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ON LANDSCAPE FUNCTIONS ASSOCIATED WITH TRANSPORT PATHS

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Abstract. In this paper, we introduce a multiple-sources version of the landscape function which was originally introduced by Santambrogio in [10]. More precisely, we study landscape functions associated with a transport path between two atomic measures of equal mass. We also study p-harmonic functions on a directed graph for nonpositive p. We show an equivalence relation between landscape functions associated with an α-transport path and p-harmonic functions on the underlying graph of the transport path for $p = \alpha/(\alpha - 1)$, which is the conjugate of α. Furthermore, we prove the Lipschitz continuity of a landscape function associated with an optimal transport path on each of its connected components.

1. Introduction.

The optimal transport problem aims at finding an optimal way to transport a given probability measure into another. In contrast to the well-known Monge-Kantorovich problem (see [11] and references therein), the ramified optimal transportation problem aims at modeling a branching transport network by an optimal transport path between two given probability measures. An essential feature of such a transport path is to favor group transportation in a large amount. Representative studies in this field can be found for instance in Bernot, Caselles and Morel [2, 3], Brancolini, Buttazzo and Santambrogio [4], Brancolini and Solimini [5], Brasco, Buttazzo and Santambrogio [6], Devillanova and Solimini [7], Maddalena, Morel and Solimini[8], Morel and Santambrogio [9], Santambrogio [10], Xia [12, 13, 14, 15, 16, 17, 18], Xia and Vershynina [19].

In particular, in [10], Santambrogio introduced a concept called "the landscape function" associated with an optimal irrigation pattern from a Dirac mass to another probability measure μ. This concept is motivated from the study of river basins, and represents the elevation of the landscape. We refer to the introduction of [10] for a detailed discussion about the geophysics background on this topic. A main result of [10] is the Hölder regularity of the landscape function when the targeting measure μ has a density bounded from below by a positive constant and its support satisfies the "type A" condition. These hypotheses on the targeting measure μ imply that the measure μ is Ahlfors regular from below in the ambient Euclidean space. Later in [5], Brancolini and Solimini extended the Hölder continuity of the

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landscape function to general Ahlfors regular measures from below. In the context of ramified optimal transportation, the landscape function at a point can be viewed as the marginal transportation cost from the point to the source. Analogous ideas of the landscape function have been used in applications such as [20] in modeling diffusion-limited aggregation driven by optimal transportation and [21] about the ramified optimal allocation problem in economics.

In the works [5, 10], it was assumed that the initial measure is a Dirac mass (i.e. a single common source). In this article, we generalize the concept of landscape function by allowing multiple sources rather than a single source. For simplicity, we will only consider transport paths between atomic measures in this article. The continuous version will be left for future exploration.

We organized this paper as follows: First, we introduce preliminary notations and concepts about graph theory and ramified optimal transportation in section 2. In particular, we introduce the concept of the incidence matrix of a directed graph and see how to relate it with concepts in ramified optimal transportation. In section 3, we generalize the concept of landscape functions associated with a transport path between atomic measures. By means of the incidence matrix, landscape functions can be solved using linear algebra. Then, we also explore some properties of landscape functions. In section 4, we consider p-harmonic functions defined on a directed graph for $p \leq 0$. These p-harmonic functions minimize the corresponding p-energy with a Dirichlet boundary condition. In section 5, we showed an equivalence relation between an α-landscape function associated with a transport path and p-harmonic function on the underlying graph of the transport path when the parameters $p \leq 0$ and $0 \leq \alpha < 1$ are Hölder conjugate. In section 6, we studied properties of a landscape function associated with an optimal transport path. In particular, we showed Lipschitz continuity of any landscape function associated with an optimal transport path on each connected component of the transport path.

2. **Preliminaries.**

We first recall the definition of the incidence matrix of a directed graph as given in [1]. Let $G = \{V(G), E(G)\}$ be a directed graph with a vertex set $V(G) = \{v_1, v_2, \cdots, v_J\}$ and an edge set $E(G) = \{e_1, e_2, \cdots, e_K\}$ consisting of directed edges e_j's. The (vertex-edge) *incidence matrix* of G, denoted by $Q(G)$, is the $J \times K$ matrix defined as follows. For any $i = 1, \cdots, J$ and $j = 1, \cdots, K$, the (i, j)-entry of $Q(G)$ is

$$Q(G)(i,j) = \begin{cases} 1, & \text{if } e_j^+ = v_i \\ -1, & \text{if } e_j^- = v_i \\ 0, & \text{otherwise} \end{cases}, \tag{2.1}$$

where e_j^- is the starting endpoint of e_j and e_j^+ is the ending point of e_j. Sometimes, we denote $Q(G)(i,j)$ by $Q(G)(v_i, e_j)$.

Here are some basic properties of incidence matrix $Q(G)$ that we will use in this article. Proofs of these properties can be found for instance in [1].

Proposition 2.1. *Suppose $Q(G)$ is the incidence matrix of a directed graph G.*

1. *If G has g connected components, then the rank of $Q(G)$ is $J - g$.*
2. *Let G be a connected graph on J vertices. Then the column space of $Q(G)$ consists of all vectors $x \in \mathbb{R}^J$ such that $\sum_{i=1}^J x_i = 0$.*
3. *Columns of $Q(G)$ are linearly independent if and only if G is an acyclic graph.*

We now recall some concepts about ramified optimal transportation as in Xia [12]. Let X be a compact convex subset of a Euclidean space $\mathbb{R}^m$, equipped with the standard Euclidean distance $\|\cdot\|$. Recall that a positive Radon measure $\mathbf{a}$ on X is *atomic* if $\mathbf{a}$ is a finite sum of Dirac measures with positive multiplicities. That is

$$\mathbf{a} = \sum_{i=1}^{k} m_i \delta_{x_i}$$

for some integer $k \geq 1$ and some points $x_i \in X$, $m_i > 0$ for each $i = 1, \cdots, k$. The mass of the measure $\mathbf{a}$ is $\sum_{i=1}^{k} m_i$.

Definition 2.2. Suppose

$$\mathbf{a} = \sum_{i=1}^{k} m_i \delta_{x_i} \text{ and } \mathbf{b} = \sum_{j=1}^{\ell} n_j \delta_{y_j} \tag{2.2}$$

are two atomic measures on X of equal mass. A *transport path* $P = \{V(P), E(P), w\}$ *from* $\mathbf{a}$ *to* $\mathbf{b}$ is a directed graph $G_P = \{V(P), E(P)\}$ together with a weight function $\omega : E(P) \to (0, +\infty)$ such that

1. $V(P) \subseteq \mathbb{R}^m$ and $\{x_1, x_2, ..., x_k\} \cup \{y_1, y_2, ..., y_\ell\} \subseteq V(P)$;
2. Each directed edge $e \in E(P)$ is a line segment from the starting endpoint $e^- \in V(P)$ to the ending endpoint $e^+ \in V(P)$;
3. The weight function $\omega : E(P) \to (0, +\infty)$ satisfies a balance equation

$$\sum_{e \in E(P), e^- = v} \omega(e) = \sum_{e \in E(P), e^+ = v} \omega(e) + \begin{cases} m_i, \text{ if } v = x_i \text{ for some } i = 1, ..., k \\ -n_j, \text{ if } v = y_j \text{ for some } j = 1, ..., \ell \\ 0, \text{ otherwise} \end{cases} \tag{2.3}$$

at any vertex $v \in V(P)$.

The directed graph $G_P = \{V(P), E(P)\}$ is called the *underlying graph* of the transport path P. Also, we denote $Path(\mathbf{a}, \mathbf{b})$ as the space of all transport paths from $\mathbf{a}$ to $\mathbf{b}$.

For each transport path $P \in Path(\mathbf{a}, \mathbf{b})$ as above and any $\alpha \in [0, 1]$, the $\mathbf{M}_\alpha$ cost of P is defined by

$$\mathbf{M}_\alpha(P) := \sum_{e \in E(P)} [\omega(e)]^\alpha \, length(e), \tag{2.4}$$

where $length(e) = \|e^+ - e^-\|$ is the Euclidean distance between endpoints e^+ and e^- of edge e. Ramified optimal transportation studies how to find a transport path to minimize the $\mathbf{M}_\alpha$ cost, i.e.,

$$\min_{P \in Path(\mathbf{a}, \mathbf{b})} \mathbf{M}_\alpha(P), \tag{2.5}$$

whose minimizer is called an $\alpha-$optimal transport path from $\mathbf{a}$ to $\mathbf{b}$.

For any $\alpha-$optimal transport path $P = \{V(P), E(P), \omega\}$, the first variation of $\mathbf{M}_\alpha(P)$ yields

$$\sum_{e^+ = v} [\omega(e)]^\alpha \, \vec{e} = \sum_{e^- = v} [\omega(e)]^\alpha \, \vec{e} \tag{2.6}$$

at any vertex $v \in V(P) \setminus \{x_1, \cdots, x_k, y_1, \cdots, y_\ell\}$, where $\vec{e}$ is the unit directional vector of the directed edge e.

According to Xia [12, Proposition 2.1], the underlying graph of an optimal transport path is *acyclic*, i.e. contains no cycles. Thus, without loss of generality, we will only consider acyclic transport paths in this article.

For any atomic measures $\mathbf{a}$ and $\mathbf{b}$ on X of equal mass, define the minimum transportation cost as

$$d_\alpha(\mathbf{a},\mathbf{b}) := \min\left\{\mathbf{M}_\alpha(P) : P \in Path(\mathbf{a},\mathbf{b})\right\}. \tag{2.7}$$

As shown in Xia [12], d_α is indeed a metric on the space of atomic measures of equal mass. Also, for each $\lambda > 0$, it holds that

$$d_\alpha(\lambda\mathbf{a},\lambda\mathbf{b}) = \lambda^\alpha d_\alpha(\mathbf{a},\mathbf{b}). \tag{2.8}$$

Let $P = \{V(P), E(P), \omega\}$ be a transport path from $\mathbf{a}$ to $\mathbf{b}$ with $V(P) = \{v_1, v_2, \cdots, v_J\}$ and $E(P) = \{e_1, e_2, \cdots, e_K\}$. Then in matrix notation, the balance equation (2.3) can be expressed as

$$\boldsymbol{w} Q(G_P)' = \boldsymbol{c} \tag{2.9}$$

where $Q(G_P)'$ denotes the transpose of the incidence matrix $Q(G_P)$ of the underlying graph G_P, $\boldsymbol{w} = [\omega(e_1), \cdots, \omega(e_K)]$ and $\boldsymbol{c} = [c(v_1), \cdots, c(v_J)]$ with for each $v \in V(G) = \{v_1, v_2, \cdots, v_J\}$,

$$c(v) = \begin{cases} -m_i, \text{ if } v = x_i \text{ for some } i = 1, ..., k \\ n_j, \text{ if } v = y_j \text{ for some } j = 1, ..., \ell \\ 0, \text{ otherwise} \end{cases}.$$

By (3) of Proposition 2.1, when G_P is acyclic, rows of $Q(G_P)'$ are linearly independent. If in addition G_P is connected, then by (2) of Proposition 2.1, $\boldsymbol{c}$ lies in the row space of $Q(G_P)'$. In this case, the linear system (2.9) has a unique solution $\boldsymbol{w}$. In other words, when G_P is acyclic and connected, then $\boldsymbol{w}$ is uniquely determined by the incidence matrix $Q(G_P)$ and $\boldsymbol{c}$.

3. **Landscape functions associated with a transport path.** Suppose the initial source is a single Dirac mass δ_S and the targeting measure μ is a probability measure on X. In [10], Santambrogio introduced the landscape function $z(x)$. It is the marginal transportation cost of the mass from the initial source S to the point x. A main result of [10] is the Hölder regularity of the landscape function when the targeting measure μ has a density bounded from below by a positive constant and its support satisfies the "type A" condition.

Let $\mathbf{a} = \delta_S$ and $\mathbf{b}$ be an atomic measure on X. Let $P = \{V(P), E(P), w\}$ be an acyclic transport path from $\mathbf{a}$ to $\mathbf{b}$. Then for any x in the support of P, there is a unique polyhedral curve γ_x on P from the initial source S to x. In this case, Santambrogio's landscape function $z(x)$ associated with P can be simply written as

$$z(x) = \int_{\gamma_x} \theta(s)^{\alpha-1}\, d\mathcal{H}^1(s)$$

where $\mathcal{H}^1$ represents the 1−dimensional Hausdorff measure, and θ is a function on P defined as follows: for any point p on the support of P, set

$$\theta(p) := \begin{cases} w(e), & \text{if } p \text{ is in the interior of some edge } e \in E(P) \\ 1, & \text{if } p = S \\ w(e), & \text{if } p \text{ is the ending endpoint } e^+ \text{ for some } e \in E(P) \end{cases}, \tag{3.1}$$

which represents the mass flowing through p.

In general, an atomic measure $\mathbf{a}$ is not necessarily a single source but of the form in (2.2) that contains multiple-sources. In this section, we will study a multiple-sources version of landscape functions associated with a transport path.

Definition 3.1. Let $P = \{V(P), E(P), \omega\}$ be an acyclic transport path from $\mathbf{a}$ to $\mathbf{b}$ with $V(P) = \{v_1, v_2, \cdots, v_J\}$, $E(P) = \{e_1, e_2, \cdots, e_K\}$, and $0 \leq \alpha < 1$. A function $Z : V(P) \to \mathbb{R}$ is called an $\alpha-$landscape function associated with P if for each edge $e \in E(P)$, it holds that

$$Z\left(e^+\right) - Z\left(e^-\right) = \omega(e)^{\alpha-1}\, length\,(e)\,. \tag{3.2}$$

Using matrix notations the system (3.2) of linear equations on Z becomes

$$ZQ\left(G_P\right) = D \tag{3.3}$$

where $Z = [Z(v_1), Z(v_2), \cdots, Z(v_J)]$, $Q(G_P)$ is the incidence matrix of the underlying graph G_P of P and

$$D = [\omega(e_1)^{\alpha-1}\, length\,(e_1), \cdots, \omega(e_K)^{\alpha-1}\, length\,(e_K)].$$

If G_P has g connected components, then by (1) of Proposition 2.1, the rank of $Q(G_P)$ is $J - g$. Since G_P is acyclic, by (3) of Proposition 2.1, columns of $Q(G_P)$ are linearly independent. As a result, the solution space of the system (3.3) has dimension g. In particular, for a connected transport path P, its α-landscape function is unique up to a constant. This agrees with the motivation that the landscape function represents the elevation of the landscape.

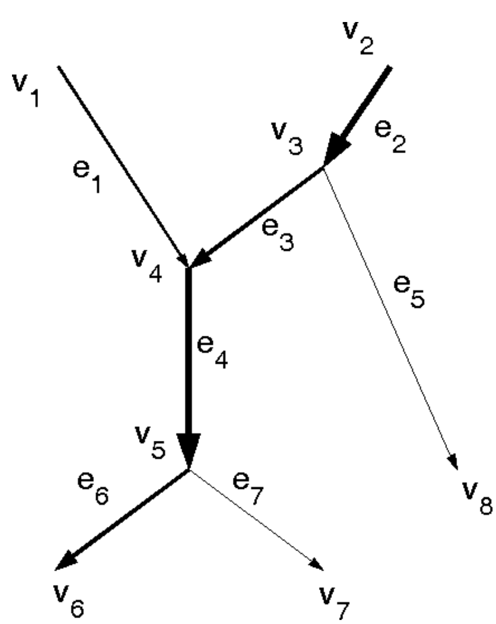

FIGURE 1. Landscape functions associated with a transport path.

Example 3.2. We consider landscape functions associated with the transport path P as shown in Figure 1. The coordinates of vertices $\{v_i\}$ in $\mathbb{R}^2$ are given by the following table:

vertex $v=(x,y)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
x	3	5.5	5	4	4	3	5	6
y	9	9	8	7	5	4	4	5

Let

$$\begin{aligned}\mathbf{a} &= \frac{1}{4}\delta_{v_1}+\frac{3}{4}\delta_{v_2}\\ \mathbf{b} &= \frac{1}{2}\delta_{v_6}+\frac{1}{4}\delta_{v_7}+\frac{1}{4}\delta_{v_8}\end{aligned}$$

Then, the incidence matrix of G_P is

$$Q\left(G_P\right)=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & -1 & 0 & 0\\ 1 & 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & -1 & -1\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{c}=\begin{pmatrix} -1/4\\ -3/4\\ 0\\ 0\\ 0\\ 1/2\\ 1/4\\ 1/4 \end{pmatrix}'$$

By calculating the reduced row echelon form corresponding to the system (2.9), the weight function $\boldsymbol{w}$ solving (2.9) is given by

$$\boldsymbol{w}=[1/4,3/4,1/2,3/4,1/4,1/2,1/4].$$

The lengths of edges are given by the vector

$$L=[\sqrt{5},\sqrt{5}/2,\sqrt{2},2,\sqrt{10},\sqrt{2},\sqrt{2}].$$

Now, we can find landscape functions associated with P by solving the system (3.3). For instance, when $\alpha=0.5$,

$$Z=[0,1.1811,2.4721,4.4721,6.7815,8.7815,9.6100,8.7967]$$

is a particular α-landscape function associated with P. When $\alpha=0.85$,

$$Z=[0,0.0164,1.1838,2.7529,4.8411,6.4103,6.5822,5.0770]$$

is a particular α-landscape function associated with P.

Now, we study properties of landscape functions as follows.

Proposition 3.3. *For any $\alpha-$landscape function Z associated with a transport path $P\in Path\left(\mathbf{a},\mathbf{b}\right)$, it holds that*

$$\int_X Zd\left(\mathbf{b}-\mathbf{a}\right)=\mathbf{M}_\alpha\left(P\right).$$

In particular, if P is an α-optimal transport path from $\mathbf{a}$ to $\mathbf{b}$, then

$$\int_X Zd\left(\mathbf{b}-\mathbf{a}\right)=d_\alpha\left(\mathbf{a},\mathbf{b}\right).$$

Proof. By (3.2) and the balance equation (2.3) on ω,

$$\begin{aligned}
\mathbf{M}_\alpha(P) &= \sum_{e\in E(P)} \omega(e)^\alpha \, length(e) \\
&= \sum_{e\in E(P)} \left(Z(e^+) - Z(e^-)\right)\omega(e) \\
&= \sum_{v\in V(P)} \left(\sum_{\substack{e\in E(P)\\ e^+=v}} \omega(e) - \sum_{\substack{e\in E(P)\\ e^-=v}} \omega(e) \right) Z(v) \\
&= -\sum_{i=1}^{k} Z(x_i)\, m_i + \sum_{j=1}^{\ell} Z(y_j)\, n_j \\
&= \int_X Z d(\mathbf{b}-\mathbf{a}).
\end{aligned}$$

□

Let Z be an α-landscape function associated with a transport path $P \in Path(\mathbf{a},\mathbf{b})$. For each edge $e \in E(P)$, define

$$\nabla Z(e) = \left(\frac{Z(e^+) - Z(e^-)}{length(e)} \right) \vec{e}$$

where $\vec{e}$ is the unit directional vector of e. Then, by (3.2),

$$\nabla Z(e) = \omega(e)^{\alpha-1}\, \vec{e} \text{ and } \omega(e) = |\nabla Z(e)|^{1/(\alpha-1)}. \tag{3.4}$$

Proposition 3.4. *For any α-landscape function Z associated with a transport path P, it holds that*

$$\mathbf{M}_\alpha(P) = \sum_{e\in E(P)} |\nabla Z(e)|\, \omega(e)\, length(e)$$

Proof. The transport cost of P is

$$\begin{aligned}
\mathbf{M}_\alpha(P) &= \sum_{e\in E(P)} \omega(e)^\alpha \, length(e) \\
&= \sum_{e\in E(P)} \omega(e)^{\alpha-1} \, length(e)\, \omega(e) \\
&= \sum_{e\in E(P)} |\nabla Z(e)| \, length(e)\, \omega(e).
\end{aligned}$$

□

Now, we will prove an estimate on the landscape function. Such an estimate will be used later in Proposition 6.3 to show the Lipschitz continuity of landscape function on an optimal transport path. This result is analogous to Theorem 3.7 in [5] by Brancolini and Solimini.

Proposition 3.5. *Let Z be an α-landscape function associated with an acyclic transport path $P \in Path(\mathbf{a},\mathbf{b})$. Then, for any $x, y \in V(P)$ on the same connected components of G_P, it holds that*

$$\alpha\epsilon\left(Z(x) - Z(y)\right) \geq \mathbf{M}_\alpha(P + \epsilon\gamma) - \mathbf{M}_\alpha(P) \tag{3.5}$$

for any ϵ with $|\epsilon| \leq \min_{e\subseteq\gamma} \omega(e)$, where γ is the unique polyhedral curve on G_P from x to y and $P+\epsilon\gamma$ is an acyclic transport path from $\mathbf{a}+\epsilon\delta_x$ to $\mathbf{b}+\epsilon\delta_y$.

Proof. Since P is acyclic, there exists a unique polyhedral curve γ on P from x to y. This polyhedral curve γ is represented by a list of vertices $\{v_0, v_1, \cdots, v_k\} \subseteq V(P)$ such that $v_0 = x$, $v_k = y$ and $[v_{i-1}, v_i] = \sigma_i e_i$ for some edge e_i in $E(P)$ for each $i = 1, 2, \cdots, k$, where

$$\sigma_i = \begin{cases} 1, & \text{if } e_i \text{ has the same direction with } \gamma \\ -1, & \text{if } e_i \text{ has the opposite direction with } \gamma \end{cases}.$$

By definition, the α-landscape function Z satisfies

$$Z(v_i) - Z(v_{i-1}) = \omega(e_i)^{\alpha-1}\, length(e_i)\, \sigma_i$$

for each i. Using the inequality

$$(1+t)^\alpha \leq 1 + \alpha t$$

when $t > -1$, it follows that whenever $\epsilon > -\omega(e_i)$,

$$(\omega(e_i)+\epsilon)^\alpha = \omega(e_i)^\alpha \left(1 + \frac{\epsilon}{\omega(e_i)}\right)^\alpha \leq \omega(e_i)^\alpha \left(1 + \alpha\frac{\epsilon}{\omega(e_i)}\right).$$

That is,

$$(\omega(e_i)+\epsilon)^\alpha - \omega(e_i)^\alpha \leq \alpha\epsilon\omega(e_i)^{\alpha-1}.$$

Now, when $|\epsilon| < \min\{\omega(e_i) : i = 1, \cdots, k\}$, it holds that $\epsilon\sigma_i > -\omega(e_i)$ for each i and

$$\begin{aligned} \alpha\epsilon(Z(x) - Z(y)) &= \alpha\epsilon \sum_{i=1}^k (Z(v_i) - Z(v_{i-1})) \\ &= \sum_{i=1}^k \alpha\epsilon\sigma_i\omega(e_i)^{\alpha-1}\, length(e_i) \\ &\geq \sum_{i=1}^k \left[(\omega(e_i)+\sigma_i\epsilon)^\alpha - \omega(e_i)^\alpha\right] length(e_i) \\ &= \mathbf{M}_\alpha(P+\epsilon\gamma) - \mathbf{M}_\alpha(P). \end{aligned}$$

□

Remark 3.6. By linear extension of a landscape function Z on each edge e in $E(G)$, we can view Z as a continuous and "edge-wise" linear function on the support of the transport path P. The above proof still work and the inequality (3.5) holds for all x, y in the support of P.

4. p-harmonic functions on directed graphs.

Let G be an acyclic directed graph with a vertex set $V(G)$ and an edge set $E(G)$ of directed edges.

Definition 4.1. For any two vertices $v, \tilde{v} \in V(G)$, we say $v \prec \tilde{v}$ if there exists a list of edges $\{e_{i_1}, e_{i_2}, \cdots, e_{i_k}\}$ in $E(G)$ with $e_{i_h}^+ = e_{i_{h+1}}^-$ for $h = 1, 2, \cdots, k-1$, and $e_{i_1}^- = v$, $e_{i_k}^+ = \tilde{v}$.

Note that since G is acyclic, the partial order $\prec$ is well-defined on $V(G)$. In particular, $e^- \prec e^+$ for any edge $e \in E(G)$.

Definition 4.2. Let $\tilde{V}$ be a subset of $V(G)$. A function $u : \tilde{V} \to \mathbb{R}$ is monotone increasing with respect to G if for any $x, y \in \tilde{V}$ with $x \prec y$, it holds that $u(x) < u(y)$.

Clearly, when $\tilde{V} = V(G)$, a function $u : V(G) \to \mathbb{R}$ is monotone increasing with respect to G if and only if for any edge $e \in E(G)$,

$$u\left(e^{+}\right) > u\left(e^{-}\right). \tag{4.1}$$

Let $\mathcal{F}_G$ be the family of all monotone increasing functions $u : V(G) \to \mathbb{R}$ with respect to G. For instance, any landscape function Z associated with an acyclic transport path P is monotone increasing with respect to the underlying graph G_P.

For each $u \in \mathcal{F}_G$ and $p \leq 0$, the *p-energy* of u is given by

$$\mathbf{E}_p(u) = \sum_{e \in E(G)} |\nabla u(e)|^p \, length(e) \tag{4.2}$$

where for each $e \in E(G)$ with unit directional vector $\vec{e}$,

$$\nabla u(e) = \frac{u(e^+) - u(e^-)}{length(e)} \vec{e}.$$

We are interested in minimizing $\mathbf{E}_p(u)$ among $u \in \mathcal{F}_G$ with a Dirichlet boundary condition.

Remark 4.3. One may also consider the minimization of

$$\tilde{\mathbf{E}}_p(u) =: \int_G |\nabla u|^p \, d\mathcal{H}^1(x) = \sum_{e \in E(G)} \int_e |\nabla u|^p \, d\mathcal{H}^1(x) \tag{4.3}$$

among all continuous real valued functions u on the support of G such that u is differentiable and monotone increasing on each directed edge e in $E(G)$. By considering the first variation of $\tilde{\mathbf{E}}_p$, the minimum of each $\int_e |\nabla u|^p \, d\mathcal{H}^1(x)$ is achieved when u is linear on e. Thus, it is sufficient to restrict u to be linear on each edge when considering the minimization problem (4.3). The corresponding energy $\tilde{\mathbf{E}}_p$ for an edge-wise linear function u is $\mathbf{E}_p(u)$ in the form of (4.2).

Example 4.4. Without the monotone increasing condition (4.1), a minimizer of $\mathbf{E}_p$ may fail to exist when $p < 0$. For instance, let G be a directed graph with $V(G) = \{-1, 0, 1\}$ and $E(G) = \{[-1, 0], [0, 1]\}$. We consider the minimization problem: minimize

$$\mathbf{E}_p(u) = |u(0) - u(-1)|^p + |u(1) - u(0)|^p$$

among $u : V(G) \to \mathbb{R}$ with boundary conditions $u(-1) = 0$ and $u(1) = 1$. Clearly, for any $u : V(G) \to \mathbb{R}$, $\mathbf{E}_p(u) > 0$. Consider $u_n : V(G) \to \mathbb{R}$ with $u_n(-1) = 0, u_n(0) = n$ and $u_n(1) = 1$. Then, $\mathbf{E}_p(u_n) = n^p + (n-1)^p \to 0$ as $n \to \infty$ when $p < 0$. This indicates that the infimum of $\mathbf{E}_p(u)$ with

$$u \in \{u : V(G) \to \mathbb{R} \text{ with } u(-1) = 0 \text{ and } u(1) = 1\}$$

is zero. Thus, without the condition (4.1), the minimizer of $\mathbf{E}_p$ does not exist in this case. Nevertheless, with the monotone increasing condition (4.1), the problem has a unique minimizer given by $u^*(-1) = 0, u^*(0) = 1/2$ and $u^*(1) = 1$. The minimum value is $\mathbf{E}_p(u^*) = 2^{1-p}$.

Let ∂G be a subset of $V(G)$ such that

$$\left\{ \begin{array}{c} v \in V(G): \text{ either there is no edge } e \in E(G) \text{ with } e^{+} = v \\ \text{or there is no edge } e \in E(G) \text{ with } e^{-} = v \end{array} \right\} \subseteq \partial G \subseteq V(G).$$

In other words, ∂G contains all source and sink vertices in $V(G)$, and may contain some other vertices. We view ∂G as the boundary set of $V(G)$. Without loss of generality, we may assume that

$$V(G) \backslash \partial G = \{v_1, v_2, \cdots, v_I\}.$$

For a given function $u_0 : \partial G \to \mathbb{R}$, we consider the map

$$g : U = \{u \in \mathcal{F}_G | u = u_0 \text{ on } \partial G\} \to \mathbb{R}^I$$

given by $g(u) = (u(v_1), u(v_2), \cdots, u(v_I))$ for each $u \in U$. This map g is injective, and we denote $\Omega \subseteq \mathbb{R}^I$ as the image of U under the map g.

Lemma 4.5. *The set Ω is a nonempty convex domain in $\mathbb{R}^I$ if and only if $u_0 : \partial G \to \mathbb{R}$ is monotone increasing with respect to G as in definition 4.2.*

Proof. For any function $u_0 : \partial G \to \mathbb{R}$, since $u \in \mathcal{F}_G$ is determined by a system of linear inequalities in the form of (4.1), Ω is a convex domain in $\mathbb{R}^I$ if it is non-empty. We will show that Ω is nonempty if and only if u_0 is monotone increasing with respect to G.

If Ω is nonempty, then U is nonempty. For any $u \in U$, by definition, $u \in \mathcal{F}_G$ and $u = u_0$ on ∂G. Since $u : V(G) \to \mathbb{R}$ is monotone increasing, its restriction $u_0 : \partial G \to \mathbb{R}$ is also monotone increasing with respect to G.

On the other hand, suppose u_0 is monotone increasing with respect to G. We construct a function u in U as follows. We first define $u = u_0$ on ∂G. If $\partial G \neq V(G)$, then the partially ordered set $(V(G) \setminus \partial G, \prec)$ has a minimal element $x_{\min}$ which is not greater than any other element in $V(G) \setminus \partial G$. Note that since G is acyclic and ∂G contains all source and sink vertices of $V(G)$, both $\{x \in \partial G, x \prec x_{\min}\}$ and $\{y \in \partial G, x_{\min} \prec y\}$ are nonempty sets. If $x \in \partial G$ with $x \prec x_{\min}$ and $y \in \partial G$ with $x_{\min} \prec y$, then $x \prec y$. By the monotonicity of u_0, it follows that $u_0(x) < u_0(y)$. Thus,

$$\max\{u_0(x) : x \in \partial G, x \prec x_{\min}\} < \min\{u_0(y) : y \in \partial G, x_{\min} \prec y\}.$$

Define $u(x_{\min})$ to be a number with

$$\max\{u_0(x) : x \in \partial G, x \prec v_1\} < u(x_{\min}) < \min\{u_0(y) : y \in \partial G, v_1 \prec y\}.$$

Now, u is defined on $\partial G \cup \{x_{\min}\}$. Continue the above process by treating $\partial G \cup \{x_{\min}\}$ as the new ∂G, we eventually define u on the set $V(G)$. By the construction process, this function u is monotone increasing with respect to G and $u = u_0$ on the original ∂G. Thus, $u \in U$ and U is nonempty. $\square$

Now we consider the following minimization problem with Dirichlet boundary condition:

Problem. *Suppose $u_0 : \partial G \to \mathbb{R}$ is monotone increasing with respect to G, and $p < 0$. Minimize*

$$\mathbf{E}_p(u) = \sum_{e \in E(G)} |\nabla u(e)|^p \, length(e)$$

among all $u \in U = \{u : V(G) \to \mathbb{R} | u \in \mathcal{F}_G \text{ and } u = u_0 \text{ on } \partial G\}$.

Equivalently, we want to minimize

$$f\left(u_1,u_2,\cdots,u_I\right) := \mathbf{E}_p\left(u\right) \tag{4.4}$$

among all $(u_1,u_2,\cdots,u_I)$ in the nonempty convex domain Ω, where

$$u = g^{-1}\left(u_1,u_2,\cdots,u_I\right).$$

i.e. $u\left(v\right) = u_0\left(v\right)$ on ∂G and $u\left(v_i\right) = u_i$ for each $i = 1, 2, \cdots, I$.

Lemma 4.6. *Suppose $u_0 : \partial G \to \mathbb{R}$ is monotone increasing with respect to G, and $p < 0$. Then, the function f given in (4.4) is strictly convex in the nonempty convex domain Ω.*

Proof. Let $Q = \left[Q\left(v,e\right)\right]$ be the incidence matrix of G. The rows and the columns of $Q\left(G\right)$ are indexed by $V\left(G\right)$ and $E\left(G\right)$ respectively. Assume also that $V\left(G\right)\setminus\partial G = \{v_1, v_2, \cdots, v_I\}$ are first I vertices in $V\left(G\right)$ and K is the total number of edges in $E\left(G\right)$.

Now, for each edge $e \in E\left(G\right)$, by (4.1),

$$\left|\nabla u\left(e\right)\right| = \frac{u\left(e^+\right) - u\left(e^-\right)}{length\left(e\right)} = \frac{1}{length\left(e\right)} \sum_{v\in V(G)} Q\left(v,e\right)u\left(v\right). \tag{4.5}$$

Thus, for each $i = 1, \cdots, I$,

$$\frac{\partial}{\partial u_i}\left(\left|\nabla u\left(e\right)\right|\right) = \frac{1}{length\left(e\right)}\frac{\partial}{\partial u_i}\left(\sum_{v\in V(G)} Q\left(v,e\right)u\left(v\right)\right) = \frac{Q\left(v_i,e\right)}{length\left(e\right)}.$$

Therefore, the partial derivative

$$\begin{aligned}\frac{\partial f}{\partial u_i} &= \frac{\partial}{\partial u_i}\left(\sum_{e\in E(G)} \left|\nabla u\left(e\right)\right|^p length\left(e\right)\right)\\ &= p\sum_{e\in E(G)} \left|\nabla u\left(e\right)\right|^{p-1} length\left(e\right)\frac{\partial}{\partial u_i}\left(\left|\nabla u\left(e\right)\right|\right)\\ &= p\sum_{e\in E(G)} \left|\nabla u\left(e\right)\right|^{p-1} Q\left(v_i,e\right).\end{aligned}$$

Moreover, for any $i, j \in \{1, 2, \cdots, I\}$, the second order partial derivative

$$\begin{aligned}\frac{\partial^2 f}{\partial u_i \partial u_j} &= \frac{\partial}{\partial u_i}\left[p\sum_{e\in E(G)} \left|\nabla u\left(e\right)\right|^{p-1} Q\left(v_j,e\right)\right]\\ &= p\left(p-1\right)\sum_{e\in E(G)} \left|\nabla u\left(e\right)\right|^{p-2} Q\left(v_j,e\right)\frac{\partial}{\partial u_i}\left|\nabla u\left(e\right)\right|\\ &= p\left(p-1\right)\sum_{e\in E(G)} \frac{\left|\nabla u\left(e\right)\right|^{p-2}}{length\left(e\right)} Q\left(v_j,e\right)Q\left(v_i,e\right).\end{aligned}$$

As a result, the Hessian matrix of f is

$$Hess\left(f\right) = p\left(p-1\right)\tilde{Q}R\tilde{Q}',$$

where $\tilde{Q}$ is the $I \times K$ matrix consisting of the first I rows of the incidence matrix $Q\left(G\right)$ of G, and R is a diagonal $K \times K$ matrix whose diagonal entries are $\left\{\frac{|\nabla u(e_1)|^{p-2}}{length(e_1)}, \cdots, \frac{|\nabla u(e_K)|^{p-2}}{length(e_K)}\right\}$.

When $p < 0$, luckily $p\,(p-1) > 0$ and the Hessian matrix $Hess\,(f)$ is positive definite. Thus, f is strictly convex on the nonempty convex domain Ω. $\square$

To find critical points of f in Ω, we set $\frac{\partial f}{\partial u_i} = 0$. That is,

$$\sum_{e \in E(G)} |\nabla u\,(e)|^{p-1} Q\,(v_i, e) = 0.$$

i.e.

$$\sum_{\substack{e \in E(G) \\ e^+ = v_i}} |\nabla u\,(e)|^{p-1} = \sum_{\substack{e \in E(G) \\ e^- = v_i}} |\nabla u\,(e)|^{p-1} \tag{4.6}$$

at each $v_i \in V\,(G) \setminus \partial G$. Using the discrete version of the divergence notation:

$$\operatorname{div}\left(\vec{V}\right)(v) = \sum_{\substack{\text{either } e^+ = v \\ \text{or } e^- = v}} \vec{V}\,(e) \cdot \vec{e}$$

for any vector field $\vec{V} : E\,(G) \to \mathbb{R}^m$, equation (4.6) can be expressed as the $p-$Laplace equation on the graph G :

$$\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)(v) = 0 \tag{4.7}$$

for any $v \in V\,(G) \setminus \partial G$.

A solution to the $p-$Laplace equation (4.7) in U is called a *$p-$harmonic function* on the graph G. By the strict convexity of f, any $p-$harmonic function u is an $\mathbf{E}_p$-minimizer in $\mathcal{F}_G$ with respect to its boundary datum.

5. Equivalence between α-landscape functions and p-harmonic functions.

In this section, we will show that an α-landscape function associated with a transport path P is equivalent to a $p-$harmonic function on the underlying graph G_P for conjugate parameters $\alpha \in [0,1)$ and $p \le 0$.

Proposition 5.1. *Let $p \le 0$ and $\alpha = \frac{p}{p-1} \in [0,1)$ be the conjugate of p. If u is a $p-$harmonic function on an acyclic graph G, then u is an α-landscape function associated with an acyclic transport path $P \in Path(\mathbf{a}_P, \mathbf{b}_P)$ with G being its underlying graph, where $\mathbf{a}_P$ and $\mathbf{b}_P$ are two atomic measures of equal mass given in (5.1). Moreover, $\mathbf{E}_p\,(u) = \mathbf{M}_\alpha\,(P)$.*

Proof. Suppose u is a $p-$harmonic function on G. For each edge $e \in E\,(G)$, set

$$\omega\,(e) = |\nabla u\,(e)|^{p-1} = \left(\frac{u\,(e^+) - u\,(e^-)}{length\,(e)}\right)^{p-1}.$$

Since u is $p-$harmonic on G, by (4.6), it follows that

$$\sum_{e^+ = v} \omega\,(e) = \sum_{e^- = v} \omega\,(e)$$

for each $v \in V\,(G) \setminus \partial G$. Now, for each $v \in V\,(G)$, define

$$\mathfrak{m}\,(v) = \sum_{e^- = v} \omega\,(e) - \sum_{e^+ = v} \omega\,(e)\,.$$

Let

$$\mathbf{a}_P = \sum_{\mathfrak{m}(v_i) > 0} \mathfrak{m}\,(v_i)\,\delta_{v_i} \text{ and } \mathbf{b}_P = \sum_{\mathfrak{m}(v_i) < 0} \left(-\mathfrak{m}\,(v_i)\right)\delta_{v_i}. \tag{5.1}$$

Then,

$$\begin{aligned}\sum_{v\in V(G)}\mathfrak{m}(v) &= \sum_{v\in V(G)}\left(\sum_{e^-=v}\omega(e)-\sum_{e^+=v}\omega(e)\right)\\ &= \sum_{e\in E(G)}(\omega(e)-\omega(e))=0.\end{aligned}$$

i.e.

$$\sum_{\mathfrak{m}(v_i)>0}\mathfrak{m}(v_i)=\sum_{\mathfrak{m}(v_i)<0}(-\mathfrak{m}(v_i)).$$

This says that the atomic measures $\mathbf{a}_P$ and $\mathbf{b}_P$ have the same mass on X. Together with the weight function ω, G becomes a transport path $P=\{V(G),E(G),\omega\}$ from $\mathbf{a}_P$ to $\mathbf{b}_P$. Moreover, the transport cost of P is

$$\begin{aligned}\mathbf{M}_\alpha(P) &= \sum_{e\in E(P)}\omega(e)^\alpha\, length\,(e)\\ &= \sum_{e\in E(P)}\omega(e)^{\frac{p}{p-1}}\, length\,(e)\\ &= \sum_{e\in E(P)}|\nabla u|^p\, length\,(e)=\mathbf{E}_p(u).\end{aligned}$$

That is, the transport cost $\mathbf{M}_\alpha(P)$ is the least $p-$energy $\mathbf{E}_p(u)$ on G.

Furthermore, for each edge e,

$$\begin{aligned}u(e^+)-u(e^-) &= |\nabla u(e)|\, length\,(e)\\ &= \omega(e)^{\frac{1}{p-1}}\, length\,(e)\\ &= \omega(e)^{\alpha-1}\, length\,(e).\end{aligned}$$

This shows that u is an α-landscape function associated with the transport path P. □

In the following proposition, we have the reverse statement of Proposition 5.1.

Proposition 5.2. *Let $\alpha\in[0,1)$ and $p=\frac{\alpha}{\alpha-1}\le 0$ be the conjugate of α. If Z is an α-landscape function associated with an acyclic transport path P, then, Z is a p-harmonic function on the underlying graph G_P. Moreover, $\mathbf{M}_\alpha(P)=\mathbf{E}_p(Z)$.*

Proof. Let $G=G_P$ be the underlying graph of P. Clearly, any α-landscape function Z of P satisfies the monotone increasing condition (4.1) with respect to G. So, $Z\in\mathcal{F}_G$. Let $\partial G=\{x_1,\cdots,x_k,y_1,\cdots,y_\ell\}$.

We need to show that

$$\mathbf{E}_p(Z)\le\mathbf{E}_p(u)$$

for any $u\in\mathcal{F}_G$ with $u=Z$ on ∂G.

Indeed, set $h=u-Z$, then $h=0$ on ∂G. For any $\epsilon\in[0,1]$, observe that $Z+\epsilon h=(1-\epsilon)Z+\epsilon u$ is still in $\mathcal{F}_G$. We now consider the function

$$f(\epsilon):=\mathbf{E}_p(Z+\epsilon h)=\sum_{e\in E(P)}|\nabla Z(e)+\epsilon\nabla h(e)|^p\, length\,(e)$$

for $\epsilon\in[0,1]$. Note that

$$|\nabla Z(e)+\epsilon\nabla h(e)|=(1-\epsilon)\left(Z(e^+)-Z(e^-)\right)+\epsilon\left(u(e^+)-u(e^-)\right)>0$$

for any edge e and any $\epsilon \in [0,1]$. Also,

$$\begin{aligned} f'(\epsilon) &= \frac{d}{d\epsilon}\left(\sum_{e\in E(P)} |\nabla Z(e)+\epsilon\nabla h(e)|^p \, length(e)\right) \\ &= \sum_{e\in E(P)} p\,|\nabla Z(e)+\epsilon\nabla h(e)|^{p-1}\frac{(\nabla Z(e)+\epsilon\nabla h(e))\cdot(\nabla h(e))}{|\nabla Z(e)+\epsilon\nabla h(e)|} length(e) \\ &= \sum_{e\in E(P)} p\,|\nabla Z(e)+\epsilon\nabla h(e)|^{p-2}(\nabla Z(e)+\epsilon\nabla h(e))\cdot(\nabla h(e))\, length(e) \\ &= \sum_{e\in E(P)} p\,|\nabla Z(e)+\epsilon\nabla h(e)|^{p-1}\left(h\left(e^+\right)-h\left(e^-\right)\right). \end{aligned}$$

Thus, by (2.3), (3.4) and $h=0$ on $\partial G = \{x_1,\cdots,x_k,y_1,\cdots,y_\ell\}$, we have

$$\begin{aligned} f'(0) &= \sum_{e\in E(P)} p\,|\nabla Z(e)|^{p-1}\left(h\left(e^+\right)-h\left(e^-\right)\right) \\ &= \sum_{e\in E(P)} p\omega(e)^{(\alpha-1)(p-1)}\left(h\left(e^+\right)-h\left(e^-\right)\right) \\ &= \sum_{e\in E(P)} p\omega(e)\left(h\left(e^+\right)-h\left(e^-\right)\right) \\ &= p\sum_{v\in V(G)}\left(\sum_{e^+=v}\omega(e)-\sum_{e^-=v}\omega(e)\right)h(v)=0. \end{aligned}$$

Moreover, since each $|\nabla Z(e)+\epsilon\nabla h(e)|^p$ is a convex function of $\epsilon\in[0,1]$ when $p=\frac{\alpha}{\alpha-1}\leq 0$, the function $f(\epsilon)$ is also convex in $[0,1]$ with $f'(0)=0$. This shows that $f(\epsilon)$ has an absolute minimum at $\epsilon=0$. Thus,

$$\mathbf{E}_p(Z) = f(0) \leq f(1) = \mathbf{E}_p(u).$$

Notice also that the transport cost

$$\begin{aligned} \mathbf{M}_\alpha(P) &= \sum_{e\in E(P)} \omega(e)^\alpha \, length(e) \\ &= \sum_{e\in E(P)} |\nabla Z(e)|^{\alpha/(\alpha-1)} \, length(e) \\ &= \sum_{e\in E(P)} |\nabla Z(e)|^p \, length(e) = \mathbf{E}_p(Z). \end{aligned}$$

□

6. **Landscape functions associated with optimal transport paths.** In this section, we study some properties of landscape functions associated with optimal transport paths.

Proposition 6.1. *Suppose $P\in Path(\mathbf{a},\mathbf{b})$ is an α-optimal transport path. For any α-landscape function Z associated with P, it holds that at each vertex $v\in V(P)\setminus\{x_1,\cdots,x_k,y_1,\cdots,y_\ell\}$,*

$$\sum_{e^+=v}\omega(e)\nabla Z(e) = \sum_{e^-=v}\omega(e)\nabla Z(e) \tag{6.1}$$

and

$$\sum_{e^+=v} |\nabla Z(e)|^{p-1} \nabla Z(e) = \sum_{e^-=v} |\nabla Z(e)|^{p-1} \nabla Z(e) \tag{6.2}$$

for $p = \frac{\alpha}{\alpha-1}$.

Proof. Since P is optimal, the equation (2.6)

$$\sum_{e^+=v} [\omega(e)]^\alpha \vec{e} = \sum_{e^-=v} [\omega(e)]^\alpha \vec{e}$$

holds at any vertex $V(P) \setminus \{x_1, \cdots, x_k, y_1, \cdots, y_\ell\}$. By (3.4),

$$[\omega(e)]^\alpha \vec{e} = \omega(e) \nabla Z(e).$$

So (2.6) becomes (6.1). Moreover,

$$[\omega(e)]^\alpha \vec{e} = [\omega(e)]^{(\alpha-1)p} \vec{e} = |\nabla Z(e)|^p \vec{e} = |\nabla Z(e)|^{p-1} \nabla Z(e).$$

Using this identity, equation (2.6) yields (6.2). □

Remark 6.2. For any transport path P, by (2.3), mass is conserved at each interior vertex of P. When the transport path P is optimal, by (6.1), momentum is also conserved at each interior vertex when viewing $\nabla Z(e) = w(e)^{\alpha-1}\vec{e}$ as the velocity of a moving particle of mass $w(e)$ on each e.

The following proposition says that any α-landscape function associated with an $\alpha-$optimal transport path is Lipschitz. Before stating the proposition, we first extend the domain of a landscape function $Z : V(P) \to \mathbb{R}$ to the support of P by linear extensions of Z on each edge of P.

Proposition 6.3. *Suppose $P \in Path(\mathbf{a}, \mathbf{b})$ is an α-optimal transport path for some $\alpha \in (0,1)$. Let Z be an α-landscape function associated with P. Then, for any x, y on the same connected components of the support of G_P, it holds that*

$$|Z(x) - Z(y)| \le \frac{1}{\alpha} \sigma^{\alpha-1} \|x - y\|$$

where $\sigma = \min_{e \subseteq \gamma} \omega(e)$, and γ is the unique curve on P from x to y. In particular, when P is connected, let $\sigma_P = \min_{e \in E(P)} \omega(e)$, then

$$|Z(x) - Z(y)| \le \frac{1}{\alpha} (\sigma_P)^{\alpha-1} \|x - y\|$$

for any x, y on the support of G_P.

Proof. As shown in Figure 2, for any ϵ with $|\epsilon| \le \sigma$, we set $\tilde{P} = P + \epsilon\gamma - \epsilon[x, y]$ where $[x, y]$ denotes the line segment from x to y. The construction of $\tilde{P}$ here is standard in this kind of problems, see for instance Proposition 3.5 in [16] and Theorem 3.7 in [5]. By Definition 2.2, $\tilde{P}$ is also a transport path in $Path(\mathbf{a}, \mathbf{b})$. Since P is optimal, it follows that

$$\mathbf{M}_\alpha(P) \le \mathbf{M}_\alpha\left(\tilde{P}\right) = \mathbf{M}_\alpha(P + \epsilon\gamma) + |\epsilon|^\alpha \|x - y\|.$$

On the other hand, by Proposition 3.4 and Remark 3.6, it follows that

$$\alpha\epsilon(Z(x) - Z(y)) \ge \mathbf{M}_\alpha(P + \epsilon\gamma) - \mathbf{M}_a(P).$$

Hence, when $|\epsilon| \le \sigma$, we have

$$\begin{aligned} |\epsilon|^\alpha \|x - y\| &\ge \mathbf{M}_\alpha(P) - \mathbf{M}_\alpha(P + \epsilon\gamma) \\ &\ge -\alpha\epsilon(Z(x) - Z(y)). \end{aligned}$$

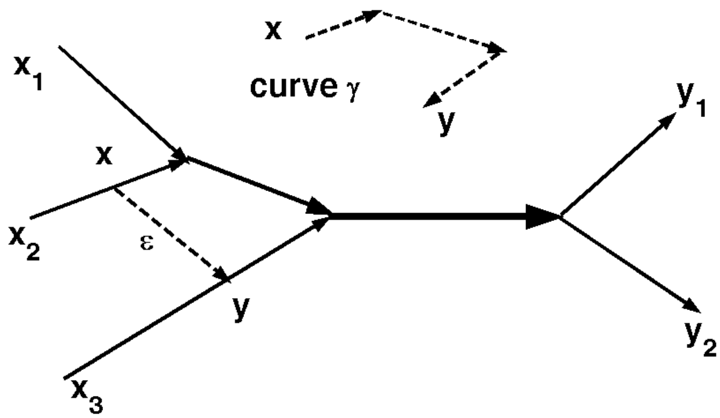

FIGURE 2. The construction for the proof of Proposition 6.3.

Picking $\epsilon = -\sigma sign\left(Z\left(x\right) - Z\left(y\right)\right)$, it follows that

$$\|x-y\| \geq \alpha\sigma^{\alpha-1}\left|Z\left(x\right)-Z\left(y\right)\right|$$

as desired. □

In the end of this section, we derive a formula of $\mathbf{M}_\alpha\left(P\right)$ of an optimal transport path P in terms of boundary values of ∇Z of a landscape function Z. Such a formula might be useful in the future works including the one considering the dual problem of ramified optimal transportation.

Proposition 6.4. *For any $P \in Path\left(\mathbf{a}, \mathbf{b}\right)$, it holds that*

$$\mathbf{M}_\alpha\left(P\right) = \sum_{v\in V(P)} \vec{\mathbf{m}}_\alpha\left(v\right)\cdot\mathbf{v}, \tag{6.3}$$

where $\mathbf{v}$ is the corresponding position vector of v in $\mathbb{R}^m$ and

$$\vec{\mathbf{m}}_\alpha\left(v\right) = \sum_{e^+=v} w\left(e\right)^\alpha \vec{e} - \sum_{e^-=v} w\left(e\right)^\alpha \vec{e}.$$

In particular, for any $\alpha-$landscape function Z associated with P, by (3.4),

$$\vec{\mathbf{m}}_\alpha\left(v\right) = \sum_{e^+=v} w\left(e\right)\nabla Z\left(e\right) - \sum_{e^-=v} w\left(e\right)\nabla Z\left(e\right), \tag{6.4}$$

i.e. the net momentum at v in the sense of Remark 6.2.

If P is an α-optimal transport path, then by (6.1), $\vec{\mathbf{m}}_\alpha\left(v\right) = 0$ for any vertex $v \in V\left(P\right)\setminus\{x_1,\cdots,x_k,y_1,\cdots,y_\ell\}$ and thus

$$\mathbf{M}_\alpha\left(P\right) = \sum_{v\in\{x_1,x_2,\cdots,x_k,y_1,\cdots,y_\ell\}} \vec{\mathbf{m}}_\alpha\left(v\right)\cdot\mathbf{v}. \tag{6.5}$$

Proof. By (2.4),

$$\begin{aligned}\mathbf{M}_\alpha(P) &= \sum_{e\in E(P)} w(e)^\alpha \, length(e) \\ &= \sum_{e\in E(P)} w(e)^\alpha \left(e^+ - e^-\right)\cdot \vec{e} \\ &= \sum_{v\in V(P)} \left(\sum_{e^+=v} w(e)^\alpha \vec{e} - \sum_{e^-=v} w(e)^\alpha \vec{e}\right)\cdot \mathbf{v} \\ &= \sum_{v\in V(P)} \vec{\mathbf{m}}_\alpha(v)\cdot \mathbf{v}.\end{aligned}$$

□

Corollary 6.5. *Suppose P is an α-optimal transport path from* $\mathbf{a}$ *to* $\mathbf{b}$*, and Z is an α-landscape function associated with P. Then, for $\vec{\mathbf{m}}_\alpha(v)$ given in (6.4), it satisfies that*

$$\sum_{v\in\{x_1,x_2,\cdots,x_k,y_1,\cdots,y_\ell\}} \vec{\mathbf{m}}_\alpha(v) = 0.$$

Proof. For any $\mathbf{c}\in\mathbb{R}^m$, we consider the translation $\mathbf{x}\to\mathbf{x}+\mathbf{c}$ in $\mathbb{R}^m$. Since $\mathbf{M}_\alpha(P)$ is translational invariant, by (6.5), it holds that

$$\begin{aligned}&\sum_{v\in\{x_1,x_2,\cdots,x_k,y_1,\cdots,y_\ell\}} \vec{\mathbf{m}}_\alpha(v)\cdot\mathbf{v} = \mathbf{M}_\alpha(P) \\ =& \sum_{v\in\{x_1,x_2,\cdots,x_k,y_1,\cdots,y_\ell\}} \vec{\mathbf{m}}_\alpha(v)\cdot(\mathbf{v}+\mathbf{c}).\end{aligned}$$

That is,

$$\sum_{v\in\{x_1,x_2,\cdots,x_k,y_1,\cdots,y_\ell\}} \vec{\mathbf{m}}_\alpha(v)\cdot\mathbf{c} = 0$$

for any $\mathbf{c}\in\mathbb{R}^m$. As a result,

$$\sum_{v\in\{x_1,x_2,\cdots,x_k,y_1,\cdots,y_\ell\}} \vec{\mathbf{m}}_\alpha(v) = 0.$$

□

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