# RAMIFIED OPTIMAL TRANSPORTATION IN GEODESIC METRIC SPACES

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Abstract. An optimal transport path may be viewed as a geodesic in the space of probability measures under a suitable family of metrics. This geodesic may exhibit a tree-shaped branching structure in many applications such as trees, blood vessels, draining and irrigation systems. Here, we extend the study of ramified optimal transportation between probability measures from Euclidean spaces to a geodesic metric space. We investigate the existence as well as the behavior of optimal transport paths under various properties of the metric such as completeness, doubling, or curvature upper boundedness. We also introduce the transport dimension of a probability measure on a complete geodesic metric space, and show that the transport dimension of a probability measure is bounded above by the Minkowski dimension and below by the Hausdorff dimension of the measure. Moreover, we introduce a metric, called "the dimensional distance", on the space of probability measures. This metric gives a geometric meaning to the transport dimension: with respect to this metric, the transport dimension of a probability measure equals to the distance from it to any finite atomic probability measure.

The optimal transportation problem aims at finding an optimal way to transport a given measure into another with the same mass. In contrast to the well-known Monge-Kantorovich problem (e.g. [1], [6], [7], [14], [15], [18], [20], [22]), the ramified optimal transportation problem aims at modeling a branching transport network by an optimal transport path between two given probability measures. An essential feature of such a transport path is to favor transportation in groups via a nonlinear (typically concave) cost function on mass. Transport networks with branching structures are observable not only in nature as in trees, blood vessels, river channel networks, lightning, etc. but also in efficiently designed transport systems such as used in railway configurations and postage delivery networks. Several different approaches have been done on the ramified optimal transportation problem in Euclidean spaces, see for instance [16], [24], [19], [25], [26], [4], [2], [27], [13], [5], [28], and [29]. Related works on flat chains may be found in [23], [12], [25] and [21].

This article aims at extending the study of ramified optimal transportation from Euclidean spaces to metric spaces. Such generalization is not only mathematically nature but also may be useful for considering specific examples of metric spaces later. By exploring various properties of the metric, we show that many results about ramified optimal transportation is not limited to Euclidean spaces, but can be extended to metric spaces with suitable properties on the metric. Some results that we prove in this article are summarized here:

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When $X$ is a geodesic metric space, we define in §1 a family of metrics $d_\alpha$ on the space $\mathcal{A}(X)$ of atomic probability measures on $X$ for a (possibly negative) parameter $\alpha < 1$. The space $(\mathcal{A}(X), d_\alpha)$ is still a geodesic metric space when $0 \leq \alpha < 1$. A geodesic, also called an optimal transport path, in this space is a weighted directed graph whose edges are geodesic segments.

Moreover, when $X$ is a geodesic metric space of curvature bounded above, we find in §2, a universal lower bound depending only on the parameter $\alpha$ for each comparison angle between edges of any optimal transport path. If in addition $X$ is a doubling metric space, we show that the degree of any vertex of an optimal transport path in $X$ is bounded above by a constant depending only on $\alpha$ and the doubling constant of $X$. On the other hand, we also provide a lower bound of the curvature of $X$ by a quantity related to the degree of vertices.

Furthermore, when $X$ is a complete geodesic metric space, we consider optimal transportation between any two probability measures on $X$ by considering the completion $\mathcal{P}_\alpha(X)$ of the metric space $(\mathcal{A}(X), d_\alpha)$ in §3. A geodesic, if it exists, in the completed metric space $(\mathcal{P}_\alpha(X), d_\alpha)$ is viewed as an $\alpha-$optimal transport path between measures on $X$. The existence of an optimal transport path is closely related to the dimensional information of the measures. As a result, we consider the dimension of measures on $X$ by introducing a concept called the *transport dimension* of measures, which is analogous to the irrigational dimension of measures in Euclidean spaces studied by [13]. We show in §3.2.3 that the transport dimension of a measure is bounded below by its Hausdorff dimension and above by its Minkowski dimension. Furthermore, we show that the transport dimension has an interesting geometric meaning: under a metric (called the *dimensional distance* in §3.2.4), the transport dimension of a probability measure equals to the distance from it to any atomic probability measure.

In §4, when $X$ is a compact geodesic doubling metric space with Assouad dimension $m$ and the parameter $\alpha > \max\left\{1 - \frac{1}{m}, 0\right\}$, then we show that the space $\mathcal{P}(X)$ of probability measures on $X$ with respect to $d_\alpha$ is a geodesic metric space. In other words, there exists an $\alpha$-optimal transport path between any two probability measures on $X$.

## 1. The $d_\alpha$ metrics on atomic probability measures on a metric space

1.1. **Transport paths between atomic measures.** We first extend some basic concepts about transport paths between measures of equal mass as studied in [24], with some necessary modifications, from Euclidean spaces to a metric space.

Let $(X, d)$ be a geodesic metric space. Recall that a (finite, positive) atomic measure on $X$ is in the form of

$$\mathbf{a} = \sum_{i=1}^{k} m_i \delta_{x_i} \tag{1.1.1}$$

with distinct points $x_i \in X$ and positive numbers $m_i$, where $\delta_x$ denotes the Dirac mass located at the point $x$. The measure $\mathbf{a}$ is a probability measure if its mass $\sum_{i=1}^{k} m_i = 1$. Let $\mathcal{A}(X)$ be the space of all atomic probability measures on $X$.

**Definition 1.1.1.** *Given two atomic measures*

$$\mathbf{a} = \sum_{i=1}^{k} m_i \delta_{x_i} \text{ and } \mathbf{b} = \sum_{j=1}^{\ell} n_j \delta_{y_j} \tag{1.1.2}$$

*on $X$ of equal mass, a* ***transport path*** *from* $\mathbf{a}$ *to* $\mathbf{b}$ *is a weighted directed graph $G$ consisting of a vertex set $V(G)$, a directed edge set $E(G)$ and a weight function $w : E(G) \to (0, +\infty)$ such that $\{x_1, x_2, ..., x_k\} \cup \{y_1, y_2, ..., y_n\} \subseteq V(G)$ and for any vertex $v \in V(G)$, there is a balance equation*

(1.1.3)

$$\sum_{e \in E(G), e^- = v} w(e) = \sum_{e \in E(G), e^+ = v} w(e) + \begin{cases} m_i, & \text{if } v = x_i \text{ for some } i = 1, ..., k \\ -n_j, & \text{if } v = y_j \text{ for some } j = 1, ..., n \\ 0, & \text{otherwise} \end{cases}$$

*where each edge $e \in E(G)$ is a geodesic segment in $X$ from the starting endpoint $e^-$ to the ending endpoint $e^+$.*

Note that the balance equation (1.1.3) simply means the conservation of mass at each vertex. In terms of polyhedral chains, we simply have $\partial G = \mathbf{b} - \mathbf{a}$.

For any two atomic measures $\mathbf{a}$ and $\mathbf{b}$ on $X$ of equal mass, let

$$Path(\mathbf{a}, \mathbf{b})$$

be the space of all transport paths from $\mathbf{a}$ to $\mathbf{b}$. Now, we define the transport cost for each transport path as follows.

**Definition 1.1.2.** *For any real number $\alpha \in (-\infty, 1]$ and any transport path $G \in Path(\mathbf{a}, \mathbf{b})$, we define*

$$\mathbf{M}_\alpha(G) := \sum_{e \in E(G)} w(e)^\alpha length(e).$$

**Remark 1.1.3.** *Note that in* [24, Definition 2.2] *(or other approaches of ramified optimal transportation, see* [5]*) we only allowed that $0 \leq \alpha \leq 1$. On the other hand, allowing negative $\alpha$ in* [29] *is a key idea for studying fractal dimension of measures (e.g. the Cantor measure) in Euclidean spaces. Another application of ramified optimal transportation with negative $\alpha$ may be found in* [30]. *Negative $\alpha$ also corresponds to many transport problems in reality. For instance, the "cost" (i.e., risk here) for one person to pass through a dangerous region may be higher than the cost for two or many persons. Negative $\alpha$ is used for such phenomenon.*

The study of transport paths for negative $\alpha$ is subtler than the one for nonnegative $\alpha$. When $\alpha$ is negative, it is possible to significantly decrease the $\mathbf{M}_\alpha$ cost of some transport paths (using the original definition) by adding some really unpleasant cycles (see [29, Example 2.2.2]). To avoid the existence of these unpleasant cycles, we will still adopt the following convention on the definition of transport paths when $\alpha$ is negative. Detailed reasons for introducing this convention were given in [29, §2.2].

**Convention 1.1.4.** *(*[29, Convention 2.2.3]*.) For any vertex $v \in V(G)$ of a transport path $G$, if there exists a list of vertices $\{v_1, v_2, \cdots, v_n\}$ such that $v_1 = v_n = v$ and $[v_i, v_{i+1}]$ is a directed edge in $E(G)$ with a positive edge length for each $i = 1, 2, \cdots, n-1$, then we will view $v_n$ as a different copy of the point $v$. In other words, $v_n$ and $v$ are two different vertices in $V(G)$ and thus the balance equation (1.1.3) must be separately hold at each of them.*

This convention plays an important role in proving proposition 1.2.4. Also, together with the balance equation (1.1.3), this convention indicates the following fact: there exists a universal upper bound (i.e. the total mass of the source) on the weight of each edge:

$$w(e) \leq ||\mathbf{a}|| \tag{1.1.4}$$

for each edge $e$ of a transport path $G$ from $\mathbf{a}$ to $\mathbf{b}$. In particular, if $\mathbf{a}$ is a probability measure, then $w(e) \leq 1$.

We now consider the following ramified optimal transport problem:

**Problem 1.** *Given two atomic measures* $\mathbf{a}$ *and* $\mathbf{b}$ *of equal mass on a geodesic metric space* $X$ *and* $-\infty < \alpha < 1$*, find a minimizer of*

$$\mathbf{M}_\alpha(G)$$

*among all transport paths* $G \in Path(\mathbf{a}, \mathbf{b})$.

An $\mathbf{M}_\alpha$ minimizer in $Path(\mathbf{a}, \mathbf{b})$ is called an $\alpha-$optimal transport path from $\mathbf{a}$ to $\mathbf{b}$.

**Definition 1.1.5.** *For any* $\alpha \in (-\infty, 1]$*, we define*

$$d_\alpha(\mathbf{a}, \mathbf{b}) = \inf\{\mathbf{M}_\alpha(G) : G \in Path(\mathbf{a}, \mathbf{b})\}$$

*for any* $\mathbf{a}, \mathbf{b} \in \mathcal{A}(X)$.

**Remark 1.1.6.** *Let* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *be two atomic measures of equal mass* $\Lambda > 0$*, and let* $\mathbf{a} = \frac{1}{\Lambda}\bar{\mathbf{a}}$ *and* $\mathbf{b} = \frac{1}{\Lambda}\bar{\mathbf{b}}$ *be the normalization of* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$*. Then, for any transport path* $\bar{G} \in Path\left(\bar{\mathbf{a}}, \bar{\mathbf{b}}\right)$*, we have* $G = \left\{V\left(\bar{G}\right), E\left(\bar{G}\right), \frac{1}{\Lambda}w\right\}$ *is a transport path from* $\mathbf{a}$ *to* $\mathbf{b}$ *with* $\mathbf{M}_\alpha(\bar{G}) = \Lambda^\alpha \mathbf{M}_\alpha(G)$*. Thus, we also set* $d_\alpha\left(\bar{\mathbf{a}}, \bar{\mathbf{b}}\right) = \Lambda^\alpha d_\alpha(\mathbf{a}, \mathbf{b})$.

1.2. **The $d_\alpha$ metrics.** It is easy to see that $d_\alpha$ is a metric on $\mathcal{A}(X)$ when $0 \leq \alpha \leq 1$. But to show that $d_\alpha$ is still a metric when $\alpha < 0$, we need some estimates on the lower bound of $d_\alpha(\mathbf{a}, \mathbf{b})$ when $\mathbf{a} \neq \mathbf{b}$.

We denote $S(p, r)$ (and $\bar{B}(p, r)$, respectively) to be the sphere (and the closed ball, respectively) centered at $p \in X$ of radius $r > 0$. Note that for any transport path $G$, the restriction of $G$ on any closed ball $\bar{B}(p, r_0)$ gives a transport path $G|_{\bar{B}(p,r_0)}$ between the restriction of measures.

**Lemma 1.2.1.** *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are two atomic measures of equal mass on a geodesic metric space* $X$*, and* $G$ *is a transport path from* $\mathbf{a}$ *to* $\mathbf{b}$*. For a point* $p \in X$*, if the intersection of* $G \cap S(p, r)$ *as sets is nonempty for almost all* $r \in [0, r_0]$ *for some* $r_0 > 0$*, then*

$$\mathbf{M}_\alpha(G|_{\bar{B}(p,r_0)}) \geq \int_0^{r_0} \sum_{e \in \mathbf{E}_r} [w(e)]^\alpha \, dr, \tag{1.2.1}$$

*where for each* $r$*, the set*

$$\mathbf{E}_r := \{e \in E(G) : e \cap S(p, r) \neq \emptyset\}$$

*is the family of all edges of* $G$ *that intersects with the sphere* $S(p, r)$.

*Proof.* For every edge $e$ of $G$, let $p^*$ and $p_*$ be the points on $e$ such that

$$d(p, p^*) = \max\{d(p, x) : x \in e\} \text{ and } d(p, p_*) = \min\{d(p, x) : x \in e\}.$$

Then, since $e$ is a geodesic segment in $X$,

$$length(e) \geq d(p^*, p^*) \geq |d(p, p^*) - d(p, p_*)| = \int_0^\infty \chi_{I_e}(r) \, dr$$

where $\chi_{I_e}(r)$ is the characteristic function of the interval $I_e := [d(p, p_*), d(p, p^*)]$. By assumption, $\mathbf{E}_r$ is nonempty for almost all $r \in [0, r_0]$. Also, observe that $e \in \mathbf{E}_r$ if and only if $\chi_{I_e}(r) = 1$. Therefore,

$$\begin{aligned}
\mathbf{M}_\alpha(G|_{\bar{B}(p,r_0)}) &= \sum_{e \in E(G|B(p,r_0))} [w(e)]^\alpha \operatorname{length}(e) \\
&\geq \sum_{e \in E(G|B(p,r_0))} [w(e)]^\alpha \int_0^\infty \chi_{I_e}(r)\, dr \\
&= \int_0^{r_0} \sum_{e \in \mathbf{E}_r} [w(e)]^\alpha\, dr.
\end{aligned}$$

□

The following corollary implies a positive lower bound on $d_\alpha(\mathbf{a}, \mathbf{b})$ when $\mathbf{a} \neq \mathbf{b}$.

**Corollary 1.2.2.** *Let the assumptions be as in Lemma 1.2.1 and $\alpha \leq 0$. Then*

$$\mathbf{M}_\alpha\left(G|_{\bar{B}(p,r_0)}\right) \geq \Lambda^\alpha r_0,$$

*where $\Lambda$ is an upper bound of the weight $w(e)$ for every edge $e$ in $G|_{\bar{B}(p,r_0)}$. In particular, for any atomic measure*

$$\mathbf{a} = \sum_{i=1}^k m_i \delta_{x_i}$$

*on $X$ with mass $||\mathbf{a}|| := \sum_{i=1}^k m_i > 0$, we have the following estimate*

$$d_\alpha\left(\mathbf{a}, ||\mathbf{a}||\, \delta_p\right) \geq ||\mathbf{a}||^\alpha \max_{1 \leq i \leq k} \{d(p, x_i)\}\ . \tag{1.2.2}$$

*Proof.* When $\alpha \leq 0$, we have for each $r$ with $\mathbf{E}_r$ nonempty,

$$\sum_{e \in \mathbf{E}_r} [w(e)]^\alpha \geq \max_{e \in \mathbf{E}_r} [w(e)]^\alpha \geq \Lambda^\alpha,$$

where $\Lambda$ is any upper bound of the weights of edges in $G|_{\bar{B}(p,r_0)}$. Therefore, by (1.2.1),

$$\mathbf{M}_\alpha\left(G|_{\bar{B}(p,r_0)}\right) \geq \Lambda^\alpha r_0.$$

Now, let $G \in Path\left(\mathbf{a}, ||\mathbf{a}||\, \delta_p\right)$. When $\alpha \leq 0$, (1.1.4) says that

$$w(e) \leq ||\mathbf{a}||$$

for every edge $e \in E(G)$. Then, (1.2.2) follows by setting $r_0 = \max_{1 \leq i \leq k} \{d(p, x_i)\}$ and $\Lambda = ||\mathbf{a}||$. □

Lemma 1.2.1 also gives a lower bound estimate for positive $\alpha$, which will be used in proposition 3.1.6.

**Corollary 1.2.3.** *Suppose $0 \leq \alpha < 1$. For any $\mathbf{a} \in \mathcal{A}(X)$ in the form of (1.1.1), $p \in X$ and $r_0 > 0$, we have*

$$\left[\sum_{d(p,x_i) > r_0} m_i\right]^\alpha \leq \frac{d_\alpha(\mathbf{a}, \delta_p)}{r_0}. \tag{1.2.3}$$

*Proof.* Let $\lambda = \sum_{d(p,x_i)>r_0} m_i$. Let $G$ be any transport path from $\mathbf{a}$ to $\delta_p$. Then, for any $0 < r \leq r_0$, we have

$$\sum_{e \in \mathbf{E}_r} w(e) \geq \lambda.$$

By lemma 1.2.1, since the function $f(x) = x^\alpha$ is concave on $[0,1]$ when $0 \leq \alpha < 1$, we have

$$\begin{aligned} \mathbf{M}_\alpha(G|_{\bar{B}(p,r_0)}) &\geq \int_0^{r_0} \sum_{e \in \mathbf{E}_r} [w(e)]^\alpha \, dr \\ &\geq \int_0^{r_0} \left[ \sum_{e \in \mathbf{E}_r} w(e) \right]^\alpha dr \geq \int_0^{r_0} \lambda^\alpha dr = \lambda^\alpha r_0. \end{aligned}$$

Therefore, we have (1.2.3). □

By means of corollary 1.2.2, the proof of [29, Proposition 2.2.3] shows the following proposition.

**Proposition 1.2.4.** *Suppose $\alpha < 1$, and $X$ is a geodesic metric space. Then $d_\alpha$ defined in definition 1.1.5 is a metric on the space $\mathcal{A}(X)$ of atomic probability measures on $X$.*

**Example 1.2.5.** *Given a finite number of points $\{x_0, x_1, \cdots, x_n\}$ in a real normed vector space $X$, we may define a metric $\tilde{\mathbf{d}}_\alpha$ on the convex hull $K$ of the given points by means of the $d_\alpha$ metrics on $\mathcal{A}(X)$. For any two points $x, y \in K$, both $x$ and $y$ can be expressed as a convex combination of the given points of the form*

$$x = \sum_{i=0}^n m_i x_i \text{ with } m_i \geq 0 \text{ and } \sum_{i=0}^n m_i = 1,$$

$$y = \sum_{i=0}^n n_i x_i \text{ with } n_i \geq 0 \text{ and } \sum_{i=0}^n n_i = 1.$$

*Then, we may define a metric on $K$ by setting*

$$\tilde{\mathbf{d}}_\alpha(x,y) := d_\alpha\left( \sum_{i=0}^n m_i \delta_{x_i}, \sum_{i=0}^n n_i \delta_{x_i} \right),$$

*where $d_\alpha$ is one of the ramified metrics defined on $\mathcal{A}(X)$. For instance, let $x_0 = 0$ and $x_1 = 1$ in $\mathbb{R}$. Then, for any $s, t \in [0,1]$, by definition,*

$$\tilde{\mathbf{d}}_\alpha(s,t) = d_\alpha\left( s\delta_{\{1\}} + (1-s)\,\delta_{\{0\}}, t\delta_{\{1\}} + (1-t)\,\delta_{\{0\}} \right).$$

*When $0 < \alpha \leq 1$, we have $\tilde{\mathbf{d}}_\alpha(s,t) = |s-t|^\alpha$. When $\alpha = 0$, $\tilde{\mathbf{d}}_0$ gives the discrete metric on $[0,1]$. When $\alpha < 0$, one may check that the formula for $\tilde{\mathbf{d}}_\alpha$ is*

$$\tilde{\mathbf{d}}_\alpha(s,t) = \begin{cases} \min\left\{ |s-t|^\alpha, s^\alpha + t^\alpha, (1-s)^\alpha + (1-t)^\alpha \right\}, & \text{if } s \neq t \\ 0 & \text{if } s = t \end{cases}.$$

1.3. **The $d_\alpha$ metric viewed as a metric induced by a quasimetric.** When $0 \le \alpha < 1$, another approach of the metric $d_\alpha$ was introduced in [28], which says that the metric $d_\alpha$ is the intrinsic metric on $\mathcal{A}(X)$ induced by a quasimetric [1] $J_\alpha$. Let us briefly recall the definition of the quasimetric $J_\alpha$ here.

Let $\mathbf{a}$ and $\mathbf{b}$ be two fixed atomic probability measures in the form of (1.1.2) on a metric space $X$, a *transport plan* from $\mathbf{a}$ to $\mathbf{b}$ is an atomic probability measure

$$\gamma = \sum_{i=1}^{m} \sum_{j=1}^{\ell} \gamma_{ij} \delta_{(x_i, y_j)} \tag{1.3.1}$$

in the product space $X \times X$ such that

$$\sum_{i=1}^{m} \gamma_{ij} = n_j \text{ and } \sum_{j=1}^{\ell} \gamma_{ij} = m_i \tag{1.3.2}$$

for each $i$ and $j$. Let $Plan(\mathbf{a}, \mathbf{b})$ be the space of all transport plans from $\mathbf{a}$ to $\mathbf{b}$.

For any atomic probability measure $\gamma$ in $X \times X$ of the form (1.3.1) and any $0 \le \alpha < 1$, we define

$$H_\alpha(\gamma) := \sum_{i=1}^{m} \sum_{j=1}^{\ell} (\gamma_{ij})^\alpha d(x_i, y_j),$$

where $d$ is the given metric on $X$.

Using $H_\alpha$, we define

$$J_\alpha(\mathbf{a}, \mathbf{b}) := \min\{H_\alpha(\gamma) : \gamma \in Plan(\mathbf{a}, \mathbf{b})\}.$$

For any given natural number $N \in \mathbb{N}$, let $\mathcal{A}_N(X)$ be the space of all atomic probability measures

$$\sum_{i=1}^{m} a_i \delta_{x_i}$$

on $X$ with $m \le N$, and then $\mathcal{A}(X) = \bigcup_N \mathcal{A}_N(X)$ is the space of all atomic probability measures on $X$.

In [28, Proposition 4.2], we showed that $J_\alpha$ defines a quasimetric on $\mathcal{A}_N(X)$. Moreover, $J_\alpha$ is a complete quasimetric on $\mathcal{A}_N(X)$ if $(X, d)$ is a complete metric space. The quasimetric $J_\alpha$ has a very nice property in the sense that this quasimetric is able to induce an intrinsic metric on $\mathcal{A}_N(X)$.

**Proposition 1.3.1.** [28, Theorem 4.17, Corollary 4.18]*Suppose $0 \le \alpha < 1$, and $X$ is a geodesic metric space. Then, the metric $d_\alpha$ defined in definition (1.1.5) is the intrinsic metric on $\mathcal{A}_N(X)$ induced by the quasimetric $J_\alpha$.*

Moreover, in [28, remark 4.16] we have a simple formula for the $\mathbf{M}_\alpha$ cost. Suppose $G \in Path(\mathbf{a}, \mathbf{b})$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{A}_N(X)$. If each edge of $G$ is a geodesic curve between its endpoints in the geodesic metric space $X$, then there exists an associated piecewise metric Lipschitz curve $g : [0,1] \to \mathcal{A}_N(X)$ such that

$$\mathbf{M}_\alpha(G) = \int_0^1 |\dot{g}(t)|_{J_\alpha} dt.$$

[1] A function $q : X \times X \to [0, +\infty)$ is a *quasimetric* on $X$ if $q$ satisfies all the conditions of a metric except that $q$ satisfies a relaxed triangle inequality $q(x,y) \le C(q(x,z) + q(z,y))$ for some $C \ge 1$, rather than the usual triangle inequality.

where the quasimetric derivative

$$|\dot{g}(t)|_{J_\alpha} := \lim_{s\to t} \frac{J_\alpha(g(t), g(s))}{|t-s|}$$

exists almost everywhere.

**Corollary 1.3.2.** *Suppose $(X,d)$ is a complete geodesic metric space. Then, the space $(\mathcal{A}_N(X), d_\alpha)$ is a complete geodesic metric space for each $0 \le \alpha < 1$.*

Since $\mathcal{A}_1(X) \subset \mathcal{A}_2(X) \subset \cdots \subset \mathcal{A}_N(X) \subset \cdots$, and $(\mathcal{A}_N(X), d_\alpha)$ is a geodesic space for each $N$, we have the following existence result of optimal transport path:

**Proposition 1.3.3.** [28, proposition 4.02] *Suppose $(X,d)$ is a complete geodesic metric space. Then, $(\mathcal{A}(X), d_\alpha)$ is a geodesic metric space for each $0 \le \alpha < 1$. Moreover, for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}(X)$, every $\alpha-$optimal transport path from $\mathbf{a}$ to $\mathbf{b}$ is a geodesic from $\mathbf{a}$ to $\mathbf{b}$ in the geodesic space $(\mathcal{A}(X), d_\alpha)$. Vice versa, every geodesic from $\mathbf{a}$ to $\mathbf{b}$ in $(\mathcal{A}(X), d_\alpha)$ is an $\alpha-$optimal transport path from $\mathbf{a}$ to $\mathbf{b}$.*

## 2. Transportation in metric spaces with curvature bounded above

In the settings of an Euclidean space $\mathbb{R}^m$, the degree (i.e. the total number of edges) at each vertex of an $\alpha-$optimal transport path is uniformly bounded above by a constant $C(m,\alpha)$ depending only on $\alpha$ and $m$ (see, for instance [25, Proposition 4.5] ). This is not necessarily true in the general metric space setting as shown in the following example.

**Example 2.0.4.** *Let $X$ be the unit disk in $\mathbb{R}^2$ with the following radial metric:*

$$d(x,y) = \begin{cases} \|x-y\|, & \text{if } x = cy \text{ for some } c \in \mathbb{R}, \\ \|x\| + \|y\|, & \text{otherwise,} \end{cases}$$

*where $\|\cdot\|$ is the standard norm on $\mathbb{R}^2$. For each $n \in \mathbb{N}$, let $\mathbf{a}_n = \sum_{k=1}^n \frac{1}{n}\delta_{x_n^k}$ with $x_n^k = \left(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}\right)$ for each $k = 1, 2, \cdots, n$. Also, let $\mathbf{b} = \delta_{\{0,0\}}$. Then, clearly, the only transport path $G_n$ from $\mathbf{a}_n$ to $\mathbf{b}$ is the weighted graph consisting of $n$ edges with weights $\frac{1}{n}$ from $\left\{x_n^k\right\}_{k=1}^n$ to the origin $O$. This $G_n$ is an $\alpha-$optimal transport path for each $\alpha$. The degree of $G_n$ at $O$ is $n$ which is obviously not uniformly bounded by any constant.*

In this section, we will show that when $X$ is a geodesic doubling metric space with curvature bounded above, then there exists a universal upper bound for the degree of every vertex of every optimal transport path on $X$.

We now recall the definition of a space of bounded curvature [3]. For a real number $k$, the model space $M_k^2$ is the simply connected surface with constant curvature $k$. That is, if $k=0$, then $M_k^2$ is the Euclidean plane. If $k>0$, then $M_k^2$ is obtained from the sphere $\mathbb{S}^2$ by multiplying the distance function by the constant $\frac{1}{\sqrt{k}}$. If $k<0$, then $M_k^2$ is obtained from the hyperbolic space $\mathbb{H}^2$ by multiplying the distance function by the constant $\frac{1}{\sqrt{-k}}$. The diameter of $M_k^2$ is denoted by $D_k := \pi/\sqrt{k}$ for $k>0$ and $D_k := \infty$ for $k \le 0$.

Let $(X,d)$ be a geodesic metric space, and let $\Delta ABC$ be a geodesic triangle in $X$ with geodesic segments as its sides. A *comparison* triangle $\Delta\bar{A}\bar{B}\bar{C}$ is a triangle in the model space $M_k^2$ such that $d(A,B) = \left|\bar{A}-\bar{B}\right|_k$, $d(B,C) = \left|\bar{B}-\bar{C}\right|_k$ and $d(A,C) = \left|\bar{A}-\bar{C}\right|_k$, where $|\cdot|_k$ denotes the distance function in the model space

$M_k^2$. Such a triangle is unique up to isometry. Also, the interior angle of $\Delta\bar{A}\bar{B}\bar{C}$ at $\bar{B}$ is called the *comparison angle* between $A$ and $C$ at $B$.

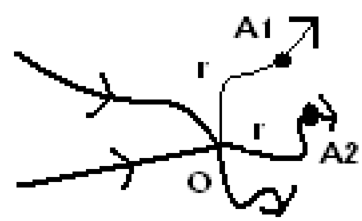
A geodesic metric space $(X, d)$ is *a space of curvature bounded above by a real number* $k$ if for every geodesic triangle $\Delta ABC$ in $X$ and every point $h$ in the geodesic segment $\gamma_{AC}$, one has

$$d(h, B) \leq \left|\bar{h} - \bar{B}\right|_k$$

where $\bar{h}$ is the point on the side $\gamma_{\bar{A}\bar{C}}$ of a comparison triangle $\Delta\bar{A}\bar{B}\bar{C}$ in $M_k^2$ such that $\left|\bar{h} - \bar{C}\right|_k = d(h, C)$.

Now, let $X$ be a geodesic metric space with curvature bounded above by a real number $k$. Suppose $\alpha < 1$ and $G$ is an $\alpha-$optimal transport path between two atomic probability measures $\mathbf{a}, \mathbf{b} \in \mathcal{A}(X)$. We will show that the comparison angle of any two edges from a common vertex of $G$ is bounded below by a universal constant depending only on $\alpha$. Moveover, when $X$ is in addition a doubling space, then the degree of any vertex $v$ of $G$ is bounded above by a constant depending only on $\alpha$ and the doubling constant of $X$.

More precisely, let $O$ be any vertex of $G$ and $e_i$ be any two distinct directed edges with $e_i^+ = O$ (or $e_i^- = O$ simultaneously) and weight $m_i > 0$ for $i = 1, 2$. Also, for $i = 1, 2$, let $A_i$ be the point on the edge $e_i$ with $d(O, A_i) = r$ for some $r$ satisfying $0 < r \leq \frac{1}{2} D_k$ and $r \leq length(e_i)$.




Now, we want to estimate the distance $d(A_1, A_2)$. To do it, we first denote

$$R = \sqrt{\frac{\left(m_1^\alpha + m_2^\alpha\right)^2 - \left(m_1 + m_2\right)^{2\alpha}}{m_1^\alpha m_2^\alpha}} \tag{2.0.3}$$

and have the following estimates for $R$:

**Lemma 2.0.5.** *For each $\alpha < 1$, the infimum of $R$ is given by*

$$R_\alpha := \begin{cases} \sqrt{2}, & \text{if } 0 < \alpha < \frac{1}{2} \\ \sqrt{4 - 4^\alpha}, & \text{if } \frac{1}{2} \leq \alpha < 1 \text{ or } \alpha \leq 0. \end{cases} \tag{2.0.4}$$

*For each $0 \leq \alpha < 1$, the supremum of $R$ is given by*

$$\bar{R}_\alpha := \begin{cases} \sqrt{2}, & \text{if } \frac{1}{2} \leq \alpha < 1 \\ \sqrt{4 - 4^\alpha}, & \text{if } 0 \leq \alpha < \frac{1}{2}. \end{cases}$$

*Also, when $\alpha = 0$, then $R \equiv \sqrt{3}$. When $\alpha = \frac{1}{2}$, then $R \equiv \sqrt{2}$. When $\alpha = 1$, then $R \equiv 0$.*

When $\alpha < 0$, we will show $R \leq 2$ later in lemma 2.0.7.

*Proof.* We first denote

$$k_1 = \frac{m_1}{m_1 + m_2},\ k_2 = \frac{m_2}{m_1 + m_2} \tag{2.0.5}$$

as in [24, Example 2.1]. Note that $k_1 + k_2 = 1$ and

$$R = \sqrt{\frac{\left(k_1^\alpha + k_2^\alpha\right)^2 - 1}{k_1^\alpha k_2^\alpha}}.$$

By considering the function

$$f_\alpha(x) = \frac{\left(x^\alpha + (1-x)^\alpha\right)^2 - 1}{x^\alpha (1-x)^\alpha}$$

for $x \in (0,1)$ and $\alpha \leq 1$, we have $R = \sqrt{f_\alpha(k_1)}$. Using Calculus, one may check that for each $x \in (0,1)$,

(1) when $\alpha \in \left(0, \frac{1}{2}\right)$, the function $f_\alpha$ is strictly concave up and

$$4 - 4^\alpha = f_\alpha\left(\frac{1}{2}\right) \leq f_\alpha(x) < \lim_{y \to 0+} f_\alpha(y) = 2;$$

(2) when $\alpha \in \left(\frac{1}{2}, 1\right)$, the function $f_\alpha$ is strictly concave down and

$$4 - 4^\alpha = f_\alpha\left(\frac{1}{2}\right) \geq f_\alpha(x) > \lim_{y \to 0+} f_\alpha(y) = 2;$$

(3) when $\alpha \in (-\infty, 0)$, the function $f_\alpha$ is strictly concave down and

$$f_\alpha(x) \geq f_\alpha\left(\frac{1}{2}\right) = 4 - 4^\alpha;$$

(4) $f_\alpha$ has constant values when $\alpha \in \left\{0, \frac{1}{2}, 1\right\}$.

Using these facts, we get the estimates for $R = \sqrt{f_\alpha(k_1)}$ for each $\alpha$. □

Now, we have the following key estimates for the distance $d(A_1, A_2)$:

**Lemma 2.0.6.** *Assume that $d(O, A_1) = d(O, A_2) = r$ and $0 < r \leq \frac{1}{2} D_k$. Then, we have the following estimates for $a := d(A_1, A_2)$:*

(1) *If $k > 0$, then*

$$\cos\left(a\sqrt{k}\right) \leq 1 - \frac{R^2}{2}\sin^2\left(r\sqrt{k}\right) \quad \textit{i.e.} \ \sin\frac{a\sqrt{k}}{2} \geq \frac{R}{2}\sin\left(r\sqrt{k}\right).$$

(2) *If $k = 0$, then*

$$a \geq Rr.$$

(3) *If $k < 0$, then*

$$\cosh\left(a\sqrt{-k}\right) \geq 1 + \frac{R^2}{2}\sinh^2\left(r\sqrt{-k}\right) \ \textit{i.e.} \ \sinh\frac{a\sqrt{-k}}{2} \geq \frac{R}{2}\sinh\left(r\sqrt{-k}\right).$$

*Proof.* Let $P$ be the point on the geodesic $\gamma_{A_1A_2}$ from $A_1$ to $A_2$ with

$$d(A_i, P) = \lambda_i d(A_1, B)$$

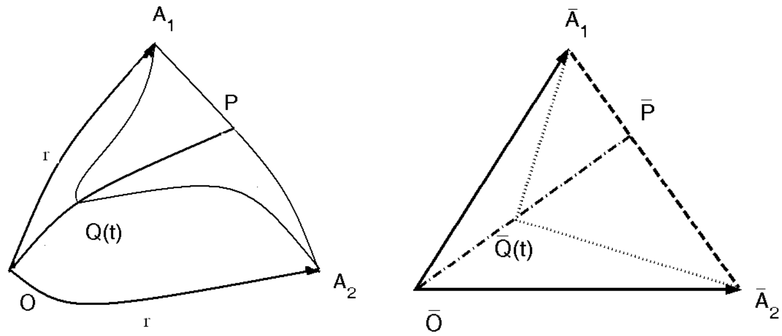
for $i = 1, 2$ and some $\lambda_i \in (0,1)$ to be chosen later in (2.0.7) with $\lambda_1 + \lambda_2 = 1$. For any $t \in [0,1]$, let $Q(t)$ be the point on the geodesic $\gamma_{OP}$ from $O$ to $P$ such that

$$d(O, Q(t)) = tb \text{ and } d(P, Q(t)) = (1-t)\, b$$

where $b = d(O, P)$. For $i = 1, 2$, let $\Delta \bar{A}_i \bar{P} \bar{O}$ be a comparison triangle of $\Delta A_i P O$ in the model space $M_k^2$.

Thus,

$$\left|\bar{A}_i - \bar{P}\right|_k = \lambda_i a, \left|\bar{A}_i - \bar{O}\right|_k = r \text{ and } \left|\bar{O} - \bar{P}\right|_k = b.$$




(A) A triangle $\Delta A_i PO$ in $X$

(B) A comparison triangle $\Delta \bar{A}_i \bar{P} \bar{O}$ in $M_k^2$

FIGURE 1. Comparison triangles

Let $\bar{Q}_i(t)$ be the point on the side $\gamma_{\bar{O}\bar{P}}$ of $\Delta \bar{A}_i \bar{P} \bar{O}$ such that $\left|\bar{O} - \bar{Q}_i(t)\right|_k = d(O, Q(t)) = tb$ and let $\sigma_i(t) = \left|\bar{A}_i - \bar{Q}_i(t)\right|_k$. Since $X$ has curvature bounded above by $k$, we have $\sigma_i(t) \geq d(A_i, Q(t))$. Let

$$\begin{aligned} H(t) \quad : &= \frac{1}{(m_1+m_2)^\alpha}\left[m_1^\alpha \sigma_1(t) + m_2^\alpha \sigma_2(t) + (m_1+m_2)^\alpha tb\right] \\ &= k_1^\alpha \sigma_1(t) + k_2^\alpha \sigma_2(t) + tb \\ &\geq \frac{1}{(m_1+m_2)^\alpha}\left[m_1^\alpha d(A_1, Q(t)) + m_2^\alpha d(A_2, Q(t)) + (m_1+m_2)^\alpha d(O, Q(t))\right] \\ &\geq \frac{m_1^\alpha d(O, A_1) + m_2^\alpha d(O, A_2)}{(m_1+m_2)^\alpha} = H(0), \end{aligned}$$

since $O$ is a vertex of an $\alpha-$optimal transport path $G$. This implies that $H'(0) \geq 0$ if $H'(0)$ exists. Now, we may calculate the derivative $H'(0) = k_1^\alpha \sigma_1'(0) + k_2^\alpha \sigma_2'(0) + b$ as follows.

When $k > 0$, by applying the spherical law of cosines to triangles $\Delta \bar{A}_i \bar{P} \bar{O}$ and $\Delta \bar{A}_i \bar{Q}(t) \bar{O}$, we have

$$\begin{aligned} \cos\left(\lambda_i a\sqrt{k}\right) &= \cos\left(r\sqrt{k}\right)\cos\left(b\sqrt{k}\right) + \sin\left(r\sqrt{k}\right)\sin\left(b\sqrt{k}\right)\cos\theta_i \\ \cos\left(\sigma_i(t)\sqrt{k}\right) &= \cos\left(r\sqrt{k}\right)\cos\left(tb\sqrt{k}\right) + \sin\left(r\sqrt{k}\right)\sin\left(tb\sqrt{k}\right)\cos\theta_i, \end{aligned}$$

where $\theta_i$ is the angle $\measuredangle \bar{A}_i \bar{O} \bar{P}$. Thus,

$$\sin\left(tb\sqrt{k}\right)\cos\left(\lambda_i a\sqrt{k}\right) - \sin\left(b\sqrt{k}\right)\cos\left(\sigma_i(t)\sqrt{k}\right) = -\cos\left(r\sqrt{k}\right)\sin\left((1-t)b\sqrt{k}\right).$$

Taking derivative with respect to $t$ at $t = 0$ and using the fact $\sigma_i(0) = r$, we have

$$\left(b\sqrt{k}\right)\cos\left(\lambda_i a\sqrt{k}\right) + \sin\left(b\sqrt{k}\right)\sin\left(r\sqrt{k}\right)\sigma_i'(0)\sqrt{k} = b\sqrt{k}\cos\left(r\sqrt{k}\right)\cos\left(b\sqrt{k}\right).$$

Therefore, for $i = 1, 2$,

$$\sigma_i'(0) = \frac{b\cos\left(r\sqrt{k}\right)\cos\left(b\sqrt{k}\right) - b\cos\left(\lambda_i a\sqrt{k}\right)}{\sin\left(b\sqrt{k}\right)\sin\left(r\sqrt{k}\right)}.$$

Applying these expressions to $H'(0) = k_1^\alpha \sigma_1'(0) + k_2^\alpha \sigma_2'(0) + b \geq 0$, we have
(2.0.6)

$$(k_1^\alpha + k_2^\alpha)\cos\left(r\sqrt{k}\right)\cos\left(b\sqrt{k}\right)+\sin\left(r\sqrt{k}\right)\sin\left(b\sqrt{k}\right) \geq k_1^\alpha \cos\left(\lambda_1 a\sqrt{k}\right)+k_2^\alpha\cos\left(\lambda_2 a\sqrt{k}\right).$$

By setting

$$Ve^{i\Theta_1} = (k_1^\alpha + k_2^\alpha)\cos\left(r\sqrt{k}\right) + i\sin\left(r\sqrt{k}\right)$$

as a complex number, we have

$$(k_1^\alpha + k_2^\alpha)\cos\left(r\sqrt{k}\right)\cos\left(b\sqrt{k}\right)+\sin\left(r\sqrt{k}\right)\sin\left(b\sqrt{k}\right) = V\cos\left(\Theta_1 - b\sqrt{k}\right).$$

On the other hand, as $\lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned}
& k_1^\alpha \cos\left(\lambda_1 a\sqrt{k}\right) + k_2^\alpha \cos\left(\lambda_2 a\sqrt{k}\right) \\
= \ & k_1^\alpha \cos\left(\lambda_1 a\sqrt{k}\right) + k_2^\alpha \cos\left(a\sqrt{k}\right)\cos\left(\lambda_1 a\sqrt{k}\right) + k_2^\alpha \sin\left(a\sqrt{k}\right)\sin\left(\lambda_1 a\sqrt{k}\right) \\
= \ & \left(k_1^\alpha + k_2^\alpha\cos\left(a\sqrt{k}\right)\right)\cos\left(\lambda_1 a\sqrt{k}\right) + k_2^\alpha \sin\left(a\sqrt{k}\right)\sin\left(\lambda_1 a\sqrt{k}\right) \\
= \ & W\cos\left(\Theta_2 - \lambda_1 a\sqrt{k}\right),
\end{aligned}$$

where

$$We^{i\Theta_2} = \left(k_1^\alpha + k_2^\alpha\cos\left(a\sqrt{k}\right)\right) + i\left(k_2^\alpha\sin\left(a\sqrt{k}\right)\right) = k_1^\alpha + k_2^\alpha e^{ia\sqrt{k}}$$

as a complex number for some $\Theta_2 \in [0, 2\pi)$. Thus, inequality (2.0.6) becomes

$$V\cos\left(\Theta_1 - b\sqrt{k}\right) \geq W\cos\left(\Theta_2 - \lambda_1 a\sqrt{k}\right).$$

Since $0 < r \leq \frac{1}{2}D_k$, we have $0 < a \leq 2r \leq \pi/\sqrt{k}$. Then it is easy to see that $0 < \Theta_2 < a\sqrt{k}$. Let

$$\lambda_1 = \frac{\Theta_2}{a\sqrt{k}} \in (0,1), \tag{2.0.7}$$

we have the inequality $V \geq W$. That is,

$$(k_1^\alpha + k_2^\alpha)^2\cos^2\left(r\sqrt{k}\right) + \sin^2\left(r\sqrt{k}\right) \geq \left(k_1^\alpha + k_2^\alpha\cos\left(a\sqrt{k}\right)\right)^2 + k_2^{2\alpha}\sin^2\left(a\sqrt{k}\right).$$

By simplifying this inequality, we get

$$\cos\left(a\sqrt{k}\right) \leq 1 - \frac{R^2}{2}\sin^2\left(r\sqrt{k}\right) \text{ and thus } \sin\frac{a\sqrt{k}}{2} \geq \frac{R}{2}\sin\left(r\sqrt{k}\right).$$

The proof for the cases $k = 0$ and $k < 0$ are similar when using the ordinary (or the hyperbolic) law of cosines in the model space $M_k^2$. □

Using lemma 2.0.6, we have the following upper bounds for $R$ defined as in (2.0.3) which is useful when $\alpha < 0$.

**Lemma 2.0.7.** *Let $R$ be defined as in (2.0.3). For any $k$ and $\alpha < 1$, we have*

$$R \leq 2.$$

*Proof.* By the triangle inequality, we have $a \leq 2r \leq D_k$. We now use the estimates in lemma 2.0.6.

When $k < 0$, then

$$\sinh\left(\sqrt{-k}r\right) \geq \sinh\frac{\sqrt{-k}a}{2} \geq \frac{R}{2}\sinh\left(\sqrt{-k}r\right).$$

This yields $R \leq 2$.

When $k = 0$, then

$$2r \geq a \geq Rr,$$

so $R \leq 2$.

When $k > 0$, then

$$\sin\left(r\sqrt{k}\right) \geq \sin\frac{a\sqrt{k}}{2} \geq \frac{R}{2}\sin\left(r\sqrt{k}\right)$$

as $0 \leq \frac{a\sqrt{k}}{2} \leq r\sqrt{k} \leq \frac{\pi}{2}$. Therefore, we still have $R \leq 2$. □

The following proposition says that when $\alpha$ is negative, the weights on any two directed edges from a common vertex of an $\alpha-$optimal transport path are comparable to each other.

**Proposition 2.0.8.** *If $\alpha < 0$, then for each $i = 1, 2$,*

$$k_i \geq \frac{1}{1 + (1 + 2^\alpha)^{-\frac{1}{\alpha}}},$$

*where $k_i$ is defined as in (2.0.5).*

*Proof.* Without losing generality, we may assume that $k_2 \geq k_1$. By proposition 2.0.7, we have $R \leq 2$. That is,

$$\frac{\left(k_1^\alpha + k_2^\alpha\right)^2 - 1}{k_1^\alpha k_2^\alpha} \leq 4.$$

Simplify it, we have

$$k_1^\alpha - k_2^\alpha \leq 1.$$

Since $k_1 \in (0, \frac{1}{2}]$ and $\alpha < 0$, we have

$$1 - \left(\frac{k_2}{k_1}\right)^\alpha \leq (k_1)^{-\alpha} \leq 2^\alpha.$$

Simplify it again using $k_2 = 1 - k_1$, we have

$$k_1 \geq \frac{1}{1 + (1 + 2^\alpha)^{-\frac{1}{\alpha}}}.$$

□

We now may investigate the comparison angle $\theta$ between $A_1$ and $A_2$ at $O$, given in figure 1a:

**Proposition 2.0.9.** *Let $X$ be a geodesic metric space with curvature bounded above by a real number $k$. Let $\theta$ be the comparison angle between $A_1$ and $A_2$ at $O$ in the model space $M_k^2$. Then*

$$\theta \geq \arccos\left(1 - \frac{R^2}{2}\right) = \arccos\left(\frac{1 - k_1^{2\alpha} - k_2^{2\alpha}}{2k_1^\alpha k_2^\alpha}\right).$$

*Thus, by (2.0.4), we have*

$$\theta \geq \theta_\alpha := \begin{cases} \frac{\pi}{2}, & \text{if } 0 < \alpha \leq \frac{1}{2} \\ \arccos\left(2^{2\alpha-1} - 1\right), & \text{if } \frac{1}{2} < \alpha < 1 \text{ or } \alpha \leq 0 \end{cases}.$$

Note that when $k = 0$, this agrees with what we have found in [24, Example 2.1] for a "Y-shaped" path. Also, when $\alpha$ approaches $-\infty$, then $\theta_\alpha$ approaches $\pi$, and when $\alpha$ approaches 1, then $\theta_\alpha$ approaches 0.

*Proof.* When $k > 0$, then by the spherical law of cosines,

$$\begin{aligned} \cos\theta &= \frac{\cos\left(a\sqrt{k}\right) - \cos^2\left(r\sqrt{k}\right)}{\sin^2\left(r\sqrt{k}\right)} \\ &\leq \frac{1 - \frac{R^2}{2}\sin^2\left(r\sqrt{k}\right) - \cos^2\left(r\sqrt{k}\right)}{\sin^2\left(r\sqrt{k}\right)} = 1 - \frac{R^2}{2}. \end{aligned}$$

When $k < 0$, then by the hyperbolic law of cosines

$$\begin{aligned} \cos\theta &= \frac{-\cosh\left(a\sqrt{-k}\right) + \cosh^2\left(r\sqrt{-k}\right)}{\sinh^2\left(r\sqrt{-k}\right)} \\ &\leq \frac{-1 - \frac{R^2}{2}\sinh^2\left(r\sqrt{-k}\right) + \cosh^2\left(r\sqrt{-k}\right)}{\sinh^2\left(r\sqrt{-k}\right)} = 1 - \frac{R^2}{2}. \end{aligned}$$

When $k = 0$, then by the law of cosines,

$$\cos\theta = \frac{r^2 + r^2 - a^2}{2r^2} \leq \frac{2r^2 - R^2 r^2}{2r^2} = 1 - \frac{R^2}{2}.$$

$\square$

Now, we want to estimate the degree (i.e. the total number of edges) at each vertex of an optimal transport path. We first rewrite lemma 2.0.6 as follows. For any real numbers $x \leq \left(\frac{\pi}{2}\right)^2$ and $0 < y \leq 2$, define

$$\Psi(x,y) := \begin{cases} \frac{1}{\sqrt{x}} \arcsin\left(\frac{y}{2}\sin\left(\sqrt{x}\right)\right), & \text{if } 0 < x \leq \left(\frac{\pi}{2}\right)^2 \\ \frac{R}{2}, & \text{if } x = 0 \\ \frac{1}{\sqrt{-x}} \sinh^{-1}\left(\frac{y}{2}\sinh\left(\sqrt{-x}\right)\right) & \text{if } x < 0 \end{cases}.$$

Then, one may check that $\Psi$ is a continuous strictly decreasing function of the variable $x$ and an increasing function of $y$. Moreover, for each fixed $y$, $\lim_{x\to-\infty}\Psi(x,y) = 1$ and

$$\Psi(x,y) \geq \Psi\left(\left(\frac{\pi}{2}\right)^2, y\right) = \frac{2}{\pi}\arcsin\left(\frac{y}{2}\right). \tag{2.0.8}$$

By means of the function $\Psi$, (2.0.8) and (2.0.4), the lemma 2.0.6 becomes

**Lemma 2.0.10.** *Assume that* $d(O, A_1) = d(O, A_2) = r$ *and* $0 < r \leq \frac{1}{2}D_k$. *Then, we have the following estimate for* $d(A_1, A_2)$*:*

$$2r \geq d(A_1, A_2) \geq 2r\Psi\left(r^2 k, R\right) \geq 2rC_\alpha$$

*where* $C_\alpha := \frac{2}{\pi}\arcsin\left(\frac{R_\alpha}{2}\right)$.

Note that since $\lim_{x\to-\infty}\Psi(x,R)=1$, we have $d(A_1,A_2)$ is nearly $2r$ when $k$ approaches $-\infty$.

For the purpose of theorem 2.0.11, we consider the function

$$\Phi(x,\alpha)=1+\frac{\ln\left(1+\frac{1}{\Psi(x,R_\alpha)}\right)}{\ln 2} \tag{2.0.9}$$

for $x\leq(\frac{\pi}{2})^2$ and $\alpha<1$. For each fixed $\alpha<1$, $\Phi_\alpha(k):=\Phi(k,\alpha)$ is a strictly increasing function of $k$ with lower bound $\lim_{k\to-\infty}\Phi(k,\alpha)=2$, upper bound

$$\Phi(k,\alpha)\leq 1+\frac{\ln\left(1+\frac{1}{C_\alpha}\right)}{\ln 2}$$

and

$$\Phi(0,\alpha)=1+\frac{\ln\left(1+\frac{2}{R_\alpha}\right)}{\ln 2}.$$

As in [17, 10.13], a metric space $X$ is called *doubling* if there is a doubling constant $C_d\geq 1$ so that every subset of diameter $r$ in $X$ can be covered by at most $C_d$ subsets of diameter at most $\frac{r}{2}$. Doubling spaces have the following covering property: there exists constants $\beta>0$ and $C_\beta\geq 1$ such that for every $\epsilon\in(0,\frac{1}{2}]$, every set of diameter $r$ in $X$ can be covered by at most $C_\beta\epsilon^{-\beta}$ sets of diameter at most $\epsilon r$. This function $C_\beta\epsilon^{-\beta}$ is called a covering function of $X$. The infimum of all numbers $\beta>0$ such that a covering function can be found is called the *Assouad dimension* of $X$. It is clear that subsets of doubling spaces are still doubling. For any subset $K$ of $X$, let $\dim_A(K)$ denote the Assouad dimension of $K$.

**Theorem 2.0.11.** *Suppose $X$ is a geodesic doubling metric space of curvature bounded above by a real number $k$. Let $\alpha<1$ and $G$ be an $\alpha-$optimal transport path between two atomic probability measures on $X$, and $O$ is a vertex of $G$. Let $\deg(O)$ be the degree of the vertex $O$ and $r(O)$ be the maximum number $r$ in $(0,\frac{1}{2}D_k]$ such that the truncated ball $B(O,r)\setminus\{O\}$ contains no vertices of $G$. Then,*

1. *for any $0<r\leq r(O)$, we have*
$$\deg(O)\leq 2(C_d)^{\Phi\left(r^2k,\alpha\right)},$$
*where $C_d$ is the doubling constant of $X$, and $\Phi$ is given in (2.0.9).*
2. *Moreover, $\deg(O)\leq 2(C_d)^{\Phi(0,\alpha)}$, which is a constant depends only on $\alpha$ and $C_d$.*
3. *If $\deg(O)\geq 2(C_d)^2$, then the curvature upper bound*
$$k\geq\frac{1}{r(O)^2}(\Phi_\alpha)^{-1}\left(\log_{C_d}^{\deg(O)/2}\right).$$
4. *In particular, if $\deg(O)=2(C_d)^{\Phi(0,\alpha)}$, then $k\geq 0$.*
5. *If $k<0$, then*
$$r(O)\leq\sqrt{\frac{(\Phi_\alpha)^{-1}\left(\log_{C_d}^{\deg(O)/2}\right)}{k}}.$$

*Proof.* For any $0<r\leq r(O)$, let $\{A_i\}$ be the intersection points of the sphere $S(O,r)$ in $X$ with all edges of $G$ that flows out of $O$. It is sufficient to show that the cardinality of $\{A_i\}$ is bounded above by $(C_d)^{\Phi\left(r^2k,\alpha\right)}$. By lemma 2.0.10, the balls

$\left\{B\left(A_i, r\Psi\left(r^2k, R_\alpha\right)\right)\right\}$ are disjoint and contained in $B\left(O,\left(1+\Psi\left(r^2k, R_\alpha\right)\right)r\right)$. Since $X$ is doubling, the cardinality of $\left\{B\left(A_i, \Psi\left(r^2k, R_\alpha\right)r\right)\right\}$ (i.e. the cardinality of $\{A_i\}$) is bounded above by $(C_d)^{\Phi\left(r^2k,\alpha\right)}$. This proves (1). By setting $r \to 0$ in (1), we have (2). Then (3) and (5) follow from (1), and (4) follows from (3). □

## 3. Optimal transport paths between arbitrary probability measures

3.1. **The completion of the metric space $(\mathcal{A}(X), d_\alpha)$.** In this section, we consider optimal transport paths between two arbitrary probability measures on a complete geodesic metric space $(X, d)$. Unlike what we did in Euclidean space [24], we use a new approach here by considering the completion of $\mathcal{A}(X)$ with respect to the metric $d_\alpha$. Note that $(\mathcal{A}(X), d_\alpha)$ is not necessarily complete for $\alpha < 1$. So, we consider its completion as follows.

**Definition 3.1.1.** *For any $\alpha \in (-\infty, 1]$, let $\mathcal{P}_\alpha(X)$ be the completion of the metric space $\mathcal{A}(X)$ with respect to the metric $d_\alpha$.*

It is easy to check that (see [29, lemma 2.2.5]) if $\beta < \alpha$, then $\mathcal{P}_\beta(X) \subseteq \mathcal{P}_\alpha(X)$, and for all $\mu, \nu$ in $\mathcal{P}_\beta(X)$ we have $d_\beta(\mu,\nu) \geq d_\alpha(\mu,\nu)$. Note that when $\alpha = 1$, the metric $d_1$ is the usual Monge's distance on $\mathcal{A}(X)$ and $\mathcal{P}_1(X)$ is just the space $\mathcal{P}(X)$ of all probability measures on $X$. Therefore, each element in $\mathcal{P}_\alpha$ can be viewed as a probability measure on $X$ when $\alpha < 1$.

By proposition 1.3.3, for $0 \leq \alpha < 1$, the concept of an $\alpha-$optimal transport path between measures in $\mathcal{A}(X)$ coincides with the concept of geodesic in $(\mathcal{A}(X), d_\alpha)$. This motivates us to introduce the following concept.

**Definition 3.1.2.** *For any two probability measures $\mu^+$ and $\mu^-$ on a complete geodesic metric space $X$ and $\alpha < 1$, if there exists a geodesic in $(\mathcal{P}_\alpha(X), d_\alpha)$ from $\mu^+$ to $\mu^-$, then this geodesic is called an $\alpha-$optimal transport path from $\mu^+$ to $\mu^-$.*

In other words, the existence of an $\alpha-$optimal transport path is the same as the existence of a geodesic in $\mathcal{P}_\alpha(\mathbf{X})$. Thus, an essential part in understanding the optimal transport problem becomes describing properties of elements of $\mathcal{P}_\alpha(\mathbf{X})$, and investigating the existence of geodesics in $\mathcal{P}_\alpha(\mathbf{X})$. Since the completion of a geodesic metric space is still a geodesic space, by proposition 1.3.3, we have

**Proposition 3.1.3.** *Suppose $X$ is a complete geodesic metric space. Then for any $0 \leq \alpha < 1$, $(\mathcal{P}_\alpha(X), d_\alpha)$ is a complete geodesic metric space.*

In other words, for any two probability measures $\mu^+, \mu^- \in \mathcal{P}_\alpha(X)$ with $0 \leq \alpha < 1$, there exists an optimal transport path (i.e. a geodesic) from $\mu^+$ to $\mu^-$. In particular, since atomic measures are contained in $\mathcal{P}_\alpha(X)$, there exists an $\alpha-$optimal transport path from any probability measure $\mu \in \mathcal{P}_\alpha(X)$ to $\delta_p$ for any $p \in X$.

A positive Borel measure $\mu$ on $X$ is said to be *concentrated* on a Borel set $A$ if $\mu(X \setminus A) = 0$. The following proposition says that if $\alpha$ is nonpositive, then any element of $\mathcal{P}_\alpha(X)$ must be bounded.

**Proposition 3.1.4.** *Suppose $\alpha \leq 0$. If $\mu \in \mathcal{P}_\alpha(X)$, then $\mu$ is concentrated on the closed ball $\bar{B}(p, d_\alpha(\mu, \delta_p))$ for any $p \in X$.*

*Proof.* If $\mu \in \mathcal{P}_\alpha(X)$, then $\mu$ is represented by a Cauchy sequence $\{\mathbf{a}_n\} \in \mathcal{A}(X)$ with respect to the metric $d_\alpha$. For any $p \in X$, by corollary 1.2.2, each $\mathbf{a}_n$ is concentrated on the ball $\bar{B}(p, d_\alpha(\mathbf{a}_n, \delta_p))$. Thus, $\mu$ is concentrated on the ball $\bar{B}(p, d_\alpha(\mu, \delta_p))$. □

When $0 < \alpha < 1$, $\mu \in \mathcal{P}_\alpha(X)$ does not necessarily imply $\mu$ is concentrated on a bounded set.

**Example 3.1.5.** *Let $X = \mathbb{R}$ and $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{\{n\}}$, which is clearly not concentrated on a bounded set. But when $0 < \alpha < 1$, we have $\mu \in \mathcal{P}_\alpha(X)$ as*

$$d_\alpha\left(\mu, \delta_{\{0\}}\right) = \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{1}{2^k}\right)^\alpha \cdot 1 = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}\right)^\alpha < \infty.$$

*This measure $\mu$ is represented by the Cauchy sequence $\{\mathbf{a}_n\}_{n=1}^{\infty}$ in $(\mathcal{A}(X), d_\alpha)$ where*

$$\mathbf{a}_n = \sum_{k=1}^{n}\frac{1}{2^k}\delta_{\{k\}} + \frac{1}{2^n}\delta_{\{n+1\}}.$$

Nevertheless, the following proposition says that the mass of every measure in $\mathcal{P}_\alpha(X)$ outside a ball decays to 0 as the radius of the ball increases.

**Proposition 3.1.6.** *Suppose $0 < \alpha < 1$ and $\tilde{\mu} = \lambda\mu$ for some $\mu \in \mathcal{P}_\alpha(X)$ and $\lambda > 0$. Then, for any point $p \in X$ and $r > 0$, we have*

$$\left[\tilde{\mu}(X \setminus \bar{B}(p,r))\right]^\alpha \le \frac{d_\alpha(\tilde{\mu}, \lambda\delta_p)}{r}.$$

*In particular, if $r \ge d_\alpha(\tilde{\mu}, \lambda\delta_p)^{1-\alpha}$, we have*

$$\tilde{\mu}(X \setminus \bar{B}(p,r)) \le d_\alpha(\tilde{\mu}, \lambda\delta_p). \tag{3.1.1}$$

*Proof.* By corollary 1.2.3,

$$\left[\mu(X \setminus \bar{B}(p,r))\right]^\alpha \le \frac{d_\alpha(\mu, \delta_p)}{r}.$$

Now, for any $\lambda > 0$,

$$\left[\tilde{\mu}(X \setminus \bar{B}(p,r))\right]^\alpha = \left[\lambda\mu(X \setminus \bar{B}(p,r))\right]^\alpha \le \lambda^\alpha\frac{d_\alpha(\mu, \delta_p)}{r} = \frac{d_\alpha(\tilde{\mu}, \lambda\delta_p)}{r}.$$

□

3.2. **Transport Dimension of measures on a metric space.** Now, a natural question is to describe properties of measures that lie in the space $\mathcal{P}_\alpha(X)$. The answer to this question crucially related to the dimensional information of the measure. When $X$ is a Euclidean space $\mathbb{R}^m$, we answered this question by considering the transport dimension of a measure in [29], which is partially motivated by the work of [13]. It turns out that results achieved in [29] still hold when $\mathbb{R}^m$ is replaced by a complete geodesic metric space $X$. Here, we will only briefly mention some main results/notations of [29] in this metric space setting without giving detailed proofs or explanations. These results/notations will play an important role in §4.

We first define dimensions of a measure on a complete geodesic metric space.

3.2.1. *Dimensions of measures.* For any Radon measure $\mu$ on a complete geodesic metric space $X$, the Hausdorff dimension of $\mu$ is defined to be

$$\dim_H(\mu) = \inf\left\{\dim_H(A) : \mu(X \backslash A) = 0\right\},$$

where $\dim_H(A)$ is the Hausdorff dimension of a set $A \subseteq X$.

3.2.2. *Minkowski dimension of a measure.* A *nested collection*

$$\mathcal{F} = \{Q_i^n : i = 1, 2, \cdots, N_n \text{ and } n = 1, 2, \cdots\} \tag{3.2.1}$$

*of cubes* in $X$ is a collection of Borel subsets of $X$ with the following properties:

(1) for each $Q_i^n$, its diameter

$$C_1 \sigma^n \leq diam\,(Q_i^n) \leq C_2 \sigma^n \tag{3.2.2}$$

for some constants $C_2 \geq C_1 > 0$ and some $\sigma \in (0,1)$;

(2) for any $k, l, i, j$ with $l \geq k$, either $Q_i^k \cap Q_j^l = \emptyset$ or $Q_i^k \subseteq Q_j^l$;

(3) for each $Q_j^{n+1}$ there exists exactly one $Q_i^n$ (parent of $Q_j^{n+1}$) such that $Q_j^{n+1} \subseteq Q_i^n$;

(4) for each $Q_i^n$ there exists at least one $Q_j^{n+1}$ (child of $Q_i^n$) such that $Q_j^{n+1} \subseteq Q_i^n$.

Each $Q_i^n$ is called a cube of generation $n$ in $\mathcal{F}$. In the next section, we will see that for each bounded subset of a complete geodesic doubling metric space, there always exists a nested collection of cubes which covers the set.

**Definition 3.2.1.** *For any nested collection $\mathcal{F}$, we define its Minkowski dimension*

$$\dim_M(\mathcal{F}) := \lim_{n \to \infty} \frac{\log(N_n)}{\log\left(\frac{1}{\sigma^n}\right)} \tag{3.2.3}$$

*provided the limit exists, where $N_n$ is the total number of cubes of generation $n$.*

**Definition 3.2.2.** *Let $\mu$ be a Radon measure on a complete geodesic metric space $X$, and $\mathcal{F}$ be a nested collection in $X$. The measure measure $\mu$ is said to be concentrated on a nested collection $\mathcal{F}$ in $X$ if for each $n$,*

$$\mu\left(X \setminus \left(\bigcup_{i=1}^{N_n} Q_i^n\right)\right) = 0.$$

*Also, the measure $\mu$ is evenly concentrated on $\mathcal{F}$ if for each cube $Q_i^n$ of generation $n$ in $\mathcal{F}$, either $Q_i^n$ has no brothers (i.e. the parent of $Q_i^n$ has only one child, namely $Q_i^n$ itself ) or $\mu(Q_i^n) \geq \frac{\lambda}{N_n}$ for some constant $\lambda > 0$.*

Some examples of evenly concentrated measures have been given in [29]. In particular, if $\mu$ is an Ahlfors regular measure concentrated on a nested collection $\mathcal{F}$ in $X$, then $\mu$ is evenly concentrated on $\mathcal{F}$.

**Definition 3.2.3.** *For any Radon measure $\mu$, we define the Minkowski dimension of the measure $\mu$ to be*

$$\dim_M(\mu) := \inf\{\dim_M(\mathcal{F})\}$$

*where the infimum is over all nested collection $\mathcal{F}$ that $\mu$ is concentrated on. Also, we define*

$$\dim_U(\mu) := \inf\{\dim_M(\mathcal{F})\}$$

*where the infimum is over all nested collection $\mathcal{F}$ that $\mu$ is evenly concentrated on.*

Obviously, $\dim_M(\mu) \leq \dim_U(\mu)$.

3.2.3. *Transport dimension of measures.* Let $\{\mathbf{a}_k\}_{k=1}^{\infty}$ be a sequence of atomic measures on a complete geodesic metric space $X$ of equal total mass in the form of

$$\mathbf{a}_k = \sum_{i=1}^{N_k} m_i^{(k)} \delta_{x_i^{(k)}}$$

for each $k$, and $\alpha < 1$. We say that this sequence is a $d_\alpha-$**admissible Cauchy sequence** if for any $\epsilon > 0$, there exists an $N$ such that for all $n > k \geq N$ there exists a partition of

$$\mathbf{a}_n = \sum_{i=1}^{N_k} \mathbf{a}_{n,i}^{(k)}$$

with respect to $\mathbf{a}_k$ as sums of disjoint atomic measures and a path

$$G_{n,i}^k \in Path(m_i^{(k)} \delta_{x_i^{(k)}}, \mathbf{a}_{n,i}^{(k)})$$

for each $i = 1, 2, \cdots, N_k$ such that

$$\sum_{i=1}^{N_k} \mathbf{M}_\alpha \left(G_{n,i}^k\right) \leq \epsilon.$$

Clearly, each $d_\alpha-$**admissible Cauchy sequence** of probability atomic measures corresponds to an element in $\mathcal{P}_\alpha(X)$. Let

$$\mathcal{D}_\alpha(X) \subseteq \mathcal{P}_\alpha(X)$$

be the set of all probability measures $\mu$ which corresponds to a $d_\alpha$ admissible Cauchy sequence of probability measures.

We now introduce the following concept:

**Definition 3.2.4.** *Suppose $X$ is a complete geodesic metric space. For any probability measure $\mu$ on $X$, we define the transport dimension of $\mu$ to be*

$$\dim_T(\mu) := \inf_{\alpha<1} \left\{ \frac{1}{1-\alpha} : \mu \in \mathcal{D}_\alpha(X) \right\}.$$

Note that if $\frac{1}{1-\alpha} > \dim_T(\mu)$, then $\mu \in \mathcal{D}_\alpha(X)$, and thus $d_\alpha(\mu, \delta_O) < +\infty$ for any fixed point $O \in X$. If in addition $\alpha \geq 0$, then there exists an $\alpha-$optimal transport path from $\mu$ to $\delta_O$.

It turns out that the same proof of many theorems in [29] still hold when the underlying space $\mathbb{R}^m$ is replaced by a complete geodesic metric space $X$.

**Theorem 3.2.5.** *Suppose $X$ is a complete geodesic metric space. Let $\mu$ be any probability measure on $X$, then*

(1) *If $\mu \in \mathcal{D}_\alpha(X)$ for some $\alpha \in (-\infty, 1)$, then $\mu$ is concentrated on a subset $A$ of $X$ with Hausdorff measure $\mathcal{H}^{\frac{1}{1-\alpha}}(A) = 0$, and thus $\dim_H(\mu) \leq \frac{1}{1-\alpha}$.*
(2) *If $\dim_M(\mu) < \frac{1}{1-\alpha}$ for some $0 \leq \alpha < 1$, then $\mu \in \mathcal{D}_\alpha(X)$.*
(3) *If $\dim_U(\mu) < \frac{1}{1-\alpha}$ for some $\alpha < 1$, then $\mu \in \mathcal{D}_\alpha(X)$.*
(4) *In general, we have*

$$\dim_H(\mu) \leq \dim_T(\mu) \leq \max\{\dim_M(\mu), 1\}.$$

*Moreover, we also have*

$$\dim_H(\mu) \leq \dim_T(\mu) \leq \dim_U(\mu).$$

Proof of this theorem comes from a straightforward extension of the proofs of [29, theorem 3.2.1] (by means of corollary 1.2.2 and (3.1.1) in this article), [29, theorem 3.3.5] and [29, theorem 3.4.6].

In [29, Example 3.5.3], we showed that for the Cantor measure $\mu$, we have

$$\dim_H(\mu) = \dim_T(\mu) = \dim_U(\mu) = \frac{\ln 2}{\ln 3}.$$

3.2.4. *The Dimensional Distance between probability measures.* What distinct the transport dimension from others is the following theorem (see: [29, Theorem 4.0.6 and Theorem 4.0.8])

**Theorem 3.2.6.** *Let $X$ be a complete geodesic metric space. There exists a pseudometric*[1] *$D$ on the space of probability measures on $X$ such that for any $\mu \in \mathcal{P}(X)$,*

$$\dim_T(\mu) = D(\mu, \mathbf{a})$$

*where $\mathbf{a}$ is any atomic probability measure.*

The pseudometric $D$ is called the *dimensional distance* on $\mathcal{P}(X)$. This theorem says that the transport dimension of a probability measure $\mu$ is the distance from $\mu$ to any atomic measure with respect to the dimensional distance. In other words, the dimension information of a measure tells us quantitatively how far the measure is from being an atomic measure.

## 4. Measures on a complete doubling metric space

In [24], we showed that for any probability measure $\mu$ on a compact convex subset $X$ of an Euclidean space $\mathbb{R}^m$ and any $\alpha > 1 - \frac{1}{m}$, there exists an $\alpha$-optimal transport path from $\mu$ to the Dirac measure $\delta_O$. In other words, $\mathcal{P}(X) = \mathcal{P}_\alpha(X)$ whenever $\alpha > 1 - \frac{1}{m}$. The following example shows that, in the general metric space setting, it may fail to exist such an $\alpha < 1$ with $\mathcal{P}(X) = \mathcal{P}_\alpha(X)$.

**Example 4.0.7.** *Let $X$ be the $\ell_2$ space of square-summable sequences of real numbers with the $\ell_2-$ metric:*

$$\ell_2(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

*for each $x = (x_1, x_2, \cdots, x_n, x_{n+1}, \cdots)$ and $y = (y_1, y_2, \cdots, y_n, y_{n+1}, \cdots)$ with $\sum_i (x_i)^2 < \infty$, $\sum_i (y_i)^2 < \infty$. Assume there exists an $\alpha^* < 1$ with $\mathcal{P}(X) = \mathcal{P}_{\alpha^*}(X)$. Let $K$ be any convex compact subset of $X$. Then, it is easy to see that $\mathcal{P}(K) = \mathcal{P}_{\alpha^*}(K)$. This contradicts with the fact that $\mathcal{P}_{\alpha^*}(K) \subsetneq \mathcal{P}(K)$ when $K$ is e.g. a closed unit ball in $\mathbb{R}^m \subset X$ when $\alpha^* < 1 - \frac{1}{m}$. Therefore, there does not exist an $\alpha < 1$ with $\mathcal{P}(X) = \mathcal{P}_\alpha(X)$.*

In this section, we will show that $\mathcal{P}(X) = \mathcal{P}_\alpha(X)$ (i.e. there exists an $\alpha-$optimal transport path between any two probability measures) on a compact doubling geodesic metric space $X$ whenever $\max\left\{1 - \frac{1}{m}, 0\right\} < \alpha < 1$, where $m$ is the Assouad dimension of $X$.

---

[1] A pseudometric $D$ means that it is nonnegative, symmetric, satisfies the triangle inequality, and $D(\mu, \mu) = 0$. But $D(\mu, \nu) = 0$ does not imply $\mu = \nu$.

Recall that a *space of homogeneous type* ([11]) is a quasimetric space $X$ equipped with a doubling measure $\nu$, which is a Radon measure on $X$ satisfying

$$\nu\left(B\left(x,2r\right)\right)\leq C\nu\left(B\left(x,r\right)\right)$$

for any ball $B\left(x,r\right)$ in $X$ and for some constant $C>0$. If $(X,d)$ is a metric space equipped with a doubling measure $\nu$, then the triple $(X,d,\nu)$ is called a metric measures space. Recently, many works (see [8], [17], etc) have been done on studying analysis on metric measure spaces, in particular, when the measure $\mu$ is doubling and satisfying the Poincaré inequality.

In [9] and [10], Christ introduced a decomposition of a space of homogeneous type as cubes and proved the following proposition:

**Proposition 4.0.8.** *Suppose $(X,\nu)$ is a space of homogeneous type. For any $k\in\mathbb{Z}$, there exists a set, at most countable $I_k$ and a family of subsets $Q_\theta^k\subseteq X$ with $\theta\in I_k$, such that*

(1) $\nu\left(X\setminus\cup_\theta Q_\theta^k\right)=0$, $\forall k\in\mathbb{Z}$;
(2) for any $k,l,\theta,\eta$ with $l\leq k$, either $Q_\theta^k\cap Q_\eta^l=\emptyset$ or $Q_\theta^k\subseteq Q_\eta^l$;
(3) for each $Q_\eta^{n+1}$ there exists exactly one $Q_\theta^n$ (parent of $Q_\eta^{n+1}$) such that $Q_\eta^{n+1}\subseteq Q_\theta^n$;
(4) for each $Q_\theta^n$ there exists at least one $Q_\eta^{n+1}$ (child of $Q_\theta^n$) such that $Q_\eta^{n+1}\subseteq Q_\theta^n$;

These open subsets of the kind $Q_\theta^k$ are called dyadic cubes of generation $k$ due to the analogous between them and the standard Euclidean dyadic cubes. A useful property regarding such dyadic cubes is: there exists a point $x_\theta^k\in X$ for each cube $Q_\theta^k$ such that

$$B\left(x_\theta^k,C_0\sigma^k\right)\subseteq Q_\theta^k\subseteq B\left(x_\theta^k,C_1\sigma^k\right)$$

for some constants $C_0,C_1$ and $\sigma\in(0,1)$. Moreover, for any $x_\theta^k$ and $x_\eta^k$, $d\left(x_\theta^k,x_\eta^k\right)\geq\sigma^k$.

Since $\nu$ is a doubling measure, from (1), we see that $X=\cup_\theta\bar{Q}_\theta^k$, where $\bar{Q}_\theta^k$ denotes the closure of $Q_\theta^k$. Then, it is easy to see that there exists a family of Borel subsets $\left\{B_\theta^k\right\}$ with $\theta\in I_k$ such that $Q_\theta^k\subseteq B_\theta^k\subseteq\bar{Q}_\theta^k$ with $X=\cup_\theta B_\theta^k$ for each $k$, and $\left\{B_\theta^k\right\}$ still satisfy conditions (2,3,4) above.

A very useful fact is pointed out in [17, theorem 13.3]: every complete doubling metric space $(X,d)$ has a nontrivial doubling measure on it. Thus, one may also construct a family of disjoint Borel subsets $\left\{B_\theta^k\right\}$ for $(X,d)$ as above. Now, for any bounded subset $K$ of $X$, we set $\mathcal{F}_K$ to be the collection of all dyadic cubes $B_\theta^k$ that has a nonempty intersection with $K$. It is easy to check that $\mathcal{F}_K$ is a nested collection of cubes as defined in (3.2.1). Moreover, for any $\beta>\dim_A(K)$, from the definition of Assouad dimension, we see that the cardinality $N_n$ of all dyadic cubes of generation $n$ that intersect with the set $K$ is bounded above by $C_\beta\left(\sigma^n\right)^{-\beta}$ for some constant $C_\beta>0$. Thus,

$$\dim_M\mathcal{F}\leq\lim\frac{\log C_\beta\left(\sigma^n\right)^{-\beta}}{\log\frac{1}{\sigma^n}}=\beta.$$

This shows that $\dim_M\mathcal{F}_K\leq\dim_A(K)$.

**Proposition 4.0.9.** *Suppose $(X, d)$ is a complete geodesic doubling metric space. If $\mu$ is a probability measure concentrated on a bounded subset $K$ of $X$, then*

$$\dim_M(\mu) \leq \dim_A(K).$$

*If in addition, $\mu$ is Ahlfors regular, then*

$$\dim_U(\mu) \leq \dim_A(K).$$

*Proof.* Since $\mu$ is concentrated on $K$, we have $\mu$ is concentrated on the associated nested collection $\mathcal{F}_K$. Thus,

$$\dim_M(\mu) \leq \dim_M \mathcal{F}_K \leq \dim_A(K).$$

When $\mu$ is Ahlfors regular, $\mu$ is evenly concentrated on $\mathcal{F}_K$, thus $\dim_U(\mu) \leq \dim_A(K)$. □

In particular, we have

**Theorem 4.0.10.** *Suppose $X$ is a complete geodesic doubling metric space with Assouad dimension $m$, and $\mu$ is any probability measure on $X$ with a compact support. Let $1 - \frac{1}{m} < \alpha < 1$. Then,*

(1) *$\mu \in \mathcal{D}_\alpha(X)$ if $\alpha > 0$. In particular, if in addition $X$ is compact, then $\mathcal{D}_\alpha(X) = \mathcal{P}_\alpha(X) = \mathcal{P}(X)$.*
(2) *$\mu \in \mathcal{D}_\alpha(X)$ if $\mu$ is Ahlfors regular.*

*Proof.* Let $K$ be the support of $\mu$. Then, $\dim_M(\mu) \leq \dim_A(K) \leq \dim_A(X) = m$. By theorem 3.2.5, $\dim_T(\mu) \leq \max\{1, \dim_M(\mu)\} \leq \max\{1, m\}$. Therefore, for any $\max\left\{1 - \frac{1}{m}, 0\right\} < \alpha < 1$, we have $\frac{1}{1-\alpha} > \max\{1, m\} \geq \dim_T(\mu)$, and thus $\mu \in \mathcal{D}_\alpha(X)$. When $\mu$ is Ahlfors regular on $X$, we have $\dim_T(\mu) \leq \dim_U(\mu) \leq \dim_A(K) \leq m$. Thus, if $\frac{1}{1-\alpha} > m$, then $\mu \in \mathcal{D}_\alpha(X)$. □

Thus, by proposition 3.1.3, we have

**Corollary 4.0.11.** *Suppose $X$ is a compact geodesic doubling metric space with Assouad dimension $m$. Then, the space $(\mathcal{P}(X), d_\alpha)$ of probability measures on $X$ is a complete geodesic metric space whenever $\max\left\{1 - \frac{1}{m}, 0\right\} < \alpha < 1$. In other words, there exists an $\alpha-$optimal transport path between any two probability measures on $X$.*

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