $\mathbf{M}^1$ mass. By a proof similar to [9, 34.5], we know that $C = \underline{\underline{\tau}}(W, \theta_C, \xi_C)$ is an $\mathbf{M}^\alpha$ mass minimizer because it is a limit of a sequence of $\mathbf{M}^\alpha$ mass minimizers $\{T_{\lambda_j}\}$. Moreover, for a subsequence, without changing notations, $\mathcal{H}^1\lfloor\theta_j \rightharpoonup \mathcal{H}^1\lfloor\theta_C$ weakly as Radon measures, where $\theta_j = \theta \circ \eta_{p,\lambda_j^{-1}}$. Now,

$$\begin{aligned}\frac{\int_{B_\rho(0)} \theta_C^\alpha(x)\, d\mathcal{H}^1(x)}{\rho} &= \lim_j \frac{\int_{B_\rho(p)} \theta_j^\alpha(x)\, d\mathcal{H}^1(x)}{\rho}\\ &= \lim_j \frac{\int_{B_{\lambda_j\rho}(p)} \theta^\alpha(x)\, d\mathcal{H}^1(x)}{\lambda_j\rho} \qquad \text{(by Corollary 3.1)}\\ &= \lim_{\rho\to 0+} \frac{\int_{B_\rho(p)} \theta^\alpha(x)\, d\mathcal{H}^1(x)}{\rho},\end{aligned}$$

which is constant in ρ. By Corollary 3.2, we have

$$\xi_C(x) = \frac{x}{|x|}$$

for any $x \in spt(C)$. By an argument similar to that in the proof of [9, 19.3], this fact implies that C is a cone, in the sense that

$$\eta_{0,\lambda\#}C = C$$

for any $\lambda > 0$. □

The above cone is called a *tangent cone* of T at p. From our main Theorem 4.10, we will see that such a cone is in fact unique, i.e. independent of the choice of the sequence $\{\lambda_j\}$.

4. Regularity of optimal transport paths

4.1. Transport paths between Radon measures

Given two fixed Radon measures $\mu^+, \mu^- \in \mathcal{M}_\Lambda(X)$ of equal total mass Λ. Let $0 \le \alpha < 1$ be fixed.

Definition 4.1. *A transport path from μ^+ to μ^- is a real flat 1-chain T with*

$$\partial T = \mu^- - \mu^+$$

as real flat chains. The $\mathbf{M}^\alpha$ mass of T is also called the $\mathbf{M}^\alpha$ cost of T.

Remark 4.2. Note that the definition here of transport paths is a restatement of a previous definition of transport paths in [12].

Let

$$Path(\mu^+, \mu^-) = \{T \in \mathcal{F}_1(X) : \partial T = \mu^- - \mu^+\}$$

be the space of all transport paths from μ^+ to μ^-.

In [12], we showed that for any polyhedral 1-chain P, there exists a polyhedral 1-chain $\tilde{P}$ with $\partial\tilde{P} = \partial P$ such that $\tilde{P}$ contains no 1-cycles in its support and $\mathbf{M}^\alpha(\tilde{P}) \le \mathbf{M}^\alpha(P)$.

Definition 4.3. *An optimal transport path T from μ^+ to μ^- is an element in $Path(\mu^+, \mu^-)$ of least $\mathbf{M}^\alpha$ mass such that there exists a sequence of polyhedral 1-chains $\{P_i\}$ containing no cycles with $\{P_i\} \to T$ in flat W^1 metric.*

In [12], for any $\alpha \in (1 - \frac{1}{m}, 1]$, we showed that there exists an optimal transport path from μ^+ to μ^- with finite $\mathbf{M}^\alpha$ cost. We also verified that

$$d_\alpha\left(\mu^+, \mu^-\right) := \min\left\{\mathbf{M}^\alpha(T) : T \in Path\left(\mu^+, \mu^-\right)\right\}, \tag{4.1}$$

defines a metric on $\mathcal{M}_\Lambda(X)$, which metrizies the weak * topology of Radon measures.

For any Radon measure $\mu \in \mathcal{M}_\Lambda(X)$ of total mass Λ, we showed in [12] that

$$d_\alpha(\mu, \Lambda\delta_c) \le K_{m,\alpha}\Lambda^\alpha \frac{diam(spt(\mu))}{2}, \tag{4.2}$$

where d_α is the distance on $\mathcal{M}_\Lambda(X)$ given in (4.1), $K_{m,\alpha}$ is the constant $\frac{1}{2^{1-m(1-\alpha)}-1}$ independent of μ, and c is the center of the cube Q containing $spt(\mu)$.

Now, suppose $T \in Path(\mu^+, \mu^-)$ is an optimal transport path of finite $\mathbf{M}^\alpha$ mass. In [12], we also verified that

$$\mathbf{M}^1(T) \le \Lambda^{1-\alpha}\mathbf{M}^\alpha(T).$$

Thus, T is automatically of finite $\mathbf{M}^1$ mass. By the rectifiability Theorem 2.7, T is rectifiable.

4.2. Tangent cone

Let $\{P_i\}$ be a sequence of polyhedral 1-chains containing no cycles and converging to an optimal transport path T in flat metric. Note that each P_i determines a polyhedral 1-current. Since

$$\mathbf{M}^1(P_i) + \mathbf{M}^1(\partial P_i) \le \Lambda^{1-\alpha}\mathbf{M}^\alpha(P_i) + 2 \to \Lambda^{1-\alpha}\mathbf{M}^\alpha(T) + 2 < +\infty$$

is uniformly bounded, by the compactness theorem of currents, $\{P_i\}$ converges to a current in flat metric. Thus, T is also a current. Since T is optimal, T becomes an $\mathbf{M}^\alpha$ mass minimizer. Put everything together, we get the following

Proposition 4.4. *Any optimal transport path $T \in Path(\mu^+, \mu^-)$ with finite $\mathbf{M}^\alpha$ cost is a rectifiable $1-$current $\underline{\underline{\tau}}(M, \theta, \xi)$ with*

$$\mathbf{M}^\alpha(T) = \int_M \theta^\alpha d\mathcal{H}^k(x) < +\infty.$$

Moreover, T is $\mathbf{M}^\alpha$ mass minimizing.

Now, we have the following proposition about the tangent cone of any optimal transport path:

Proposition 4.5. *Suppose $T \in Path\,(\mu^+, \mu^-)$ is an optimal transport path. For any $p \in spt\,(T) \setminus (spt\,(\mu^+) \cup spt\,(\mu^-))$, there exists a tangent cone C_p of T at p. Moreover, the intersection of C_p with the closed unit ball is a finite union of line segments:*

$$C_p = \sum_{i=1}^{k} m_i[|p_i, 0|]$$

for some $\{p_i\} \subset S_1^{m-1}(0)$, some suitable multiplicities $m_i \in \mathbb{R}$ satisfying a balance equation

$$\sum_{i=1}^{k} \frac{m_i}{|m_i|^{1-\alpha}} p_i = 0, \tag{4.3}$$

and some positive integer $k \leq C$ for some constant $C = C\,(m, \alpha)$ depends only on m and α. Here k and the minimum angle between distinct p_i depend only on α and m.

Proof. The existence of a tangent cone C_p comes from Proposition 3.3. Let l_1 and l_2 be any two outward (or inward) segments on C_p of weights m_1 and m_2 from the cone vertex O. Since C_p is also optimal, by simple computation as in the example 1 of [12], one can see that the angle between l_1 and l_2 is at least

$$\arccos\left(\frac{(m_1+m_2)^{2\alpha} - m_1^{2\alpha} - m_2^{2\alpha}}{2m_1^\alpha m_2^\alpha}\right) \geq \arccos\left(2^{2\alpha-1} - 1\right),$$

which is positive for $\alpha < 1$. This uniform lower bound forces the number of segments on C_p from the same cone vertex O to be finite. Therefore, C_p must be a finite union of line segments. Since C_p is optimal, these line segments must satisfy the balance equation (4.3). □

4.3. The normalized maximum atomic mass of an infinite atomic measure

Now, we study an important quantity of infinite atomic measures, which plays a key role in the following subsections for comparing the cost of different transport paths.

For any infinite atomic measure $\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i} \in \mathcal{A}_\Lambda\,(X)$, let

$$\chi\,(\mu) := \frac{\max\left\{|a_i| : \mu = \sum_{i=1}^{\infty} a_i \delta_{x_i}\right\}}{\mathbf{M}\,(\mu)}$$

Note that $0 < \chi\,(\mu) \leq 1$ and $spt\,(\mu)$ has at least $1/\chi\,(\mu)$ points. Also, χ is scalar invariant since $\chi\,(\lambda\mu) = \chi\,(\mu)$ for any $\lambda \neq 0$. The following two lemmas about χ will be used later.

Lemma 4.6.

$$\mathbf{M}^{\alpha}(\mu) \geq \chi(\mu)^{\alpha-1} \Lambda^{\alpha}$$

for any $\mu \in \mathcal{A}_{\Lambda}(X)$.

Proof. By choosing an order on $\{a_i\}$, one may assume that $\{|a_i|\}$ is nonincreasing. Thus,

$$\begin{aligned}\mathbf{M}^{\alpha}(\mu) &= \sum_{i=1}^{\infty} |a_i|^{\alpha} = |a_1|^{\alpha} \sum_{i=1}^{\infty} \left|\frac{a_i}{a_1}\right|^{\alpha} \\ &\geq |a_1|^{\alpha} \sum_{i=1}^{\infty} \left|\frac{a_i}{a_1}\right| = |a_1|^{\alpha-1} \Lambda \\ &= \chi(\mu)^{\alpha-1} \Lambda^{\alpha}.\end{aligned}$$

□

Lemma 4.7. *Given* $c \in (0,1]$ *and a small number* $0 < \epsilon < \frac{c}{3}$ *with*

$$(1 - \frac{\epsilon^2}{c})^{\alpha} > 1 - \frac{2\alpha}{c}\epsilon^2.$$

For any infinite atomic measure $\mu \in \mathcal{A}_{\Lambda}(X)$, *if*

$$\begin{aligned}c - \epsilon^2 < \mathbf{M}(\mu) < c + \epsilon^2 \\ c^{\alpha} - \epsilon^2 < \mathbf{M}^{\alpha}(\mu) < c^{\alpha} + \epsilon^2,\end{aligned}$$

then

$$\chi(\mu) \geq (1 - \epsilon)^{\frac{1}{1-\alpha}}.$$

Proof. By Lemma 4.6,

$$\begin{aligned}\chi(\mu)^{1-\alpha} &\geq \frac{[\mathbf{M}(\mu)]^{\alpha}}{\mathbf{M}^{\alpha}(\mu)} \geq \frac{(c - \epsilon^2)^{\alpha}}{c^{\alpha} + \epsilon^2} \\ &\geq \frac{c^{\alpha}(1 - \frac{2\alpha}{c}\epsilon^2)}{c^{\alpha} + \epsilon^2} \\ &> 1 - \epsilon,\end{aligned}$$

by some easy calculation. □

4.4. Regularity of optimal transport paths

Let S_ρ^{m-1} denotes the sphere in $\mathbb{R}^m$ with radius ρ and centered at the origin.

Lemma 4.8. *For any $\mu \in \mathcal{A}_\Lambda\left(S_\rho^{m-1}\right)$ and any optimal transport path*

$$T \in Path\left(\mu, \Lambda\delta_0\right),$$

we have

$$\chi\left(T\lfloor S_{r\rho}^{m-1}\right) \geq \left(\frac{1-r}{K_{m,\alpha}}\right)^{\frac{1}{1-\alpha}}$$

for every $r \in [0,1]$. In particular,

$$\chi\left(T\lfloor S_{\rho/2}^{m-1}\right) \geq \left(2K_{m,\alpha}\right)^{\frac{-1}{1-\alpha}}$$

has a universal lower bound for any μ, ρ and T.

Proof. We have known from (4.2) that

$$d_\alpha\left(\mu, \delta_0\right) \leq K_{m,\alpha}\Lambda^\alpha\rho$$

for any $\mu \in \mathcal{M}_\Lambda\left(S_\rho^{m-1}\right)$.

Now, let

$$f\left(r\right) = \chi\left(T\lfloor S_{r\rho}^{m-1}\right).$$

Then, since the optimal transport path T contains no loops, we know f is an decreasing function of $r \in [0,1]$. Also, by Lemma 4.6,

$$\mathbf{M}^\alpha\left(T\lfloor S_{r\rho}^{m-1}\right) \geq \left(f\left(r\right)\right)^{\alpha-1}\Lambda^\alpha.$$

Now,

$$\begin{aligned}
K_{m,\alpha}\Lambda^\alpha\rho &\geq \mathbf{M}^\alpha\left(T\right) \\
&\geq \int_0^\rho \mathbf{M}^\alpha\left(T\lfloor S_s^{m-1}\right) ds \\
&\geq \int_0^1 \mathbf{M}^\alpha\left(T\lfloor S_{r\rho}^{m-1}\right) \rho dr \\
&\geq \int_0^1 \left(f\left(r\right)\right)^{\alpha-1}\Lambda^\alpha\rho dr \\
&\geq \int_r^1 \left(f\left(s\right)\right)^{\alpha-1}\Lambda^\alpha\rho ds \\
&\geq \left(f\left(r\right)\right)^{\alpha-1}\left(1-r\right)\Lambda^\alpha\rho.
\end{aligned}$$

Therefore,

$$f\left(r\right) \geq \left(\frac{1-r}{K_{m,\alpha}}\right)^{\frac{1}{1-\alpha}}.$$

□

The following lemma is the key lemma to get our regularity result:

Lemma 4.9. *Suppose* $\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i} \in \mathcal{A}_1\left(S_1^{m-1}\right)$ *with*

$$\sum_{i=k}^{\infty} |a_i|^{\alpha} \leq \frac{1}{2} \min_{i=1,\cdots,k} \{|a_i|\}.$$

Let $T \in Path\,(\mu, \delta_0)$ *be an optimal transport path. Thus, for any* i*, there is a real flat chain* $T_i \in Path\,(a_i\delta_{x_i}, a_i\delta_0)$ *such that*

$$T = \sum_{i=1}^{\infty} T_i.$$

Suppose also that $\{T_i\}_{i=1}^{k}$ *are disjoint in* $B_1(0) \setminus \{0\}$*, then there exists a number* $\rho > 0$ *such that*

$$T \lfloor B_\rho (0)$$

is a cone consists of k *line segments with* $\{0\}$ *being a common endpoint.*

Proof. Let the set Γ be the support of the flat chain $\sum_{i=1}^{k} T_i$ and consider a map

$$g : \{k+1, k+2, \cdots\} \to \Gamma$$

by sending each $i > k$ to the first intersection point of T_i and Γ.

Now, it is sufficiently to show that the set

$$A = B_{1/8}(0) \cap \{g(i) : i = k+1, k+2, \cdots\}$$

is bounded away from $\{0\}$ by some small positive number ρ.

In fact, for any $p \in A$, let

$$\Gamma_p = \sum_{i \in g^{-1}(p)} T_i \lfloor (B_1(0) \setminus \Gamma).$$

Then, $\Gamma_p \in Path\left(\sum_{i \in g^{-1}(p)} a_i \delta_{x_i}, (\sum_{i \in g^{-1}(p)} a_i)\delta_p\right)$ is an optimal transport path. Since

$$B_{3/4}(p) \subset B_1(0) \text{ and } B_{1/4}(0) \subset B_{3/8}(p),$$

by Lemma 4.8, we have

$$\chi\left(\Gamma_p \cap S_{1/4}(0)\right) \geq \chi\left(\Gamma_p \cap S_{3/8}(p)\right) \geq D$$

where $D = (2K_{m,\alpha})^{\frac{-1}{1-\alpha}}$.

Choose $\epsilon < 1/8$ small enough so that

$$\left(1+\frac{1}{D}\right)^{\alpha} - \left(\frac{1}{D}\right)^{\alpha} \geq 64\epsilon.$$

For any $p \in A$, let $\mu_p = \Gamma_p \cap S_{1/4}(0)$ be the slicing of Γ_p with the sphere $S_{1/4}(0)$. Suppose $\mu_p = \lambda_p \delta_{y_p} + \mu_p^{'}$ with $y_p \in S_{1/4}(0)$ and

$$|\lambda_p| = \chi(\mu_p)\mathbf{M}(\mu_p) \geq D\mathbf{M}(\mu_p).$$

If A is not bounded away from 0, then one of Γ's branch, say T_1, must contain a sequence $\{p_i\}$ of disjoint points in $A \cap T_1$ such that $\{p_i\}$ converges to 0 and $\{y_{p_i}\}$ converges to some point $y \in S_{1/4}(0)$. One may also choose $\{\lambda_{p_i}\}$ to be of the same sign. For convenience, we assume $\{\lambda_{p_i}\}$ to be positive.

From this sequence, we first fix a point $p \in B_\epsilon(0) \cap \{p_i\} \subset A \cap T_1$ with $y_p \in B_\epsilon(y) \cap S_{1/4}(0)$. Since $\mathbf{M}(T) < +\infty$, we may assume $\mathbf{M}(T \cap S_{1/4}(0)) < +\infty$, otherwise one can replace $1/4$ by some number very close to $1/4$. Since there are infinitely many $\{y_{p_i}\}$ contained in $B_\epsilon(y) \cap S_{1/4}(0)$, and

$$\sum_i \lambda_{p_i} \leq \sum_i \mathbf{M}(\mu_p) \leq \mathbf{M}(T \cap S_{1/4}(0)) < +\infty,$$

there exists a y_q among them corresponding to some $q \in B_\epsilon(0) \cap \{p_i\}$ with

$$\lambda_q^{1-\alpha} < \frac{4U\epsilon}{\mathbf{M}^\alpha(T)},$$

where $U = \chi(u_p)$.

Now, we construct a new path $R \in Path(\mu, \delta_0)$ as follows. Namely, let γ_p be the unique flat chain from y_p to q and γ_q be the unique flat chain from y_q to q. Both γ_p and γ_q are of multiplicity 1. Now let

$$R = T + \lambda_q(\gamma_p - \gamma_q) + \lambda_q[[y_p, y_q]]$$

where $[[y_p, y_q]]$ denotes the line segment from y_p to y_q. We still have $R \in Path(\mu, \delta_0)$. We begin to compare the $\mathbf{M}^\alpha$ costs of R and T as follows. Since

$$(1+x)^\alpha - 1 \leq \alpha x$$

on $x \in [0,1]$, by the Definition (2.1) of $\mathbf{M}^\alpha$ mass, we easily have

$$\mathbf{M}^\alpha(\Gamma_p + \lambda_q\gamma_p) - \mathbf{M}^\alpha(\Gamma_p) \leq \alpha\left(\frac{\lambda_q}{U}\right)\mathbf{M}^\alpha(\Gamma_p).$$

Also, since $(x+1)^\alpha - x^\alpha$ is a decreasing function on $x \in [0, \frac{1}{D}]$, we get

$$\mathbf{M}^\alpha(\Gamma_q) - \mathbf{M}^\alpha(\Gamma_q - \lambda_q\gamma_q) \geq \left[\left(1+\frac{1}{D}\right)^\alpha - \left(\frac{1}{D}\right)^\alpha\right]\lambda_q^\alpha length(\gamma_q) \geq 8\lambda_q^\alpha\epsilon$$

Thus,

$$\begin{aligned}&\mathbf{M}^{\alpha}(R)-\mathbf{M}^{\alpha}(T)\\ &\leq \lambda_q^{\alpha}\epsilon+\mathbf{M}^{\alpha}(\varGamma_p+\lambda_q\gamma_p)+\mathbf{M}^{\alpha}(\varGamma_q-\lambda_q\gamma_q)+|p-q|\lambda_q^{\alpha}-\mathbf{M}^{\alpha}(\varGamma_p)-\mathbf{M}^{\alpha}(\varGamma_q)\\ &\leq 2\lambda_q^{\alpha}\epsilon+\alpha\mathbf{M}^{\alpha}(\varGamma_p)\left(\frac{\lambda_q}{U}\right)-8\lambda_q^{\alpha}\epsilon\\ &\leq \mathbf{M}^{\alpha}(T)\left(\frac{\lambda_q}{U}\right)-6\lambda_q^{\alpha}\epsilon<0\,.\end{aligned}$$

A contradiction to the optimality of T. □

Now, we have our main theorem about the regularity of optimal transport paths:

Theorem 4.10. *Suppose $T \in Path\,(\mu^+,\mu^-)$ is an optimal transport path with finite $\mathbf{M}^{\alpha}$ cost. For any point $p \in spt\,(T) \setminus spt\,(\mu^+ \cup \mu^-)$, there is an open neighborhood B_p of p, such that*

$$T\lfloor B_p$$

is a cone consists of finitely many line segments with suitable multiplicities.

Proof. For any $p \in spt(T) \setminus [spt(\mu^+) \cup spt(\mu^-)]$, by Proposition 3.3, there exists a sequence $\{\lambda_j\}$, $\lambda_j \to 0$ such that

$$T_j = \eta_{p,\lambda_j\#}(T\lfloor \bar{B}_{\lambda_j}(p)) \to C_p, \tag{4.4}$$

the tangent cone of T at p, in both W^{α} flat metric and W^1 flat metric. By Proposition 4.5, the cone C_p must be of the form

$$C_p = \varSigma_{i=1}^k m_i[|p_i, 0|]$$

satisfying a balance equation

$$\varSigma_{i=1}^k \frac{m_i p_i}{|m_i|^{1-\alpha}} = 0,$$

where $\{p_i\} \subset S_1^{m-1}(0)$ and $[|p_i, 0|]$ denotes the line segment from p_i to 0.

Let $I_i \subset S_1^{m-1}$ be a fixed small open neighborhood of p_i... Choose

$$0 < \epsilon < \frac{1}{3}\min_{i=1,\cdots,k}\{|m_i|\}$$

small enough so that

$$(1-\frac{\epsilon^2}{|m_i|})^{\alpha} > 1-\frac{2\alpha}{|m_i|}\epsilon^2,$$

for each i.

Let

$$\mu_i = T_i \cap S_1^{m-1}(0)$$

be an infinite atomic measure. By (4.4), when j large enough, we have

$$\mathbf{M}(m_i\delta_{p_i}) - \epsilon^2 < \mathbf{M}(\mu_i) < \mathbf{M}(m_i\delta_{p_i}) + \epsilon^2$$
$$\mathbf{M}(m_i\delta_{p_i})^\alpha - \epsilon^2 < \mathbf{M}^\alpha(\mu_i) < \mathbf{M}(m_i\delta_{p_i})^\alpha + \epsilon^2.$$

Thus, by Lemma 4.7,

$$\chi(\mu_i) \geq (1-\epsilon)^{\frac{1}{1-\alpha}}$$

for each i. This means that I_i contains a dominated Dirac measure $\delta_{p'_i}$ of μ_i with multiplicity $\chi(\mu_i)\mathbf{M}(\mu_i)$. From each p'_i, there is a unique path Γ_i from p'_i to 0. Choose j large enough, we may assume $\Gamma_i \setminus \{0\}$ are disjoint (except at 0). Thus, by our key Lemma 4.9, T_j is a finite sum of line segments nearby a small neighborhood of 0. Thus, we proved the theorem. □

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