RAMIFIED OPTIMAL TRANSPORTATION WITH PAYOFF ON THE BOUNDARY

Qinglan Xia

Department of Mathematics
University of California at Davis
Davis, CA 95616, USA

Shaofeng Xu

School of Economics
Renmin University of China
Beijing, 100872, China

Abstract. This paper studies a variant of ramified/branched optimal transportation problems. Given the distributions of production capacities and market sizes, a firm looks for an allocation of productions over factories, a distribution of sales across markets, and a transport path that delivers the product to maximize its profit. Mathematically, given any two measures μ and ν on X, and a payoff function h, the planner wants to minimize $\mathbf{M}_\alpha(T) - \int_X h d(\partial T)$ among all transport paths T from $\tilde{\mu}$ to $\tilde{\nu}$ with $\tilde{\mu} \leq \mu$ and $\tilde{\nu} \leq \nu$, where $\mathbf{M}_\alpha$ is the standard cost functional used in ramified transportation. After proving the existence result, we provide a characterization of the boundary measures of the optimal solution. They turn out to be the original measures restricted on some Borel subsets up to a Delta mass on each connected component. Our analysis further finds that as the boundary payoff increases, the corresponding solution of the current problem converges to an optimal transport path, which is the solution of the standard ramified transportation.

1. Introduction

1.1. The ROTPB problem. Transportation is an important force shaping the spatial distribution of economic activities. Consider a firm that produces and sells a product in various regions. Given the locations and capacities of these regions and the associated production costs and sale prices of the product, the firm looks for a distribution of productions over factories, a distribution of sales across markets, and a transport path that delivers the product to maximize its profit. The firm's optimal plan over productions and sales depends on its choice of transport path, and vice versa. The interactions between location and transport choices, however, often render these problems difficult to analyze.

In this paper, we address some of these interactions in the framework of the ramified optimal transportation. More precisely, we consider the following resource allocation problem: let μ and ν be two Radon measures on a convex compact subset X of the Euclidean space $\mathbb{R}^m$, $\mathbf{M}_\alpha$ be the standard cost functional used in ramified transportation [30] for $\alpha \in [0, 1)$, and h be a continuous function on the support of the signed measure $\nu - \mu$. We consider the problem:

2010 *Mathematics Subject Classification.* 49Q20, 49Q10, 90B10.

Key words and phrases. branch transportation, economies of scale, optimal transport path, branching structure, rectifiable current, partial transportation, firm location.

Problem (ROTPB(μ,ν))**.** *Minimize*

$$\mathbf{E}^h_\alpha(T) := \mathbf{M}_\alpha(T) - \int_X h d(\partial T) \tag{1.1}$$

among all rectifiable 1-current T with $\partial T \preceq \nu - \mu$ as signed measures.[1]

In the context of the above example, measures μ and ν represent, respectively, the distributions of production capacities and market sizes. The function h represents the payoff associated with moving mass from μ to ν, and it captures the production cost of the product over μ and its sale price over ν. The firm aims to maximize its profit defined as sale revenues minus costs involved in transportation and production. We call this problem as *ramified optimal transportation with payoff on the boundary* (ROTPB).

1.2. **Background.** This paper is related to the literature of optimal transport problems that concerns efficient mass transportation. These problems are studied early on by Monge and Kantorovich, and has been extensively analyzed in recent years. Classical references can be found in the books [23, 24] by Villani, [19] by Santambrogio, and the user's guide [1] by Ambrosio and Gigli. Our paper is most closely related to the ramified optimal transportation (ROT) (also called branched transportation) literature, which models branching transport structures thanks to the efficiency in group transportation. In contrast to the Monge-Kantorovich problems where the transportation cost is solely determined by a transport map, the cost in ramified transport problems is determined by the actual transport path. The Eulerian formulation of the ROT problem is proposed by the first author in [26], with related motivations, frameworks and applications surveyed in [30]. An equivalent Lagrangian formulation of the problem is established by Maddalena, Morel, and Solimini in [14]. One may refer to [2] for detailed discussions of the research in this direction. Some interesting recent developments on ROT can be found for example in [4, 6, 8, 13, 18, 20].

Our paper differentiates itself from the existing ROT literature in two main regards. First, in the literature both measures μ and ν are fixed and of equal mass, and the problem only involves finding a cost-minimizing transport path. By contrast, the planner in this paper optimizes over all possible combinations of $(\tilde{\mu}, \tilde{\nu})$ with $\tilde{\mu} \leq \mu$, $\tilde{\nu} \leq \nu$ and $||\tilde{\mu}|| = ||\tilde{\nu}||$. Similar kind of optimal partial mass transport has been studied for instance by Caffarelli and McCann [5] and also Figalli [12] for the scenario of Monge-Kantorovich problems with a particular attention to the quadratic cost. Second, the planner faces a reward for relocating mass at the boundary, and thus the solution relies on the payoff function h. This element has been absent in the literature up to our best knowledge.

1.3. **Main results.** Our main results include three parts: the existence theorem (Theorem 3.1), the characterization theorems (Theorem 1.1, Theorem 4.19), and the approximation theorem (Theorem 5.4).

We first prove the existence of an $\mathbf{E}^h_\alpha$-minimizer T^* for the ROTPB(μ,ν) problem in Theorem 3.1. This optimal solution T^* is an α-optimal transport path of finite $\mathbf{M}_\alpha$ cost from μ^* to ν^* for some measures $\mu^* \leq \mu$ and $\nu^* \leq \nu$. As such, T^* automatically inherits many nice geometric properties of optimal transport paths as

[1] The notation $\preceq$ is introduced in (2.4).

described previously in [30]. We next characterize the optimal allocation measures μ^* and ν^*. In the case that they are finitely supported atomic measures, we show

Theorem 1.1. *Suppose μ and ν are two atomic measures on X with finite supports, $0 < \alpha < 1$, and $T^* \in Path(\mu^*, \nu^*)$ is a solution to the ROTPB(μ, ν) problem. Let $\{K_k : k = 1, 2, \cdots, \ell\}$ be the set of the connected components of the support of T^*. Then, for each $k = 1, 2, \cdots, \ell$, it holds that*

$$\mu^* \llcorner K_k = \mu \llcorner K_k - m_k \delta_{p_k} \text{ and } \nu^* \llcorner K_k = \nu \llcorner K_k - n_k \delta_{q_k}, \tag{1.2}$$

for some points $p_k \in K_k \cap spt(\mu^*)$ and $q_k \in K_k \cap spt(\nu^*)$ with

$$m_k := \max\{\mu(K_k) - \nu(K_k), 0\} \text{ and } n_k := \max\{\nu(K_k) - \mu(K_k), 0\}.$$

As a result, we have the decomposition

$$\mu^* = \mu \llcorner A - \mathbf{a} \text{ and } \nu^* = \nu \llcorner B - \mathbf{b}, \tag{1.3}$$

for $A = spt(\mu^*), B = spt(\nu^*)$, and

$$\mathbf{a} = \sum_{k=1}^{\ell} m_k \delta_{p_k}, \ \mathbf{b} = \sum_{k=1}^{\ell} n_k \delta_{q_k}. \tag{1.4}$$

Note that in (1.2), at least one of m_k and n_k is zero for each k. It says that on each connected component K_k, all existing resources in the optimal allocation source measure μ^* will be used up, and all demands in the optimal allocation destination measure ν^* will be met with at most one exception at either a source node or a destination node. There are three scenarios:

- In the balanced case with $\mu(K_k) = \nu(K_k)$, then
$$\mu^* \llcorner K_k = \mu \llcorner K_k \text{ and } \nu^* \llcorner K_k = \nu \llcorner K_k.$$
All source and destination nodes are fully in use.
- In the over-supply case with $\mu(K_k) > \nu(K_k)$, then
$$\mu^* \llcorner K_k = \mu \llcorner K_k - (\mu(K_k) - \nu(K_k))\delta_{p_k} \text{ and } \nu^* \llcorner K_k = \nu \llcorner K_k.$$
All source nodes excluding the one at p_k and all destination nodes are fully in use.
- In the over-demand case with $\mu(K_k) < \nu(K_k)$, then
$$\mu^* \llcorner K_k = \mu \llcorner K_k \text{ and } \nu^* \llcorner K_k = \nu \llcorner K_k - (\nu(K_k) - \mu(K_k))\delta_{q_k}.$$
All source nodes and all destination nodes except for the one at q_k are fully in use.

In Theorem 4.19, we extend the results of Theorem 1.1 to general cases.

The third part of the main results highlights an important implication of the current study for solving an optimal transport path. We consider a version of ROTPB problems, where the measures μ and ν are disjointly supported and the payoff function h_c takes a constant value $2c$ on the support of ν, and vanishes on the support of μ. In the early example, the parameter c represents (half of) the gap between the sale price and the production cost, and it effectively determines the payoff from relocating a unit of mass. Intuitively, the larger the payoff, the more incentive the planner has to relocate the mass from sources to destinations. When the payoff is sufficiently large, it is in the best interest of the planner to move as much mass as possible. We prove in Theorem 5.4 that an optimal transport path, which solves the standard ramified transportation problem, can be obtained as a limit of the solutions to a sequence of ROTPB problems associated with a series of increasing boundary payoff.

2. Preliminaries

2.1. **Basic notations in geometric measure theory.** We first recall some terminology about rectifiable currents as in [11] or [21].

Let $\Omega \subseteq \mathbb{R}^m$ be an open domain and for any integer $k \geq 0$ let $\mathcal{D}^k(\Omega)$ be the set of all C^∞ differential k-forms in Ω with compact support with the usual Fréchet topology [11]. A *k-dimensional current S in Ω* is a continuous linear functional on $\mathcal{D}^k(\Omega)$. Denote $\mathcal{D}_k(\Omega)$ as the set of all k-dimensional currents in Ω. The *mass* of a current $T \in \mathcal{D}_k(\Omega)$ is defined by

$$\mathbf{M}(T) := \sup\{|T(\omega)| : ||\omega|| \leq 1, \omega \in \mathcal{D}^k(\Omega)\}.$$

Motivated by the Stokes' theorem, the *boundary* of a current $S \in \mathcal{D}_k(\Omega)$ for $k \geq 1$ is the current ∂S in $\mathcal{D}_{k-1}(\Omega)$ defined by

$$\partial S\,(\omega) := S\,(d\omega)$$

for any $\omega \in \mathcal{D}^{k-1}(\Omega)$. A current $T \in \mathcal{D}_k(\Omega)$ is called *normal* if $\mathbf{M}(T) + \mathbf{M}(\partial T) < +\infty$. A sequence of currents $\{S_i\}$ in $\mathcal{D}_k(\Omega)$ is said to be weakly convergent to another current $S \in \mathcal{D}_k(\Omega)$, denoted by $S_i \rightharpoonup S$, if

$$S_i(\omega) \to S(\omega)$$

for any $\omega \in \mathcal{D}^k(\Omega)$.

As in [21], a subset $M \subseteq \mathbb{R}^m$ is called (countably) *$k-$rectifiable* if $M = \bigcup_{i=0}^{\infty} M_i$, where $\mathcal{H}^k\,(M_0) = 0$ under the $k-$dimensional Hausdorff measure $\mathcal{H}^k$ and each M_i, for $i = 1, 2, \cdots$, is a subset of an $k-$dimensional C^1 submanifold in $\mathbb{R}^m$. A *rectifiable k-current* S is a $k-$dimensional current coming from an oriented $k-$rectifiable set with multiplicities. More precisely, $S \in \mathcal{D}_k(\Omega)$ is a *rectifiable k-current* if it can be expressed as

$$S\,(\omega) = \int_M \langle \omega\,(x)\,, \xi\,(x)\rangle \theta\,(x)\,d\mathcal{H}^k\,(x)\,,\ \forall \omega \in \mathcal{D}^k(\Omega)$$

where

- M is an $\mathcal{H}^k$ measurable and $k-$rectifiable subset of Ω.
- θ is an $\mathcal{H}^k \llcorner M$ integrable positive function and is called the multiplicity function of S.
- ξ is an $\mathcal{H}^k$-measurable k-vector valued function on M such that at $\mathcal{H}^k$-a.e. $x \in M$, $\xi(x) = \tau_1 \wedge \cdots \wedge \tau_k$, where $\{\tau_1, \cdots, \tau_k\}$ is an orthonormal basis for the approximate tangent space $T_x M$. ξ is called the orientation of S.

The rectifiable current S described as above is often denoted by

$$S = \underline{\underline{\tau}}(M, \theta, \xi).$$

In this case, the mass of S is expressed as

$$\mathbf{M}(S) = \int_M \theta(x) d\mathcal{H}^k(x).$$

Since θ is $\mathcal{H}^k \llcorner M$ integrable, each rectifiable current S here is assumed to have finite mass.

2.2. **Basic notations in ramified optimal transportation.** Let X be a convex compact subset of the Euclidean space $\mathbb{R}^m$. The ramified optimal transport problem (also called branched optimal transportation problem in the literature) consists in the following Plateau-type problem:

Problem (ROT)**.** *Given two (positive) measures μ^+ and μ^- on X of equal mass and $\alpha < 1$, minimize*

$$\mathbf{M}_\alpha(T) := \int_M \theta^\alpha d\mathcal{H}^1$$

among all rectifiable 1-current $T = \underline{\underline{\tau}}(M, \theta, \xi)$ in $\mathbb{R}^m$ with $\partial T = \mu^- - \mu^+$ in the sense of currents.

Each rectifiable 1-current $T = \underline{\underline{\tau}}(M, \theta, \xi)$ such that $\partial T = \mu^- - \mu^+$ is called a *transport path* from μ^+ to μ^-. Let

$$Path(\mu^+, \mu^-) = \{T \text{ is a rectifiable 1-current} : \partial T = \mu^- - \mu^+\}$$

be the collection of all transport paths from μ^+ to μ^-.

For the ROT problem, the existence of an $\mathbf{M}_\alpha$-minimizer in $Path(\mu^+, \mu^-)$ is shown in [26]. Each $\mathbf{M}_\alpha$-minimizer is called an *α-optimal transport path.* One shall note that for some combinations of exponent α and pair of measures $\mu^\pm$, it is possible the $\mathbf{M}_\alpha$ cost of any transport path $T \in Path(\mu^+, \mu^-)$ is infinite, and thus the existence of a solution to the ROT problem is trivial in that case.

When $1 - \frac{1}{m} < \alpha < 1$, it is shown in [26] that for any pair of measures $\mu^\pm$ of equal (finite) mass there exists an α-optimal transport path of finite $\mathbf{M}_\alpha$-cost from μ^+ to μ^-. Moreover, a distance is defined by setting

$$d_\alpha(\mu^+, \mu^-) := \min\{\mathbf{M}_\alpha(T) : \partial T = \mu^- - \mu^+\} \tag{2.1}$$

between μ^+ and μ^-. This distance metrizes the weak convergence of measures. By [26, Theorem 3.1], it holds that

$$d_\alpha(\mu^+, \mu^-) \leq C_{m,\alpha} diam(X) ||\mu^+||^\alpha, \tag{2.2}$$

where the constant

$$C_{m,\alpha} = \frac{\sqrt{m}}{2(2^{1-m(1-\alpha)} - 1)}. \tag{2.3}$$

In general, the existence of finite cost α-optimal transport path between μ^+ and μ^- depends on the dimensional information of the measures (see [9], [28]). In [28], the d_α-metric is defined on the space of finite atomic probability measures for any real number $\alpha < 1$.

The following notations are also employed in the analysis:

- Let μ and ν be two (positive) measures on X. We say $\mu \leq \nu$ if $\nu - \mu$ is still a (positive) measure on X. In this case, we say that μ is *feasible* relative to ν.
- Let $\mu_1 = \mu_1^+ - \mu_1^-$ and $\mu_2 = \mu_2^+ - \mu_2^-$ be the Jordan decompositions of two signed measures. We say

$$\mu_1 \preceq \mu_2 \tag{2.4}$$

  if $\mu_1^+ \leq \mu_2^+$ and $\mu_1^- \leq \mu_2^-$.
- For any signed measure $\bar{\mu}$, let $spt(\bar{\mu})$ denote its support, $||\bar{\mu}||$ denote its total variation, and $|\bar{\mu}|$ denote its total variation measure.

For each rectifiable normal 1-current T, its boundary ∂T can be viewed as a signed measure.

3. The ROTPB problem

This section analyzes the spatial resource allocation problem ROTPB(μ,ν) as stated in the introduction. For simplicity, when both the parameter α and the function h are clear from the context, we simply write $\mathbf{E}^h_\alpha$ given in (1.1) as $\mathbf{E}$.

The ROTPB(μ,ν) problem is indeed a double-minimizing problem

$$\min\left\{\min\left\{\mathbf{E}^h_\alpha(T):\partial T=\tilde\nu-\tilde\mu\right\}:\ \tilde\mu\le\mu,\tilde\nu\le\nu \text{ with } ||\tilde\mu||=||\tilde\nu||\right\}.$$

For each fixed $\tilde\mu\le\mu,\tilde\nu\le\nu$ with $||\tilde\mu||=||\tilde\nu||$, the inner minimization problem

$$\min\left\{\mathbf{E}^h_\alpha(T)=\mathbf{M}_\alpha(T)-\int_X hd(\partial T):\partial T=\tilde\nu-\tilde\mu\right\}$$

can be re-written as

$$\min\left\{\mathbf{M}_\alpha(T):\ \partial T=\tilde\nu-\tilde\mu\right\}-\int_X hd\tilde\nu+\int_X hd\tilde\mu. \tag{3.1}$$

Thus, under the d_α metric as given in (2.1), the ROTPB(μ,ν) problem can also be expressed as: minimize

$$\varepsilon(\tilde\mu,\tilde\nu):=d_\alpha(\tilde\mu,\tilde\nu)-\int_X hd\tilde\nu+\int_X hd\tilde\mu \tag{3.2}$$

among all feasible measures $\tilde\mu\le\mu$ and $\tilde\nu\le\nu$ with $||\tilde\mu||=||\tilde\nu||$.

From the perspective of the firm in the example given in Introduction, the ROTPB(μ,ν) problem can be interpreted as follows. Given the distributions of production capacities (μ) and market sizes (ν), the firm chooses an operation plan $\tilde\mu\le\mu$ and $\tilde\nu\le\nu$ to minimize the total costs incurred in production ($\int_X hd\tilde\mu$) and transportation ($d_\alpha(\tilde\mu,\tilde\nu)$) net the sale revenue ($\int_X hd\tilde\nu$).

We now state the existence theorem for the ROTPB(μ,ν) problem.

Theorem 3.1 (Existence)**.** *Let μ and ν be two Radon measures on X, $0\le\alpha<1$ and h be a continuous function on the support of the signed measure $\nu-\mu$. Then there exists a rectifiable 1-current T^* of finite $\mathbf{M}_\alpha$ cost that minimizes*

$$\mathbf{E}^h_\alpha(T):=\mathbf{M}_\alpha(T)-\int_X hd(\partial T)$$

among all rectifiable 1-current T with $\partial T\preceq\nu-\mu$ as signed measures.[2]

Proof. We prove this result by using the direct method of calculus of variations. Let $\{T_i\}$ be any $\mathbf{E}$-minimizing sequence of rectifiable 1-currents. That is,

$$\lim_{i\to\infty}\mathbf{E}(T_i)=\inf\{\mathbf{E}(T):\partial T\preceq\nu-\mu\},$$

and $\partial T_i\preceq\nu-\mu$ for each i. With no loss of generality, we may assume $\mathbf{E}(T_i)\le\mathbf{E}(0)=0$. Thus,

$$\mathbf{M}_\alpha(T_i)=\mathbf{E}(T_i)+\int_X hd(\partial T_i)\le\int_X hd(\partial T_i)\le\int_X |h|d(|\nu-\mu|)<\infty \tag{3.3}$$

[2]When no ambiguity occurs, we also write $\mathbf{E}^h_\alpha(T)$ as $\mathbf{E}(T)$ for notational simplicity.

as h is continuous on the compact set $spt(\nu-\mu)$, the support of $\nu-\mu$. Now suppose $T_i \in Path(\tilde{\mu}_i, \tilde{\nu}_i)$. Since $\mathbf{M}_\alpha(T_i)$ is finite, there exists an $\mathbf{M}_\alpha$-minimizer with finite cost for the minimization problem

$$\min\left\{\mathbf{M}_\alpha(T):\ \partial T = \partial T_i\right\}.$$

Note this minimizer is also an $\mathbf{E}$-minimizer for the inner minimization problem (3.1) with $\tilde{\mu} = \tilde{\mu}_i$ and $\tilde{\nu} = \tilde{\nu}_i$. Without loss of generality, we may assume that T_i is such a minimizer, which is an α-optimal transport path of finite cost. By (3.3), the sequence $\{\mathbf{M}_\alpha(T_i)\}$ is bounded. Employing Lemma 3.2 below shows that the sequence $\{\mathbf{M}(T_i)\}$ is also bounded. As a result, we get a sequence of normal 1-currents $\{T_i\}$ with equi-bounded mass and boundary mass. By the compactness of normal 1-currents ([11]), and taking a subsequence if necessary, we may assume that the sequence $\{T_i\}$ converges to a normal 1-current T^* with respect to flat convergence. Since $\mathbf{M}_\alpha$ is lower semi-continuous with respect to flat convergence ([7, 15]), we have

$$\mathbf{M}_\alpha(T^*) \leq \liminf_{i\to\infty} \mathbf{M}_\alpha(T_i) < \infty.$$

According to the rectifiability theorem (e.g., Theorem 2.7 in [27], Theorem 7.1 in [25]), finite mass and finite $\mathbf{M}_\alpha$ mass together imply that T^* is also 1-rectifiable. Since $\{T_i\}$ converges to T^* in flat convergence, the sequence $\{\partial T_i\}$, which has bounded mass, is weak-* convergent to ∂T^* as signed measures.

Since h is continuous on the support $spt(\nu-\mu)$, $spt(\partial T_i) \subseteq spt(\nu-\mu)$, and ∂T_i is weak-$*$ convergent to ∂T^*, we have

$$\int_X h d(\partial T) = \lim_{i\to\infty} \int_X h d(\partial T_i).$$

As a result,

$$\mathbf{E}(T^*) = \mathbf{M}_\alpha(T^*) - \int_X h d(\partial T^*) \leq \liminf_{i\to\infty}\{\mathbf{M}_\alpha(T_i) - \int_X h d(\partial T_i)\} = \lim_{i\to\infty} \mathbf{E}(T_i).$$

When each $\partial T_i \preceq \nu-\mu$, its limit $\partial T^* \preceq \nu-\mu$ holds as well. This shows that T^* is a solution to the ROTPB(μ,ν) problem. □

The proof of the theorem takes advantage of the following lemma:

Lemma 3.2. *Suppose T is an α-optimal transport path with $\mathbf{M}_\alpha(T) < \infty$, then*

$$\mathbf{M}(T) \leq \left(\frac{\mathbf{M}(\partial T)}{2}\right)^{1-\alpha} \mathbf{M}_\alpha(T). \tag{3.4}$$

Proof. Suppose $T = \underline{\underline{\tau}}(M,\theta,\xi)$ is an α-optimal transport path from μ^+ to μ^-, where $\partial T = \mu^- - \mu^+$ is the Jordan decomposition of ∂T as a signed measure. Since T is an α-optimal transport path of finite cost, it follows (from (4.13) for instance) that $\theta(x) \leq \mu^+(X) = \frac{1}{2}\mathbf{M}(\partial T)$ for $\mathcal{H}^1$- a.e. $x \in M$. Thus,

$$\begin{aligned}
\mathbf{M}(T) &= \int_M \theta(x) d\mathcal{H}^1(x) = \int_M \theta(x)^\alpha \theta(x)^{1-\alpha} d\mathcal{H}^1(x) \\
&\leq \int_M \theta(x)^\alpha (\mu^+(X))^{1-\alpha} d\mathcal{H}^1(x) = \left(\frac{\mathbf{M}(\partial T)}{2}\right)^{1-\alpha} \mathbf{M}_\alpha(T).
\end{aligned}$$

□

Remark 3.3. By the Jordan decomposition theorem, for any signed measure $\bar{\mu}$, there exist unique positive measures μ^+ and μ^- such that $\bar{\mu} = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. Thus, without loss of generality, we may assume that μ and ν are mutually singular when studying the ROTPB(μ, ν) problem.

In the rest of the analysis, we assume that μ and ν are mutually singular, and h is continuous on the support of $\nu - \mu$.

Proposition 3.4. *If* $\min\{h(x) : x \in spt(\mu)\} \geq \max\{h(x) : x \in spt(\nu)\}$, *then* $T^* = 0$ *is the unique solution to the ROTPB*(μ, ν) *problem.*

Proof. Suppose T^* is a solution to the ROTPB(μ, ν) problem with $\partial T^* = \tilde{\nu} - \tilde{\mu}$. Since $\tilde{\mu}$ and $\tilde{\nu}$ have the same mass,

$$\begin{aligned} & \mathbf{E}(T^*) = \mathbf{M}_\alpha(T^*) - \int_X h d\tilde{\nu} + \int_X h d\tilde{\mu} \\ \geq \quad & \mathbf{M}_\alpha(T^*) - \int_X \max\{h(x) : x \in spt(\nu)\} d\tilde{\nu} + \int_X \min\{h(x) : x \in spt(\mu)\} d\tilde{\mu} \\ = \quad & \mathbf{M}_\alpha(T^*) + \left(\min\{h(x) : x \in spt(\mu)\} - \max\{h(x) : x \in spt(\nu)\}\right) \tilde{\mu}(X) \geq 0 \end{aligned}$$

where the equality holds if and only if $T^* = 0$. □

The condition in the proposition implies that it is impossible to obtain positive net payoff from relocating mass, needless to mention the incurred transportation cost. It is thus in the best interest of the planner to not move any mass at all. This proposition illustrates the role of boundary payoff played in the problem, which we will further examine in Section 5.

Suppose that the ROTPB(μ, ν) problem has a solution $T^* \in Path(\mu^*, \nu^*)$. Then, T^* is inherently an α-optimal transport path in $Path(\mu^*, \nu^*)$ with finite $\mathbf{M}_\alpha$ cost. Thus, T^* itself exhibits some nice regularity properties (acyclic, uniform upper-bound on the degree of vertices, uniform lower-bound on the angles between edges at each vertex, boundary and interior regularity, etc) as stated in [30] for being $\mathbf{M}_\alpha$ optimal.

4. Properties of the optimal allocation measures

This section is devoted to characterizing the optimal allocation measures μ^* and ν^*. Let

$$\mathcal{E}(\mu, \nu) := \min \left\{ d_\alpha(\tilde{\mu}, \tilde{\nu}) - \int_X h d\tilde{\nu} + \int_X h d\tilde{\mu} \;\middle|\; \tilde{\mu} \leq \mu \text{ and } \tilde{\nu} \leq \nu \text{ with } ||\tilde{\mu}|| = ||\tilde{\nu}|| \right\}$$

denote the minimum value of the ROTPB(μ, ν) problem. We first observe some basic properties of $\mathcal{E}$.

Proposition 4.1. *Suppose* $0 \leq \tilde{\mu} \leq \mu$ *and* $0 \leq \tilde{\nu} \leq \nu$. *Then,*

$$0 \geq \mathcal{E}(\tilde{\mu}, \tilde{\nu}) \geq \mathcal{E}(\mu, \nu). \tag{4.1}$$

In particular, if $\mathcal{E}(\mu, \nu) = 0$, *then for all* $(\tilde{\mu}, \tilde{\nu})$ *with* $0 \leq \tilde{\mu} \leq \mu$ *and* $0 \leq \tilde{\nu} \leq \nu$, *it holds that* $\mathcal{E}(\tilde{\mu}, \tilde{\nu}) = 0$.

Proof. The results follow from the definition of $\mathcal{E}(\mu, \nu)$. □

Here, $\mathcal{E}(\mu, \nu)$ is non-positive and monotonic since $-\mathcal{E}(\mu, \nu)$ represents the overall possible profit generated for the planner from the pair (μ, ν). When $\mathcal{E}(\mu, \nu) = 0$, there is no way to generate a non-zero $\mathcal{E}(\tilde{\mu}, \tilde{\nu})$ from some part $(\tilde{\mu}, \tilde{\nu})$ of (μ, ν).

Proposition 4.2. *Suppose for each $i = 1, 2$, T_i^* is a solution to the ROTPB(μ_i, ν_i) problem, and T_{1+2}^* is a solution to the ROTPB($\mu_1 + \mu_2, \nu_1 + \nu_2$) problem, then*

$$\mathbf{E}(T_{1+2}^*) \leq \mathbf{E}(T_1^*) + \mathbf{E}(T_2^*), \tag{4.2}$$

and hence

$$\mathcal{E}(\mu_1 + \mu_2, \nu_1 + \nu_2) \leq \mathcal{E}(\mu_1, \nu_1) + \mathcal{E}(\mu_2, \nu_2). \tag{4.3}$$

Proof. By assumption, for each $i = 1, 2$, $T_i^* \in Path(\mu_i^*, \nu_i^*)$ with $\mu_i^* \leq \mu_i$ and $\nu_i^* \leq \nu_i$. Then, $T_1^* + T_2^* \in Path(\mu_1^* + \mu_2^*, \nu_1^* + \nu_2^*)$ with $\mu_1^* + \mu_2^* \leq \mu_1 + \mu_2$ and $\nu_1^* + \nu_2^* \leq \nu_1 + \nu_2$. Since T_{1+2}^* is a solution to the ROTPB($\mu_1 + \mu_2, \nu_1 + \nu_2$) problem, we have

$$\begin{aligned} \mathbf{E}(T_{1+2}^*) &\leq \mathbf{E}(T_1^* + T_2^*) = \mathbf{M}_\alpha(T_1^* + T_2^*) - \int_X hd(\partial T_1^* + \partial T_2^*) \\ &\leq \mathbf{M}_\alpha(T_1^*) + \mathbf{M}_\alpha(T_2^*) - \int_X hd(\partial T_1^*) - \int_X hd(\partial T_2^*) \\ &= \mathbf{E}(T_1^*) + \mathbf{E}(T_2^*). \end{aligned}$$

□

Remark 4.3. Following from the above proof, if the equality in (4.2) holds, then

$$\mathbf{M}_\alpha(T_1^* + T_2^*) = \mathbf{M}_\alpha(T_1^*) + \mathbf{M}_\alpha(T_2^*).$$

Suppose $T_i = \underline{\underline{\tau}}(M_i, \theta_i, \xi_i)$ with $\theta_i(x) > 0$ for $\mathcal{H}^1$-a.e. $x \in M_i$, $i = 1, 2$. Since $\alpha < 1$,

$$\begin{aligned} &\mathbf{M}_\alpha(T_1^* + T_2^*) - \mathbf{M}_\alpha(T_1^*) - \mathbf{M}_\alpha(T_2^*) \\ &\leq \int_{M_1 \cap M_2} (\theta_1(x) + \theta_2(x))^\alpha - \theta_1(x)^\alpha - \theta_2(x)^\alpha d\mathcal{H}^1(x) \leq 0, \end{aligned}$$

where the equalities hold only if $\mathcal{H}^1(M_1 \cap M_2) = 0$.

We now give a necessary condition on the solution to the ROTPB(μ, ν) problem.

Corollary 4.4. *Suppose that $T^* \in Path(\mu^*, \nu^*)$ is a solution to the ROTPB(μ, ν) problem. Then $\mathcal{E}(\mu - \mu^*, \nu - \nu^*) = 0$.*

Proof. Since $T^* \in Path(\mu^*, \nu^*)$ is a solution to the ROTPB(μ, ν) problem, $\mathcal{E}(\mu, \nu) = \mathcal{E}(\mu^*, \nu^*)$. By (4.1) and (4.3),

$$0 \geq \mathcal{E}(\mu - \mu^*, \nu - \nu^*) \geq \mathcal{E}(\mu, \nu) - \mathcal{E}(\mu^*, \nu^*) = 0.$$

Therefore, $\mathcal{E}(\mu - \mu^*, \nu - \nu^*) = 0$. □

The corollary says that the mass left unmoved by the solution would not generate further gains for the planner.

Proposition 4.5. *Suppose that the ROTPB(μ, ν) problem has a non-zero solution $T^* \in Path(\mu^*, \nu^*)$ and $\alpha < 1$. Then there exists no real number $\sigma > 1$ such that $\sigma\mu^* \leq \mu$ and $\sigma\nu^* \leq \nu$.*

Proof. Otherwise, assume that there exists a real number $\sigma > 1$ such that $\sigma\mu^* \leq \mu$ and $\sigma\nu^* \leq \nu$. We consider the function

$$g(\lambda) := \varepsilon(\lambda\mu^*, \lambda\nu^*) = \lambda^\alpha d_\alpha(\mu^*, \nu^*) - \lambda \int_X |h| d\nu^* + \lambda \int_X |h| d\mu^*$$

for $\lambda \in [0, \sigma]$, where $\varepsilon(\cdot, \cdot)$ is defined in (3.2). Since $T^* \in Path(\mu^*, \nu^*)$ is a non-zero solution to the ROTPB(μ, ν) problem, $d_\alpha(\mu^*, \nu^*) = \mathbf{M}_\alpha(T^*) > 0$. Thus, given $\alpha < 1$,

$$\begin{aligned} g'(1) &= \alpha d_\alpha(\mu^*, \nu^*) - \int_X |h| d\nu^* + \int_X |h| d\mu^* \\ &< d_\alpha(\mu^*, \nu^*) - \int_X |h| d\nu^* + \int_X |h| d\mu^* \\ &= \mathbf{E}(T^*) \leq \mathbf{E}(0) = 0. \end{aligned}$$

As a result, there exists a $\lambda^* \in (1, \sigma)$ such that $g(\lambda^*) < g(1)$. Because $\sigma\mu^* \leq \mu$ and $\sigma\nu^* \leq \nu$, we also have $\lambda^*\mu^* \leq \sigma\mu^* \leq \mu$ and $\lambda^*\nu^* \leq \sigma\nu^* \leq \nu$. Hence $\varepsilon(\lambda^*\mu^*, \lambda^*\nu^*) = g(\lambda^*) < g(1) = \varepsilon(\mu^*, \nu^*)$, which contradicts with T^* being a solution to the ROTPB(μ, ν) problem. $\square$

At a solution to the ROTPB(μ, ν) problem, the planner might only move out a portion of the mass held at one source or ship in mass less than registered at a single destination. However, the above proposition shows that this can not happen at all the involved sources and destinations. Otherwise, an improvement can be achieved by a proportional increase of the transported mass at these locations. This is because the resulting marginal payoff from moving more mass outweighs the marginal transportation cost thanks to the transport economy of scale when $\alpha < 1$.

The remainder of this section focuses on characterizing the optimal allocation measures μ^* and ν^*, with the main result stated in Theorem 4.19. We first set up some technical bases.

Definition 4.6. Let $T = \underline{\underline{\tau}}(M, \theta, \xi)$ and $S = \underline{\underline{\tau}}(N, \rho, \eta)$ be two rectifiable 1-currents. We say S is on T if $\mathcal{H}^1(N \setminus M) = 0$, and $\rho(x) \leq \theta(x)$ for $\mathcal{H}^1$ almost all $x \in N$.

Note that when $S = \underline{\underline{\tau}}(N, \rho, \eta)$ is on $T = \underline{\underline{\tau}}(M, \theta, \xi)$, then $\xi(x) = \pm\eta(x)$ for $\mathcal{H}^1$ almost all $x \in N$, since two rectifiable sets have the same tangent a.e. on their intersection.

Theorem 4.7. *Suppose that $T^* \in Path(\mu^*, \nu^*)$ is a solution to the ROTPB(μ, ν) problem, and $0 < \alpha < 1$. If there exists a rectifiable 1-current S on T^* with*

$$\partial(T^* + S) \preceq \nu - \mu \text{ and } \partial(T^* - S) \preceq \nu - \mu,$$

then $S = 0$.

Proof. Assume that $S = \underline{\underline{\tau}}(N, \rho, \eta)$ is a non-zero rectifiable 1-current on $T^* = \underline{\underline{\tau}}(W, \theta, \xi)$. One may assume that $N = W$ by extending $\rho(x) = 0$ and $\eta(x) = \xi(x)$ for $x \in W \setminus N$. Since T^* is a solution to the ROTPB(μ, ν) problem and $\partial(T^* \pm S) \preceq \nu - \mu$, the function $g(t) := \mathbf{E}(T^* + tS)$ defined on the interval $[-1, 1]$ achieves its minimum value at $t = 0$. Nevertheless,

$$\begin{aligned} g(t) &= \mathbf{E}(T^* + tS) = \mathbf{M}_\alpha(T^* + tS) - \int_X h d(\partial(T^* + tS)) \\ &= \mathbf{E}(T^*) + \int_W (\theta(x) + t\rho(x)\langle \xi(x), \eta(x) \rangle)^\alpha - \theta(x)^\alpha d\mathcal{H}^1(x) - t \int_X h d(\partial S). \end{aligned}$$

Here, the value of the inner product $\langle \xi(x), \eta(x) \rangle = \pm 1$ for $\mathcal{H}^1 - a.e.\ x \in W$. Then, since $\mathbf{M}_\alpha(T) = \int_W \theta^\alpha d\mathcal{H}^1 < \infty$ and $\rho(x) \le \theta(x)$ for $\mathcal{H}^1$ almost all $x \in W$, we have

$$g''(0) = \alpha(\alpha-1)\int_W \theta(x)^{\alpha-2}\rho(x)^2 d\mathcal{H}^1(x) < 0,$$

because $0 < \alpha < 1$ and S is non-zero. This says that g cannot achieve a local minimum at $t = 0$, a contradiction. □

4.1. **Atomic case.** In the context of finitely supported atomic measures, Theorem 4.7 has important implications for the structure of the optimal transport path T^* as demonstrated by the following results.

Proposition 4.8. *Suppose both*

$$\mu = \sum_{i=1}^{\ell} a_i \delta_{x_i} \text{ and } \nu = \sum_{j=1}^{n} b_j \delta_{y_j}$$

are atomic measures on X with finite supports, $0 < \alpha < 1$, and $T^* \in Path(\mu^*, \nu^*)$ is a solution to the ROTPB(μ,ν) problem. Also, let

$$P := spt(\mu - \mu^*) \cup spt(\nu - \nu^*)$$

denote the union of the supports of the measures $\mu - \mu^*$ and $\nu - \nu^*$. Then each connected component of the support of T^* contains at most one element of P.

Proof. Since $T^* \in Path(\mu^*, \nu^*)$ is a solution to the ROTPB(μ, ν) problem, it is an α-optimal transport path from μ^* and ν^*. Because both μ^* and ν^* are atomic measures with finite support, T^* is simply a finite acyclic graph (see [26, Proposition 2.1] and [2, Proposition 7.8]). Without loss of generality, we may assume that the support of T^* is connected, and we want to show that the set

$$\begin{aligned}(4.4)\quad P &= \{x_i : \mu^*(\{x_i\}) < \mu(\{x_i\})\} \bigcup \{y_j : \nu^*(\{y_j\}) < \nu(\{y_j\})\} \\ &= \{p \in \{x_1, \cdots, x_\ell, y_1, \cdots, y_n\} : (\nu - \nu^*)(\{p\}) + (\mu - \mu^*)(\{p\}) > 0\}\end{aligned}$$

contains at most one element. Assume that P has at least two distinct elements p_1 and p_2. Also, we may assume that $(\mu - \mu^*)(\{p_1\}) > 0$ and $(\mu - \mu^*)(\{p_2\}) > 0$ (the proofs for the other cases are similar). Since T^* is an acyclic finite graph, there exists a unique simple oriented curve γ on the support of T^* from p_1 to p_2, and set $S = \sigma I_\gamma$ with

$$\sigma = \min\left(\{\theta(x) : x \in \gamma\}, (\mu - \mu^*)\left(\{p_1\}\right), (\mu - \mu^*)\left(\{p_2\}\right)\rangle\, 0\right),$$

and I_γ being the rectifiable 1-current associated with γ (see (4.10) for the precise definition). Then, S is non-zero and on T in the sense of Definition 4.6. Moreover, by the choice of σ,

$$\mu^* \pm \sigma(\delta_{p_2} - \delta_{p_1}) \le \mu.$$

Thus,

$$\partial(T \pm S) = \nu^* - \mu^* \pm \sigma(\delta_{p_2} - \delta_{p_1}) \preceq \nu - \mu.$$

According to Theorem 4.7, S must be zero, a contradiction. □

The set P in Proposition 4.8 represents the collection of boundary nodes on which the amount of mass involved in the optimal transport path T^* is smaller than its counterpart specified initially. The proof hinges on the fact that if a connected component of the support of T^* contains two elements in P, one would be able to cut cost by reallocating the mass transported along T^*, which however is precluded by Theorem 4.7.

According to Proposition 4.5, in the finitely supported atomic case, there exists at least one point p on the support of μ^* or one point q on the support of ν^*, such that either

$$\mu^*(\{p\}) = \mu(\{p\}) \text{ or } \nu^*(\{q\}) = \nu(\{q\}). \tag{4.5}$$

Proposition 4.8 says that with at most one exception on each connected component, equation (4.5) holds for all points p or q on the supports of μ^* or ν^*, respectively. Consequently, with the help of Proposition 4.8 and Corollary 4.4, we may prove Theorem 1.1 as follows.

Proof of Theorem 1.1. By Proposition 4.8, each K_k contains at most one element of the set P. Thus, one of the following two cases holds:

Case 1:

$$\mu^* \llcorner K_k = \mu \llcorner K_k - m_k \delta_{p_k} \text{ and } \nu^* \llcorner K_k = \nu \llcorner K_k$$

for some point $p_k \in K_k \cap spt(\mu^*)$ and some real number $m_k \geq 0$.

Case 2:

$$\mu^* \llcorner K_k = \mu \llcorner K_k \text{ and } \nu^* \llcorner K_k = \nu \llcorner K_k - n_k \delta_{q_k}$$

for some point $q_k \in K_k \cap spt(\nu^*)$ and some real number $n_k \geq 0$.

In the first case,

$$\mu^*(K_k) = \mu(K_k) - m_k \text{ and } \nu^*(K_k) = \nu(K_k).$$

Since $\mu(K_k) \geq \mu^*(K_k) = \nu^*(K_k) = \nu(K_k)$, it follows that

$$m_k = \mu(K_k) - \mu^*(K_k) = \mu(K_k) - \nu(K_k) = \max\{\mu(K_k) - \nu(K_k), 0\},$$

and the desired relation (1.2) holds with $n_k = 0 = \max\{\nu(K_k) - \mu(K_k), 0\}$. Analogously, (1.2) also holds in the second case. $\square$

If the measure of mass at each source node is sufficiently large, all source nodes would fall into the set P, yielding a natural partition of the transport path T^* as stated in the following corollary. In this case, destination nodes can be classified by the source node from which they receive the mass. Under a symmetric condition, a similar decomposition exists for destination nodes.

Corollary 4.9. *Suppose both*

$$\mu = \sum_{i=1}^{\ell} a_i \delta_{x_i} \text{ and } \nu = \sum_{j=1}^{n} b_j \delta_{y_j}$$

are (positive) finitely supported atomic measures on X, and T^* is a solution to the $ROTPB(\mu, \nu)$ problem.

(a) If

$$\min_{1 \leq i \leq \ell} a_i \geq \sum_{j=1}^{n} b_j, \tag{4.6}$$

then T^* can be decomposed as $T^* = T_1 + T_2 + \cdots + T_\ell$, where for each $i = 1, \cdots, \ell$, T_i is an α-optimal transport path from a single source located at x_i.

(b) Similarly, if

$$\min_{1\leq j\leq n} b_j \geq \sum_{i=1}^{\ell} a_i, \tag{4.7}$$

then T^* can be decomposed as $T^* = T_1 + T_2 + \cdots + T_n$, where for each $j = 1, \cdots, n$, T_j is an α-optimal transport path to a single destination located at y_j.

Proof. We only need to prove case (a) as (b) follows from a symmetric argument. To do so, it is sufficient to show that each connected component of the support of T^* contains only one source point in $\{x_1, x_2, \cdots, x_\ell\}$. We prove it by contradiction. Assume that there exists a connected component of the support of T^* that contains at least two sources, say x_1 and x_2. Then

$$\mu^*(\{x_1\}) > 0 \text{ and } \mu^*(\{x_2\}) > 0.$$

As a result,

$$\mu^*(\{x_1\}) < \mu^*(\{x_1\}) + \mu^*(\{x_2\}) \leq ||\mu^*|| = ||\nu^*|| \leq \sum_{j=1}^{n} b_j \leq a_1 = \mu(\{x_1\}),$$

by (4.6). This shows that x_1 belongs to the set P in (4.4). Similar argument leads to $x_2 \in P$. This contradicts Proposition 4.8. Let $\{K_i : i = 1, 2, \cdots, \ell\}$ be the connected components of the support of T^*, and set $T_i = T \llcorner K_i$. Since T is α-optimal, and $\{K_i\}$ are pairwise disjoint, each T_i is also α-optimal. $\square$

4.2. **General case.** In what follows, we generalize the results of Theorem 1.1 for μ and ν being any two Radon measures, not necessarily finite atomic. To do so, we adopt a Lagrangian approach, and follow some notations used in [8].

By Theorem 3.1, the ROTPB(μ, ν) problem has a solution

$$T^* = \underline{\underline{\tau}}(W, \varphi, \zeta) \in Path(\mu^*, \nu^*). \tag{4.8}$$

We denote by Γ the space of 1-Lipschitz curves $\gamma : [0, \infty) \to \mathbb{R}^m$, which are eventually constant (and hence of finite length). For $\gamma \in \Gamma$, we denote the values

$$t_0(\gamma) := \sup\{t : \gamma \text{ is constant on } [0, t]\}$$

and

$$t_\infty(\gamma) := \inf\{t : \gamma \text{ is constant on } [t, \infty)\},$$

and denote $\gamma(\infty) := \lim_{t\to\infty} \gamma(t)$. Given $\gamma \in \Gamma$, the projections of γ onto its starting and stopping points are

$$p_0(\gamma) := \gamma(0) \text{ and } p_\infty(\gamma) := \gamma(\infty). \tag{4.9}$$

We say that a curve $\gamma \in \Gamma$ is *simple* if $\gamma(s) \neq \gamma(t)$ for every $t_0(\gamma) \leq s < t \leq t_\infty(\gamma)$. In particular, γ is non-constant in any non-trivial sub-interval $[s, t] \subseteq [t_0(\gamma), t_\infty(\gamma)]$.

For each simple curve $\gamma \in \Gamma$, we may canonically associate it with the rectifiable 1-current

$$I_\gamma := \underline{\underline{\tau}}\left(\mathrm{Im}(\gamma), \frac{\gamma'}{|\gamma'|}, 1\right), \tag{4.10}$$

where $\mathrm{Im}(\gamma)$ denotes the image of the curve γ in $\mathbb{R}^m$. It is easy to check that $\mathbf{M}(I_\gamma) = \mathcal{H}^1(\mathrm{Im}(\gamma))$ and $\partial I_\gamma = \delta_{\gamma(\infty)} - \delta_{\gamma(0)}$; since γ is simple, if it is also non-constant, then $\gamma(\infty) \neq \gamma(0)$ and $\mathbf{M}(\partial I_\gamma) = 2$.

A normal current $T \in \mathcal{D}_1(\mathbb{R}^m)$ is said *acyclic* if there exists no non-trival current S such that $\partial S = 0$ and $\mathbf{M}(T) = \mathbf{M}(T - S) + \mathbf{M}(S)$.

Now, we recall a fundamental result of Smirnov in [22], which establishes that every acyclic normal 1-current can be written as a weighted average of simple Lipschitz curves in the following sense.

Definition 4.10. Let T be a normal 1-current in $\mathbb{R}^m$ represented as a vector-valued measure $\vec{T}|T|$, and let η be a finite positive measure on Γ such that

$$T = \int_\Gamma I_\gamma d\eta(\gamma) \tag{4.11}$$

in the sense that for every smooth compactly supported 1-form $\omega \in \mathcal{D}^1(\mathbb{R}^m)$, it holds that

$$T(\omega) = \int_\Gamma I_\gamma(\omega) d\eta(\gamma). \tag{4.12}$$

We say that η is a *good decomposition* of T (see [6], [8], [22]) if η is supported on non-constant, simple curves and satisfies the following equalities:

(a) $\mathbf{M}(T) = \int_\Gamma \mathbf{M}(I_\gamma) d\eta(\gamma) = \int_\Gamma \mathcal{H}^1(Im(\gamma)) d\eta(\gamma)$;
(b) $\mathbf{M}(\partial T) = \int_\Gamma \mathbf{M}(\partial I_\gamma) d\eta(\gamma) = 2\eta(\Gamma)$.

It has been shown in [16, Theorem 10.1] that optimal transport paths T^* with finite $\mathbf{M}_\alpha$ cost are acyclic, and hence they admit such a good decomposition.

In the next result, we collect some useful properties of good decompositions, whose proof can be found in [6, Proposition 3.6].

Theorem 4.11. *(Existence and properties of good decompositions)*[17, Theorem 5.1] *and* [6, Proposition 3.6]. *Let T be an α-optimal transport path from μ^- to μ^+ with finite $\mathbf{M}_\alpha$ cost. Then T is acyclic and there is a Borel finite measure η on Γ such that η is a good decomposition of T. Moreover, if η is a good decomposition of T, the following statements hold:*

- $\mu^- = \int_\Gamma \delta_{\gamma(0)} d\eta(\gamma), \mu^+ = \int_\Gamma \delta_{\gamma(\infty)} d\eta(\gamma)$.
- *If $T = \underline{\underline{\tau}}(M, \theta, \xi)$ is rectifiable, then*
$$\theta(x) = \eta(\{\gamma \in \Gamma : x \in Im(\gamma)\}) \tag{4.13}$$
for $\mathcal{H}^1$-a.e. $x \in M$.
- *For every $\tilde{\eta} \leq \eta$, the representation*
$$\tilde{T} = \int_\Gamma I_\gamma d\tilde{\eta}(\gamma)$$
is a good decomposition of $\tilde{T}$. Moreover, if $T = \underline{\underline{\tau}}(M, \theta, \xi)$ is rectifiable, then $\tilde{T}$ can be written as $\tilde{T} = \underline{\underline{\tau}}\left(M, \tilde{\theta}, \xi\right)$ with $\tilde{\theta}(x) \leq \min\{\theta(x), \tilde{\eta}(\Gamma)\}$ for $\mathcal{H}^1$-a.e. $x \in M$.

We now introduce the following notations. Let $T^* = \underline{\underline{\tau}}(W, \varphi, \zeta) \in Path(\mu^*, \nu^*)$ be given as in (4.8), and η be a good decomposition of T^*. Denote

$$\tilde{\mu} = \mu - \mu^* \text{ and } \tilde{\nu} = \nu - \nu^*.$$

For any $x \in W$, let us denote by $\Gamma(x)$ the set of simple curves $\gamma \in \Gamma$ such that $x \in Im(\gamma)$. By equation (4.13), $\varphi(x) = \eta(\Gamma(x))$ for $\mathcal{H}^1$-a.e. $x \in W$.

Proposition 4.12. *For any $x \in W$ with $\eta(\Gamma(x)) > 0$, denote*

$$\mu_x^* = (p_0)_\# \left(\eta \llcorner \Gamma(x)\right) \text{ and } \nu_x^* = (p_\infty)_\# \left(\eta \llcorner \Gamma(x)\right)$$

where p_0 and p_∞ are projections given in (4.9). Let

$$\tilde{\mu} = \tilde{\mu}_x^{ac} + \tilde{\mu}_x^s,\ \tilde{\mu}_x^{ac} \ll \mu_x^*,\ \tilde{\mu}_x^s \perp \mu_x^*$$

be the Lebesgue-Radon-Nikodým decomposition of $\tilde{\mu}$ with respect to μ_x^*, and

$$\tilde{\nu} = \tilde{\nu}_x^{ac} + \tilde{\nu}_x^s,\ \tilde{\nu}_x^{ac} \ll \nu_x^*,\ \tilde{\nu}_x^s \perp \nu_x^*$$

be the Lebesgue-Radon-Nikodým decomposition of $\tilde{\nu}$ with respect to ν_x^*. Then

(a) There exist $p_x \in W$, $q_x \in W$, $m_x \geq 0$, and $n_x \geq 0$ such that

$$\tilde{\mu}_x^{ac} = m_x \delta_{p_x} \text{ and } \tilde{\nu}_x^{ac} = n_x \delta_{q_x}.$$

(b) If $m_x > 0$, then $\mu_x^*(\{p_x\}) > 0$, $m_x = \tilde{\mu}(\{p_x\})$ and there exists a Lipschitz curve γ_x^- from p_x to x such that $\varphi(y) \geq \mu_x^*(\{p_x\})$ for $\mathcal{H}^1$-a.e. $y \in Im(\gamma_x^-)$.

(c) If $n_x > 0$, then $\nu_x^*(\{q_x\}) > 0$, $n_x = \tilde{\nu}(\{q_x\})$ and there exists a Lipschitz curve γ_x^+ from x to q_x such that $\varphi(y) \geq \nu_x^*(\{q_x\})$ for $\mathcal{H}^1$-a.e. $y \in Im(\gamma_x^+)$.

(d) At least one of m_x and n_x is zero.

Proof. Note that

$$\mu_x^*(X) = (p_0)_\# \left(\eta \llcorner \Gamma(x)\right)(X) = (\eta \llcorner \Gamma(x))(p_0^{-1}(X)) = \eta(\Gamma(x)) = \varphi(x) > 0,$$

and $\mu_x^* \leq (p_0)_\# \eta = \mu^*$. For the sake of contradiction, we assume that $\tilde{\mu}_x^{ac}$ is non-zero and not a multiple of a Dirac mass. Then there exists a Borel measurable set R such that

$$\tilde{\mu}_x^{ac}(R) > 0 \text{ and } \tilde{\mu}_x^{ac}(X \setminus R) > 0.$$

Since $\tilde{\mu}_x^{ac} \ll \mu_x^*$, there exists a non-negative function $g \in \mathcal{L}^1(X, \mu_x^*)$ such that $\tilde{\mu}_x^{ac} = g \cdot \mu_x^*$. We define

$$\mu_0 := \min\{g(\cdot), 1\} \cdot \mu_x^*.$$

Still we have $\mu_0(R) > 0$ and $\mu_0(X \setminus R) > 0$. Without loss of generality, we may assume that $0 < \mu_0(R) \leq \mu_0(X \setminus R)$. Setting

$$\mu_1 = \mu_0 \llcorner R \text{ and } \mu_2 = \frac{\mu_0(R)}{\mu_0(X \setminus R)} \left(\mu_0 \llcorner (X \setminus R)\right)$$

yields two positive measures μ_1 and μ_2 such that μ_1 is concentrated on R and μ_2 is concentrated on $X \setminus R$ with equal mass. Moreover, since both μ_1 and μ_2 are absolutely continuous with respect to μ_x^*, there exist two non-negative μ_x^*-integrable functions ρ_1 and ρ_2 such that

$$\mu_1 = \rho_1 \mu_x^* \text{ and } \mu_2 = \rho_2 \mu_x^*.$$

Let $\rho = \rho_1 - \rho_2$. Note that $|\rho| \leq 1$ holds $\bar{\mu}_x$-a.e. From the construction of μ_1 and μ_2, we have

$$\mu_x^*(\rho) = \int_X \rho d\mu_x^* = \int_X \rho_1 d\mu_x^* - \int_X \rho_2 d\mu_x^* = 0,$$

and

$$\int_X |\rho|\, d\mu_x^* = 2 \int_X \rho_1 d\mu_x^* = 2\mu_0(R) > 0.$$

Now, for any $\gamma \in \Gamma(x)$, let γ^- be the part of γ from $\gamma(0)$ to x. Define

$$S := \int_{\Gamma(x)} \rho(\gamma(0)) I_{\gamma^-} d\eta(\gamma).$$

Then for any smooth function f with a compact support, we have

$$\begin{aligned}
\partial S(f) &= S(df) \\
&= \int_{\Gamma(x)} \rho(\gamma(0)) I_{\gamma^-}(df) d\eta(\gamma) \\
&= \int_{\Gamma(x)} \rho(\gamma(0)) \partial(I_{\gamma^-})(f) d\eta(\gamma) \\
&= \int_{\Gamma(x)} \rho(\gamma(0))(f(x) - f(\gamma(0)) d\eta(\gamma) \\
&= \int_{\Gamma(x)} \rho(\gamma(0)) f(x) d\eta(\gamma) - \int_{\Gamma(x)} \rho(\gamma(0)) f(\gamma(0)) d\eta(\gamma) \\
&= f(x)\mu_x^*(\rho) - (\rho\mu_x^*)(f) = -(\rho\mu_x^*)(f).
\end{aligned}$$

Therefore, $\partial S = -\rho\mu_x^* \neq 0$.

Since T^* is rectifiable, by construction S is also rectifiable. We may write it as $S = \underline{\tau}(M_S, \theta_S, \xi_S)$ for some $M_S \subseteq W$. At $\mathcal{H}^1$-a.e. $y \in M_S$,

$$\theta_S(y) \leq \int_{\Gamma(x)\cap\Gamma(y)} |\rho(\gamma(0))| \, d\eta(\gamma) \leq \int_{\Gamma(x)\cap\Gamma(y)} 1 d\eta(\gamma) \leq \int_{\Gamma(y)} 1 d\eta(\gamma) = \varphi(y).$$

This shows that S is on T^* in the sense of Definition 4.6.

We now show that $\partial(T^* \pm S) \leq \nu - \mu$. Given

$$\partial(T^* \pm S) = \nu^* - \mu^* \mp (\rho\mu_x^*) = \nu^* - (\mu^* \pm (\rho\mu_x^*)),$$

it is sufficient to show that $\mu^* \pm (\rho\mu_x^*)$ is positive, which is actually the case since $\mu_x^* \leq \mu^*$ and $|\rho| \leq 1$, and that $\mu^* \pm (\rho\mu_x^*) \leq \mu$, which also holds since

$$\mu - (\mu^* \pm (\rho\mu_x^*)) = \mu - \mu^* \mp (\rho\mu_x^*) = \tilde{\mu} \mp (\mu_1 - \mu_2)$$

are positive measures because $\mu_1 \leq \tilde{\mu}_x^{ac} \leq \tilde{\mu}$ and similarly $\mu_2 \leq \tilde{\mu}$. As a result, $\partial(T^* \pm S) \preceq \nu - \mu$. By Theorem 4.7, S is zero which contradicts $\partial S \neq 0$. Therefore, $\tilde{\mu}_x^{ac}$ must be in the form of $m_x\delta_{p_x}$ for some $m_x \geq 0$ and $p_x \in W$. Similarly, we have $\tilde{\nu}_x^{ac} = n_x\delta_{q_x}$ for some $n_x \geq 0$ and $q_x \in W$. This proves part (a).

Now assume that $m_x > 0$. Since $\tilde{\mu}_x^{ac} = m_x\delta_{p_x} \ll \mu_x^*$, we have $\mu_x^*(\{p_x\}) > 0$. Given $\tilde{\mu}_x^s \perp \mu_x^*$ and $\mu_x^*(\{p_x\}) > 0$, it follows that $\tilde{\mu}_x^s(\{p_x\}) = 0$. Thus, $\tilde{\mu}(\{p_x\}) = \tilde{\mu}_x^{ac}(\{p_x\}) + \tilde{\mu}_x^s(\{p_x\}) = m_x + 0 = m_x$. Moreover, since $\mu_x^*(\{p_x\}) > 0$, we have

$$0 < (p_0)_\# \left(\eta\lfloor_{\Gamma(x)}\right)(\{p_x\}) = \eta(\{\gamma \in \Gamma(x) : \gamma(0) = p_x\}).$$

Because T^* is acyclic and η is a good decomposition of T^*, by the single-path property as described in [2, Proposition 7.4], for η-a.e. $\gamma \in \Gamma(x)$ with $\gamma(0) = p_x$, the image $Im(\gamma)$ of γ shares a common Lipschitz curve γ_x^- in W from p_x to x. For $\mathcal{H}^1$-a.e. y on $Im(\gamma_x^-)$,

$$\varphi(y) = \eta(\{\gamma \in \Gamma(y)\}) \geq \eta(\{\gamma \in \Gamma(x) : \gamma(0) = p_x\}) = \mu_x^*(\{p_x\}).$$

This proves part (b). Similar arguments lead to part (c).

Suppose by contradiction that both $m_x > 0$ and $n_x > 0$. Then, by parts (b) and (c), we consider the rectifiable 1-current

$$S_x := \sigma \left(I_{\gamma_x^-} + I_{\gamma_x^+} \right),$$

for $\sigma = \min\{\mu_x^*(\{p_x\}), \nu_x^*(\{q_x\}), m_x, n_x\} > 0$. Clearly, S_x is on T^*, non-zero, and $\partial(T^* \pm S_x) \preceq \nu - \mu$. This contradicts Theorem 4.7. Therefore, at least one of m_x and n_x is zero. □

To derive the generalized version of Theorem 1.1, we introduce the concept of path-connectivity on rectifiable 1-currents as follows.

Definition 4.13. Let $T = \underline{\underline{\tau}}(M, \theta, \xi)$ be a rectifiable 1-current. For any two points $x_1, x_2 \in X$, we say x_1 and x_2 are T-path-connected if there exists a Lipschitz curve $\gamma : [0,1] \to X$ such that $\gamma(0) = x_1, \gamma(1) = x_2$, $\mathcal{H}^1(Im(\gamma) \setminus M) = 0$, and there exists a number $c > 0$ such that $\theta(z) \geq c$ for $\mathcal{H}^1$-a.e. $z \in Im(\gamma) \cap M$.

The T-path-connectivity defines an equivalence relation on X. Each equivalence class for this relation is called a T-path-connected component of M. A T-path-connected component of M is called degenerate if it contains only one point, and non-degenerate otherwise. Note that each non-degenerate T-path-connected component of M is a subset of the closure of the 1-rectifiable set M.

For any T-path-connected component M' of M, we consider the restriction $T \llcorner M'$ of T on M'. When M' is degenerate, $T \llcorner M'$ is simply zero. When M' is non-degenerate, i.e., it contains at least two distinct points x_1 and x_2, we have

$$\mathbf{M}(T \llcorner M') = \int_{M'} \theta d\mathcal{H}^1 \geq c\mathcal{H}^1(Im(\gamma)) > 0$$

using the notations given in Definition 4.13. Since $\mathbf{M}(T) < \infty$, M has at most countably many non-degenerate T-path-connected components.

Observe that non-degenerate components may fail to exist even if $\mathbf{M}(T) > 0$. For instance, let C (e.g., a fat-Cantor set) be a nowhere dense subset of $[0,1]$ with $0 < \mathcal{H}^1(C) < 1$. Then, for $S = \underline{\underline{\tau}}(C, \chi_C, 1)$, each S-path-connected component is degenerate. Luckily, the following lemma indicates that each non-zero α-optimal transport path has at least one non-degenerate path-connected component.

Lemma 4.14. *Let $T = \underline{\underline{\tau}}(M, \theta, \xi)$ be a non-zero α-optimal transport path for some $0 < \alpha < 1$. Then*

$$T = \sum_{i \in J} T \llcorner M_i,$$

where $\{M_i : i \in J\}$ are the collection of all non-degenerate T-path-connected components of M, and J is a non-empty countable set.

To prove Lemma 4.14, we first recall the notation of superlevel set as introduced in [29]: For any $\lambda > 0$, the λ-superlevel set of a rectifiable current $T = \underline{\underline{\tau}}(M, \theta, \xi)$ is the set

$$M_\lambda := \{p \in M : \theta(p) \geq \lambda\}.$$

When $T = \underline{\underline{\tau}}(M, \theta, \xi)$ is an α-optimal transport path, let η be a good decomposition of T. As given in (4.13), $\theta(x) = \eta(\{\gamma \in \Gamma : x \in Im(\gamma)\}) := \theta^*(x)$ for $\mathcal{H}^1 - a.e.\ x \in M$. Since both $\underline{\underline{\tau}}(M, \theta, \xi)$ and $\underline{\underline{\tau}}(M, \theta^*, \xi)$ represent T, without loss of generality, we assume that

$$\theta(x) = \eta(\{\gamma \in \Gamma : x \in Im(\gamma)\}),\ \forall x \in M. \tag{4.14}$$

Lemma 4.15. *([29, Proposition 4.3]) Let $T = \underline{\underline{\tau}}(M,\theta,\xi)$ be any α-optimal transport path with θ satisfying (4.14). Then for any $\sigma_1 > \sigma_2 > 0$ and any $p \in M_{\sigma_1}$, there exists an open ball neighborhood $B_r(p)$ of p such that*

$$M_{\sigma_1} \cap B_r(p) \subseteq spt(Q_p) \subseteq M_{\sigma_2} \cap B_r(p), \tag{4.15}$$

where $spt(Q_p)$ is the support of a bi-Lipschitz chain $Q_p = \sum_{i=1}^{K} m_i \Gamma_i$.

Here, as stated in [29, Corollary 4.2], each Γ_i is a bi-Lipschitz curve from p. These bi-Lipschitz curves Γ_i are pairwise disjoint except at their common endpoint p, and K is a universal constant. Moreover, $\sum_{i=1}^{K} m_i \mathcal{H}^1(\Gamma_i) > 0$.

Remark 4.16. In general, without the assumption that θ satisfying (4.14), the inclusions (4.15) hold up to $\mathcal{H}^1$-negligible sets for $\mathcal{H}^1$-almost every $p \in M_{\sigma_1}$. In the proof of [29, Proposition 4.3], the multiplicity function θ was taken for granted to satisfy (4.14).

Proof of Lemma 4.14: Let $\{M_i : i \in J\}$ be the collection of all non-degenerate T-path-connected components of M, where J is countable. For any $p \in M$ with $\theta(p) > 0$, let $\sigma_1 = \theta_p$ and $\sigma_2 = \frac{1}{2}\sigma_1$. By (4.15), any point on the support of the bi-Lipschitz curve Q_p is T-path-connected with p. Hence, p belongs to some non-degenerate T-path-connected component M_i for some $i \in J$. As a result, we decompose $M_+ = \{x \in M : \theta(x) > 0\}$ as the disjoint union of $M_i \cap M_+$ with $i \in J$. Thus, $T = \underline{\underline{\tau}}(M_+,\theta,\xi) = \sum_{i\in J} T \llcorner M_i$. □

We now go back to the study of T^*, which is also an α-optimal transport path. Consequently, one can write

$$T^* = \sum_{i\in J} T^* \llcorner W_i,$$

where $\{W_i : i \in J\}$ are the collection of all non-degenerate T^*-path-connected components of W, and J is a non-empty countable set if T^* is non-zero.

Lemma 4.17. *Suppose x_1 and x_2 are two distinct points belonging to the same non-degenerate T^*-path-connected component of W. Then*

(a) If $m_{x_1} > 0$, then $n_{x_1} = n_{x_2} = 0$;
(b) If $n_{x_1} > 0$, then $m_{x_1} = m_{x_2} = 0$;
(c) If both $m_{x_1} > 0$ and $m_{x_2} > 0$, then $p_{x_1} = p_{x_2}$ and $m_{x_1} = m_{x_2}$;
(d) If both $n_{x_1} > 0$ and $n_{x_2} > 0$, then $q_{x_1} = q_{x_2}$ and $n_{x_1} = n_{x_2}$.

Proof. Since x_1 and x_2 are T^*-path-connected, there exists a Lipschitz curve $\gamma_{x_1}^{x_2}$ from x_1 to x_2 such that $\mathcal{H}^1(Im(\gamma_{x_1}^{x_2}) \setminus W) = 0$, and there exists a number $c > 0$ such that $\varphi(z) \geq c$ for $\mathcal{H}^1$-a.e. $z \in Im(\gamma_{x_1}^{x_2})$.

Proof of (a): Since $m_{x_1} > 0$, by part (d) of Proposition 4.12, we have $n_{x_1} = 0$. Suppose $n_{x_2} > 0$. Then consider the rectifiable 1-current

$$S := \sigma\left(I_{\gamma_{x_1}^-} + I_{\gamma_{x_1}^{x_2}} + I_{\gamma_{x_2}^+}\right),$$

for $\sigma = \min\{\mu_{x_1}^*(\{p_{x_1}\}), \nu_{x_2}^*(\{q_{x_2}\}), m_{x_1}, n_{x_2}, c\} > 0$. Clearly, S is on T^*, non-zero, and

$$\partial(T^* \pm S) = (\nu^* - \mu^*) \pm \sigma(\delta_{q_{x_2}} - \delta_{p_{x_1}}) = (\nu^* \pm \sigma\delta_{q_{x_2}}) - (\mu^* \pm \sigma\delta_{p_{x_1}}) \preceq \nu - \mu.$$

This contradicts Theorem 4.7. The proof of (b) follows similarly.

Proof of (c): Assume $m_{x_1} > 0$ and $m_{x_2} > 0$, but $p_{x_1} \neq p_{x_2}$. Then we consider the rectifiable 1-current

$$S := \sigma\left(I_{\gamma_{x_1}^-} + I_{\gamma_{x_1}^{x_2}} - I_{\gamma_{x_2}^-}\right),$$

for $\sigma = \min\{\mu^*_{x_1}(\{p_{x_1}\}), \mu^*_{x_2}(\{p_{x_2}\}), m_{x_1}, m_{x_2}, c\} > 0$. Clearly, S is on T^* with $\partial S = \sigma(\delta_{p_{x_2}} - \delta_{p_{x_1}}) \neq 0$, which implies that S is also non-zero, and

$$\partial(T^* \pm S) = (\nu^* - \mu^*) \pm \sigma(\delta_{p_{x_2}} - \delta_{p_{x_1}}) = \nu^* - (\mu^* \pm \sigma\delta_{p_{x_1}} \mp \sigma\delta_{p_{x_2}}) \preceq \nu - \mu.$$

This contradicts Theorem 4.7. Therefore, $p_{x_1} = p_{x_2}$. Employing part (b) of Proposition 4.12 gives $m_{x_1} = \tilde{\mu}(\{p_{x_1}\}) = \tilde{\mu}(\{p_{x_2}\}) = m_{x_2}$. This completes the proof of (c). The proof of (d) proceeds in a similar fashion. □

For each $i \in J$, if there exists one point $x \in W_i$ such that $m_x > 0$, then by part(b) of Proposition 4.12, the associated point p_x is also T^*-path-connected with x and hence $p_x \in W_i$. By Lemma 4.17, $m_x\delta_{p_x}$ is independent of the choice of $x \in W_i$, and thus can be represented by $m_i\delta_{p_i}$ with $m_i = \tilde{\mu}(\{p_i\})$. If $m_x = 0$ for all $x \in W_i$, we simply pick p_i to be any fixed point in W_i and set $m_i = 0$. Analogously, when $n_x > 0$, we denote $n_x\delta_{q_x}$ by $n_i\delta_{q_i}$ with $n_i = \tilde{\nu}(\{q_i\})$ for each $i \in J$. As a result, we arrive at two atomic measures

$$\mathbf{a} = \sum_{i\in J} m_i\delta_{p_i} \text{ and } \mathbf{b} = \sum_{i \in J} n_i\delta_{q_i}, \tag{4.16}$$

where either $m_i = 0$ or $n_i = 0$ for each $i \in J$. Note that $\mathbf{a} \leq \tilde{\mu}$ and $\mathbf{b} \leq \tilde{\nu}$.

Lemma 4.18. *It holds that*

$$(\tilde{\mu} - \mathbf{a}) \perp \mu^*, \mathbf{a} \ll \mu^* \textit{ and } (\tilde{\nu} - \mathbf{b}) \perp \nu^*, \mathbf{b} \ll \nu^*.$$

Proof. Let $\hat{\mu} = \tilde{\mu} - \mathbf{a}$. Then for each $x \in W$ with $\eta(\Gamma(x)) > 0$, by Proposition 4.12,

$$\hat{\mu} \perp \mu^*_x.$$

Thus, there exists a $\hat{\mu}$-negligible set A_x such that $\mu^*_x(A_x) = \mu^*_x(X)$. Observe that one may pick (see for instance [3, Lemma 3.11]) countably many points $\{x_k : \eta(\Gamma(x_k)) > 0\}_{k=1}^\infty \subset W$ so that for η-a.e. $\gamma \in \Gamma$, γ passes at least one of these points. Now for each k,

$$\eta\left(\Gamma(x_k)\right) = \bar{\mu}_{x_k}(X) = \bar{\mu}_{x_k}(A_{x_k}) = \eta\left(\{\gamma : \gamma(0) \in A_k\}\right),$$

and thus

$$\eta(\Gamma) = \eta\left(\bigcup_k \Gamma(x_k)\right) = \eta\left(\{\gamma \in \Gamma : \gamma(0) \in \bigcup_k A_{x_k}\}\right).$$

As a result,

$$\mu^*(X) = \mu^*(\bigcup_k A_{x_k}) \text{ and } \hat{\mu}(\bigcup_k A_{x_k}) \leq \sum_k \hat{\mu}(A_{x_k}) = 0.$$

Therefore, $\mu^* \perp \hat{\mu}$ as desired. For each $i \in J$, assume $m_i\delta_{p_i} = m_x\delta_{p_x}$, then

$$m_i\delta_{p_i} \ll \mu^*_x \ll \mu^*.$$

Thus, $\mathbf{a} \ll \mu^*$. Similarly, we have $(\tilde{\nu} - \mathbf{b}) \perp \nu^*$ and $\mathbf{b} \ll \nu^*$. □

Now, we have the following theorem which is a generalized version of Theorem 1.1.

Theorem 4.19. *Suppose μ and ν are two mutually singular Radon measures on X, $0<\alpha<1$, and $T^* = \underline{\underline{\tau}}(W,\varphi,\zeta) \in Path(\mu^*,\nu^*)$ is a solution to the ROTPB(μ,ν) problem. Then*

(1) *There exist two atomic measures $\mathbf{a}$ and $\mathbf{b}$, two disjoint Borel sets A and B such that*
$$\mu \llcorner A = \mu^* + \mathbf{a} \text{ and } \nu \llcorner B = \nu^* + \mathbf{b}.$$

(2) *Let $\{W_i : i \in J\}$ be the collection of all non-degenerate T^*-path-connected components of W. Then, for each $i \in J$, there exist $m_i \geq 0$ and $n_i \geq 0$ with either $m_i = 0$ or $n_i = 0$; and two points $p_i, q_i \in W_i$ such that*
$$\mathbf{a} = \sum_{i\in J} m_i \delta_{p_i} \text{ and } \mathbf{b} = \sum_{i\in J} n_i \delta_{q_i}.$$

The theorem says that on each T^*-path-connected component, at most one atom is not fully in use. In particular, when both μ and ν are atom-free[3] measures, it follows that
$$\mu^* = \mu \llcorner A \text{ and } \nu^* = \nu \llcorner B.$$

Proof. The atomic measures $\mathbf{a}$ and $\mathbf{b}$ are obtained by (4.16). By Lemma 4.18, $(\tilde{\mu} - \mathbf{a}) \perp \mu^*$, where $\tilde{\mu} = \mu - \mu^*$ and $\mathbf{a} \ll \mu^*$. Since $(\tilde{\mu} - \mathbf{a}) \perp \mu^*$, by definition of being mutually singular, there exist a Borel set A such that $\mu^* = \mu^* \llcorner A$ and $(\tilde{\mu} - \mathbf{a}) = (\tilde{\mu} - \mathbf{a}) \llcorner (\mathbb{R}^m \setminus A)$. Hence
$$\mu^* = \mu^* \llcorner A + (\tilde{\mu} - \mathbf{a}) \llcorner A = (\mu^* + \tilde{\mu} - \mathbf{a}) \llcorner A = (\mu - \mathbf{a}) \llcorner A.$$
Since μ^* concentrates on A and $\mathbf{a} \ll \mu^*$, we have $\mathbf{a} \llcorner A = \mathbf{a}$. As a result, $\mu \llcorner A = \mu^* + \mathbf{a}$ as desired. Similarly, we have $\nu \llcorner B = \nu^* + \mathbf{b}$ for some Borel set B. Because μ and ν are mutually singular, we may assume that A and B are disjoint Borel sets. □

In light of the theorem, on locations involving mass transportation, the measure $\mathbf{a}$, which represents the mass left unmoved by the solution T^*, must be atomic, so is measure $\mathbf{b}$ which summarizes the distribution of excess demand at destinations. This is because if not the planner can exploit further gains by relocating the mass moved along the path T^* due to the efficiency in group transportation.

5. The impact of boundary payoff

An important deviation of the ROTPB problem from the literature is the dependence of its solution on the boundary payoff as exemplified by Proposition 3.4. To gain further insights, in what follows we examine the implications of the payoff function h for the problem. For the sake of expositional tractability, we assume that μ and ν are disjointly supported (i.e., $spt(\mu) \cap spt(\nu) = \emptyset$) and the function h takes the form

$$h(x) = \begin{cases} c_\mu, & \text{if } x \in spt(\mu) \\ c_\nu, & \text{if } x \in spt(\nu) \end{cases} \tag{5.1}$$

where c_μ and c_ν are constants. In this case, for any $T \in Path(\tilde{\mu}, \tilde{\nu})$,
$$\mathbf{E}^h_\alpha(T) = \mathbf{M}_\alpha(T) - \int_X c_\nu d\tilde{\nu} + \int_X c_\mu d\tilde{\mu} = \mathbf{M}_\alpha(T) - 2c||\tilde{\mu}|| = \mathbf{M}_\alpha(T) - c\mathbf{M}(\partial T),$$

[3] A measure μ is called atom-free if $\mu(\{p\}) = 0$ for every $p \in X$.

where $c = \frac{c_\nu - c_\mu}{2}$. The corresponding ROTPB(μ,ν) problem in this case becomes: Minimize

$$\mathbf{E}^c_\alpha(T) := \mathbf{M}_\alpha(T) - c\mathbf{M}(\partial T) \tag{5.2}$$

among all transport paths T with $\partial T \preceq \nu - \mu$. Without loss of generality, we may assume that $c_\nu = 2c$ and $c_\mu = 0$ in equation (5.1).

For each c, by Theorem 3.1, the ROTPB(μ,ν) problem has a solution T^*_c that minimizes $\mathbf{E}^c_\alpha$. When $c \le 0$, by Proposition 3.4, the problem has a unique solution $T^*_c = 0$. Thus, in the following context, we only need to investigate T^*_c for $c > 0$.

Proposition 5.1. *Suppose μ and ν are two disjointly supported measures on X of equal mass, and $T^*_c \in Path(\mu^*_c, \nu^*_c)$ is a solution to the ROTPB(μ,ν) problem associated with $c > 0$. Then, for any transport path $T \in Path(\mu,\nu)$,*

$$\mathbf{M}_\alpha(T) - \mathbf{M}_\alpha(T^*_c) \ge c(\mathbf{M}(\partial T) - \mathbf{M}(\partial T^*_c)) \ge 0,$$

and hence $\mathbf{M}_\alpha(T^*_c) \le d_\alpha(\mu,\nu)$.

Proof. Indeed, for any transport path $T \in Path(\mu,\nu)$,

$$\begin{aligned}
\mathbf{M}_\alpha(T) - \mathbf{M}_\alpha(T^*_c) &= (\mathbf{E}^c_\alpha(T) + c\mathbf{M}(\partial T)) - (\mathbf{E}^c_\alpha(T^*_c) + c\mathbf{M}(\partial T^*_c)) \\
&= (\mathbf{E}^c_\alpha(T) - \mathbf{E}^c_\alpha(T^*_c)) + c(\mathbf{M}(\partial T) - \mathbf{M}(\partial T^*_c)) \\
&\ge c(\mathbf{M}(\partial T) - \mathbf{M}(\partial T^*_c)) \ge 0.
\end{aligned}$$

□

Proposition 5.1 shows that the transportation cost $\mathbf{M}_\alpha(T^*_c)$ associated with the solution T^*_c is bounded from above. The following proposition derives an upper bound as well as the decay rate for the amount of mass left unmoved by T^*_c. The intuition is that since the transportation cost is bounded, a large enough value of the parameter c, which measures the profitability for relocating the mass, would induce the planner to move as much mass as possible.

Proposition 5.2. *Suppose $||\mu|| = ||\nu||$, $c > 0$, $1 - \frac{1}{m} < \alpha < 1$ and $T^*_c \in Path(\mu^*_c, \nu^*_c)$ denotes the solution to the ROTPB(μ,ν) problem. Then*

$$||\mu - \mu^*_c|| = ||\nu - \nu^*_c|| \le \left(\frac{C_{m,\alpha} diam(X)}{2c}\right)^{\frac{1}{1-\alpha}}, \tag{5.3}$$

where $C_{m,\alpha}$ is the constant given in (2.3).

Proof. Let $\tilde{T} \in Path(\mu - \mu^*_c, \nu - \nu^*_c)$ be an α-optimal transport path, and denote $T = T^*_c + \tilde{T} \in Path(\mu,\nu)$. By (2.2),

$$\begin{aligned}
0 &\le \mathbf{E}^c_\alpha(T) - \mathbf{E}^c_\alpha(T^*_c) \\
&= (\mathbf{M}_\alpha(T) - c\mathbf{M}(\partial T)) - (\mathbf{M}_\alpha(T^*_c) - c\mathbf{M}(\partial T^*_c)) \\
&= (\mathbf{M}_\alpha(T) - \mathbf{M}_\alpha(T^*_c)) - c\,(\mathbf{M}(\partial T) - \mathbf{M}(\partial T^*_c)) \\
&\le \mathbf{M}_\alpha(\tilde{T}) - c(||\mu - \mu^*_c|| + ||\nu - \nu^*_c||) \\
&\le C_{m,\alpha} diam(X)||\mu - \mu^*_c||^\alpha - 2c||\mu - \mu^*_c||,
\end{aligned}$$

which leads to inequality (5.3). □

The next proposition characterizes the monotonicity properties of the solution. Intuitively, as c rises, the planner tends to move more mass between sources and destinations, resulting in larger transportation costs.

Proposition 5.3. *Suppose $||\mu|| = ||\nu||$, $c > 0$, $1 - \frac{1}{m} < \alpha < 1$ and $T_c^* \in Path(\mu_c^*, \nu_c^*)$ denotes the solution to the ROTPB(μ,ν) problem. Then, as a function of $c \in \mathbb{R}$,*

1. *$\mathbf{E}_\alpha^c(T_c^*)$ is nonincreasing;*
2. *$\mathbf{M}_\alpha(T_c^*)$ is nondecreasing with $\lim_{c\to\infty} \mathbf{M}_\alpha(T_c^*) = d_\alpha(\mu,\nu)$;*
3. *$\mathbf{M}(\partial T_c^*)$ is nondecreasing with $\lim_{c\to\infty} ||\partial T_c^* - (\nu - \mu)|| = 0$.*

Proof. Indeed, for any $c_1 < c_2$,

$$\mathbf{E}_\alpha^{c_1}(T_{c_1}^*) = \mathbf{M}_\alpha(T_{c_1}^*) - c_1\mathbf{M}(\partial T_{c_1}^*) \geq \mathbf{M}_\alpha(T_{c_1}^*) - c_2\mathbf{M}(\partial T_{c_1}^*) = \mathbf{E}_\alpha^{c_2}(T_{c_1}^*) \geq \mathbf{E}_\alpha^{c_2}(T_{c_2}^*).$$

Also, the inequalities $\mathbf{E}_\alpha^{c_1}(T_{c_1}^*) \leq \mathbf{E}_\alpha^{c_1}(T_{c_2}^*)$ and $\mathbf{E}_\alpha^{c_2}(T_{c_2}^*) \leq \mathbf{E}_\alpha^{c_2}(T_{c_1}^*)$ imply that

$$\mathbf{M}_\alpha(T_{c_1}^*) - c_1\mathbf{M}(\partial T_{c_1}^*) \leq \mathbf{M}_\alpha(T_{c_2}^*) - c_1\mathbf{M}(\partial T_{c_2}^*)$$
$$\mathbf{M}_\alpha(T_{c_2}^*) - c_2\mathbf{M}(\partial T_{c_2}^*) \leq \mathbf{M}_\alpha(T_{c_1}^*) - c_2\mathbf{M}(\partial T_{c_1}^*).$$

Rewriting them gives

$$c_2\left(\mathbf{M}(\partial T_{c_2}^*) - \mathbf{M}(\partial T_{c_1}^*)\right) \geq \mathbf{M}_\alpha(T_{c_2}^*) - \mathbf{M}_\alpha(T_{c_1}^*) \geq c_1\left(\mathbf{M}(\partial T_{c_2}^*) - \mathbf{M}(\partial T_{c_1}^*)\right).$$

Since $c_1 < c_2$, we have $\mathbf{M}(\partial T_{c_2}^*) \geq \mathbf{M}(\partial T_{c_1}^*)$ and $\mathbf{M}_\alpha(T_{c_2}^*) \geq \mathbf{M}_\alpha(T_{c_1}^*)$. This shows that both $\mathbf{M}_\alpha(T_c^*)$ and $\mathbf{M}(\partial T_c^*)$ are nondecreasing functions of c. Moreover,

$$\lim_{c\to\infty} ||\partial T_c^* - (\nu - \mu)|| = \lim_{c\to\infty} ||(\nu_c^* - \mu_c^*) - (\nu - \mu)|| = 0$$

by the triangle inequality and (5.3). Since $||\mu_c^* - \mu|| \to 0$, $||\nu_c^* - \nu|| \to 0$ and d_α metrizes the weak convergence of measures (as in [26, Theorem 4.2]), we have $\mathbf{M}_\alpha(T_c^*) = d_\alpha(\mu_c^*, \nu_c^*) \to d_\alpha(\mu,\nu)$ as $c \to \infty$. □

Theorem 5.4. *Suppose μ and ν are two disjointly supported measures on X of equal mass, $1 - \frac{1}{m} < \alpha < 1$, and let $T_c^* \in Path(\mu_c^*, \nu_c^*)$ denote the solution to the ROTPB(μ,ν) problem corresponding to parameter c. If for some sequence $\{c_n\}$ converging to ∞, the associated sequence $\{T_{c_n}^*\}$ is subsequentially convergent to T as rectifiable normal 1-currents with respect to flat convergence, then T is an α-optimal transport path from μ to ν.*

Proof. By the lower semi-continuity of $\mathbf{M}_\alpha$ and Proposition 5.3,

$$\mathbf{M}_\alpha(T) \leq \liminf_{n\to\infty} \mathbf{M}_\alpha(T_{c_n}^*) = d_\alpha(\mu,\nu).$$

Since $\partial T = \nu - \mu$, T itself is also a transport path from μ to ν, and it holds that $d_\alpha(\mu,\nu) \leq \mathbf{M}_\alpha(T)$. As a result, T is an optimal transport path. □

References

[1] L. Ambrosio, N. Gigli. *A User's Guide to Optimal Transport. In: Modelling and Optimisation of Flows on Networks.* Lecture Notes in Mathematics, vol 2062. (2013) Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-32160-3-1.

[2] M. Bernot, V. Caselles, and J.-M. Morel. *Optimal transportation networks. Models and theory.* Lecture Notes in Mathematics, 1955. Springer, Berlin, 2009.

[3] M. Bernot, V. Caselles, and J.-M. Morel. *The structure of branched transportation networks.* Calc. Var. Partial Differential Equations, 32(3): 279-317 (2008)

[4] A. Brancolini and B. Wirth. *General transport problems with branched minimizers as functionals of 1-currents with prescribed boundary.* Calc. Var. Partial Differential Equations, 57(3):Art. 82, 39, 2018.

[5] L.A. Caffarelli and R.J. McCann, *Free boundaries in optimal transport and Monge–Ampère obstacle problems.* Ann. of Math. 171 (2010) 673–730.

[6] M. Colombo, A. De Rosa, and A. Marchese. *Improved stability of optimal traffic paths.* Calc. Var. Partial Differential Equations, 57:28, 2018.

[7] M. Colombo, A. De Rosa, A. Marchese, and S. Stuvard. *On the lower semicontinuous envelope of functionals defined on polyhedral chains.* Nonlinear Analysis, Volume 163, November 2017, Pages 201-215

[8] M. Colombo, A. De Rosa, and A. Marchese. *On the well-posedness of branched transportation.* Comm. Pure Appl. Math. 2020.

[9] G. Devillanova and S. Solimini, *On the dimension of an irrigable measure* Rend. Sem. Mat. Univ. Padova, 117 (2007), pp. 1–49.

[10] D. P. Thierry, R. Hardt. *Size minimization and approximating problems*, Calc. Var. Partial Differential Equations 17 (2003), 405-442.

[11] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York 1969.

[12] A. Figalli. *The optimal partial transport problem* Arch. Ration. Mech. Anal. 195 (2010) 533–560

[13] A. Marchese, A. Massaccesi, S. Stuvard, R. Tione. *A multi-material transport problem with arbitrary marginals.* arXiv:1807.10969v3.

[14] F. Maddalena, S. Solimini, and J.M. Morel. *A variational model of irrigation patterns.* Interfaces Free Bound., 5(4):391–415, 2003.

[15] A. Marchese and B. Wirth. *Approximation of rectifiable 1-currents and weak-$*$ relaxation of the h-mass.* Journal of Mathematical Analysis and Applications, 5(4):391–415, 2003.

[16] E. Paolini and E. Stepanov. *Optimal transportation networks as at chains.* Interfaces Free Bound., 8(4):393-436,2006.

[17] E. Paolini and E. Stepanov. *Decomposition of acyclic normal currents in a metric space.* J. Funct. Anal., 263 (11):3358-3390, 2012.

[18] P. Pegon, F. Santambrogio, and Q. Xia. *A fractal shape optimization problem in branched transport.* J. Math. Pures Appl. (9), 123:244–269, 2019.

[19] F. Santambrogio. *Optimal Transport for Applied Mathematicians- Calculus of Variations, PDEs, and Modeling*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, 2015.

[20] B. Schiffman. *α-balanced and efficient branched transport systems and their landscape functions.* PhD thesis. University of California at Davis, 2020.

[21] L. Simon. *Lectures on geometric measure theory*, volume 3 of Proceedings of the Centre for Mathematical Analysis, Australian National University. Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[22] S. K. Smirnov. *Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows.* Algebra i Analiz, 5:206-238, 1993.

[23] C. Villani. *Topics in mass transportation.* AMS Graduate Studies in Math. 58 (2003).

[24] C. Villani. *Optimal transport: old and new.* Grundlehren der mathematischen Wissenschaften, (2009)

[25] B. White, *Rectifiability of Flat Chains.* Annals of Mathematics, Vol. 150, No. 1 (Jul., 1999), pp. 165-184.

[26] Q. Xia, *Optimal paths related to transport problems*, Communications in Contemporary Mathematics, 5 (2003), 251-279.

[27] Q. Xia, *Interior Regularity of Optimal Transport Paths*, Calculus of Variations and Partial Differential Equations, 20, No. 3 (2004), 283–299.

[28] Q. Xia, A. Vershynina. *On the transport dimension of measures*, SIAM J. MATH. ANAL., Vol. 41, No. 6, (2010) pp. 2407–2430.

[29] Q. Xia, *Boundary regularity of optimal transport paths*, Adv. Calc. Var. Volume 4, Issue 2, (2011), 153–174.

[30] Q. Xia, *Motivations, ideas and applications of ramified optimal transportation*, ESAIM Math. Model. Numer. Anal. Vol. 49 Number 6 (2015), 1791–1832.

Email address: qlxia@math.ucdavis.edu
Email address: xushaofeng@ruc.edu.cn