Research Statement of Qinglan Xia

My primary interests lie in the field of geometric variational problems. So far, I have applied geometric measure theory to optimal transport problems, intersection homology theory of singular varieties, and isoperimetric minimizers with a transporting volume constraint. I have also worked on problems coming from Mathematical Biology such as description of the dynamic formation of a tree leaf, as well as modeling blood vessel structures in placentas of babies.

1 Ramifying structures in optimal transport

The transport problem introduced by Monge in 1781 has been studied in many works in the last 10 years ([1], [5], [6], [8], [10], [17], [19], [21] and others). In these works, the cost of a transport mapping or a transport plan is usually an integral of some function of the distance. However, in many real applications, the actual cost of the transport procedures is not necessarily determined by just knowing some optimal mapping from the starting position to the target position. For example in shipping two items from nearby cities to the same far away city, it may be less expensive to first bring them to a common location and put them on a single truck for most of the transport. In this case, a "Y shaped" path is preferable to a "V shaped" path. In both cases, the transport mapping is trivially the same, but the actual transport path naturally gives the total cost. In general, a ramifying structure is more cost efficient than a "linear" structure. The phenomenon of ramifying structure is very common in nature. Trees, railways, airlines, lightning, electric power supply, the circulatory system, the river channel networks, and cardiovascular systems are some common examples. This subject deserves a more general theoretical treatment and I built a model for it in a series of papers [23] [24] [25] [27] [28] [30]. In the case that it has only a single source supply, this problem is also called the irrigation problem. It has been studied independently by Maddalena, Solimini and Morel in [18] with a different (Lagrangian) formulation. In this simple case, our approaches are essentially same. Recently, many others have followed our approaches and get some nice results.

Now, I will describe my approach briefly as follows. In the general case, we consider the following problem:

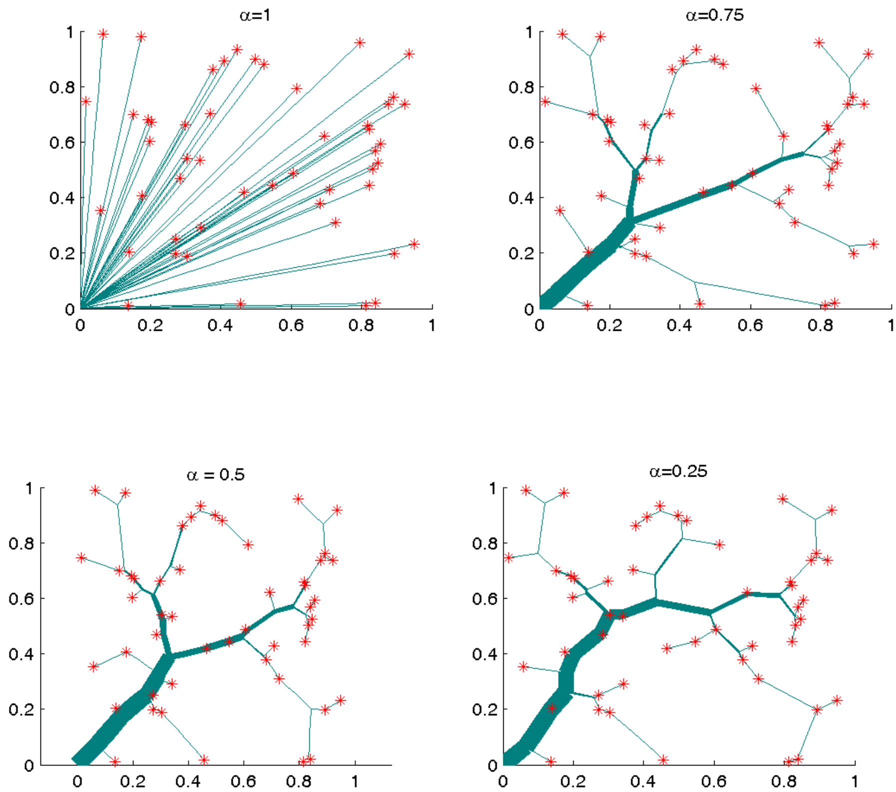
Find an optimal transport path joining two given probability measures.

To solve this problem, one needs to find a suitable category of transport paths as well as a cost functional acting on these paths. Such a category should be broad enough to give existence of an optimal transport path. Also, an optimal transport path should allow the possibility that some parts overlap in a cost efficient (maybe complicated) fashion but still enjoy some nice regularity properties. If possible, one may hope to visualize such an optimal transport path using numerical analysis and computer graphics.

The family of paths we choose in [23] is a subset of the space of vector measures with divergence being the difference of the given two measures in the sense of distributions. A transport path between two atomic measures is just a directed graph with balanced weighting at interior vertices. For arbitrary measures, a transport path between them is a vector measure given by a limit of some weighted directed graphs. The cost on each transport path is a suitably modified weighted mass of the vector measure, similar to the H mass of integral currents in [7] corresponding to a concave function H of the density. Unlike in [7], we work with vector measures (or 1 dimensional flat currents in the language of geometric measure theory) whose multiplicities are not necessarily integer valued. With this category and cost functional, the original optimal transport path problem becomes a Plateau type problem as in the study of minimal surfaces. Luckily, we have a theorem on the *existence* of an optimal transport path joining any given probability measure to another. Moreover, we consider a new distance on the space of probability measures on a fixed convex set. Such a distance is different from any of the Wasserstein distances [21] but still metrizes the weak * topology of the space of probability measures. Under this distance, the space of probability measures becomes a length space. In the end of [23], we discuss the relationship between our transport paths and transport "plans" as considered in previous works on Monge-Kantorovich problem. A compatible pair of a transport plan and a transport path contains necessary information about the actual transportation such as how, where and when the original measure is decomposed into the targeting measure on the road.

An optimal transport path has also very nice *regularity* as shown in [24], [25]. In [24], we first used a result of Brian White [22] to show that any optimal transport path with finite modified cost is rectifiable. Then we used further comparisons to establish a local finiteness property at every interior point on the support of the optimal transport path away from the boundary. Namely, in a neighborhood of such points, *the path is simply a cone consisting of a finite union of segments with suitable multiplicities.* These segments are balanced by a simple balance equation, and thus are understood very well. In general, the support of an optimal transport path may not necessarily be 1 dimensional nearby its boundary, which is the difference of the given two measures. This is because the boundary itself may even be dense in the space, as demonstrated by letting the initial Radon measure to be the Lebesgue measure. To study the regularity of an optimal transport path near its boundary, a suitable approach is to study its level sets instead. Here a level set of an optimal transport path means the set where the multiplicity on the transport path is no less than a given positive number. This idea comes from my observation of the leaf vein. The main result in [25] says that *each level set of an optimal transport path is locally concentrated on a finite union of bilipchitz curves*, which satisfies some nice properties similar to those satisfied by segments near an interior point. As a result, we conclude that optimal transport paths are indeed having very nice regularities.

Furthermore, we develop some algorithms in [30] to visualize optimal transport paths. For instance, by varying a parameter $\alpha \in [0, 1]$, an optimal transport path from 50 random points (representing an atomic probability measure) to the origin is simulated as follows



Some complicated but even nicer figures can be generated using this algorithm.

The theory of optimal transport path has applications in both pure and applied mathematics. For instance, it provides me a motivational example in [31] to study the geodesic problem in nearmetric spaces. A nearmetric space is a generalized metric space, in which the distance satisfy a relaxed triangle inequality $d(x, y) \leq c(d(x, z) + d(z, y))$ for some $c \geq 1$, rather than the usual triangle inequality. In [31], I showed that many well-known results in metric spaces (e.g. Ascoli-Arzelà theorem) still hold in those nearmetric spaces. As we know, the intrinsic metric induced by a metric has played an important role in the study of metric geometry. A nature question is: when a distance satisfies only a relaxed triangle inequality rather than the usual triangle inequality, will it still be able to induce an intrinsic *metric*? In [31], I gave a positive answer to this question. Also, using a suitable functional on "transport plans", I introduce a family of nearmetrics on the space of atomic probability measures. The associated intrinsic metrics induced by these nearmetrics coincide with the d_α metric studied early in [23]. Moreover, optimal transport paths between atomic probability measures turn out to be exactly *geodesics* in these special intrinsic metric spaces in the usual sense of metric geometry.

As optimal transport paths behave similarly to many branching structures seen in living and nonliving systems, this theory is expected to have some applications in both natural science and social science. Motivated by the idea here, we are able to study the dynamic formation of a tree leaf in [28]. More information about it will be discussed later. Also, as discussed in [29], the model of optimal transport paths provide a model for urban transport network. Under this model, we can set up an optimal urban transport network of finite total cost which provide access to all residents from their home to their destinations. The quality of the road depends on the traffic density it carries, which make it necessary to build a large highway for heavy traffic. Moreover, we provide a reasonable pricing system for an optimal transport network, under which all residents will prefer to travel to their destinations by the network.

Currently, I am exploring further applications of optimal transport paths in science with experts from various disciplines. For instance, I am collaborating with Bjorn Birnir (UC Santa Barbara), Don Turcotte (a famous geologist from UC Davis) and William Newman (UCLA) on river channel networks. Also, I am collaborating with Dr. Carolyn M Salafia(an obstetrician, NYU) and Simon Morgan on placentas. Moreover, I am looking for applications of this theory to economics with Xu from department of Economics of UC Davis. Furthermore, I am also looking for chances to collaborate with people working in other related areas such as lungs, leaves, lightning and so on.

To continue my research on this topic, I will consider the following problems.

1. River Channel Networks modeled by optimal transport paths. (In preparation, Joint with Bjorn Birnir)

2. A mathematical model of placentas using optimal transport paths. (In preparation, Joint with Simon Morgan, Carolyn M Salafia)

3. Flow of surfaces driven by optimal transportation

4. On optimal fractals

5. On the dimensional distance between probability measures

6. On the formation of trees

7. Ramified optimal transportation in metric spaces

8. Optimal transportation of rectifiable currents or varifolds (Joint with a graduate student)

2 Variational problems on singular spaces

In [26], I develop a setting for treating variational problems on stratified pseudo manifolds with singularities, such as complex projective varieties. Rather than using the ordinary homology theory on the base space, we instead use a generalized "homology theory"—the intersection homology theory, introduced by McPherson and Goresky ([12], [13]) in the early 1980's. Such a theory turns out to be more suitable than ordinary homology theory for such spaces. In variational problems, one need to take various limits of e.g. minimizing sequences, but a basic problem is that a limit of geometric intersection chains may fail to be a geometric chain; and even if it is, it may not satisfy the important *perversity* conditions of the approximating chains. This motivates our use of the rectifiable currents of geometric measure theory ([9]) with a suitably modified mass norm. We first showed how to express the intersection homology groups in terms of integer multiplicity rectifiable currents. We prove these groups are isomorphic to the usual intersection homology groups defined by geometric or subanalytic chains with the corresponding perversity conditions. Then, we use integral geometry to construct the modified mass norm. Under this modified mass norm, we found a mass minimizer in each intersection homology class. Moreover, using a result of Almgren, we prove a partial regularity theorem for these suitable mass minimizers.

To continue this project, I am considering the following problems.

1. Regularity of modified mass minimizers.

2. Detailed studies of some concise singular varieties or polyhedral pseudomanifolds.

3. Connection with Hodge theory.

4. Transport theory and intersection homology

3 Quasiperimeters with a volume constraint

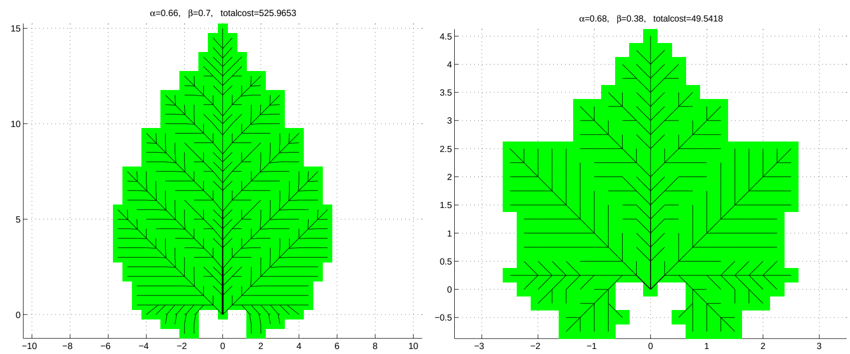
In [29], we study the regularity of the boundary of sets minimizing a quasi perimeter $T(E) = P(E, \Omega) + G(E)$ with a volume constraint. Here Ω is any open subset of $\mathbb{R}^n$ with $n \geq 2$, G is a lower semicontinuous function on sets of finite perimeter satisfies a condition that $G(E) \leq G(F) + C|E\Delta F|^{\beta}$ among all sets of finite perimeter with equal volume. The case that $\mathbf{G}(E) \equiv 0$ corresponds to a well known problem, and has been studied extensively in [14]. This problem is often encountered in the field of Capillarity Theory: liquid drops, resting on or hanging from a given surface, are the simplest ones among them. Another example is given by $\mathbf{G}(E) = \int_E H(x)\,dx$ where $H \in \mathcal{L}^p(\Omega)$, for some $p > n$, is a given function. Without a volume constraint, this is the problem of finding sets with prescribed mean curvature H, and has been studied by many authors, see e.g. in [16] by Massari. In our case, we will impose an additional volume constraint on it. Our main motivation is given by $\mathbf{G}(E) = \lambda W_p(\mathcal{L}^n\lfloor_E, \sigma\mathcal{L}^n\lfloor_\Omega)$ for some $\lambda > 0$, $\sigma \in (0,1)$ and some Wasserstein distance W_p between Radon measures. This problem comes from the study of the formation of mud cracking with Ω representing a piece of mud and E representing a crack of volume $\sigma|\Omega|$.

In [29], we showed that under the condition $\beta > 1 - \frac{1}{n}$ (which is true for all examples listed above), any volume constrained minimizer E of the quasi perimeter T has both interior points and exterior points, and E is indeed a quasi minimizer of perimeter without the volume constraint. Using a well known regularity result about quasi minimizers of perimeter, we get the classical $C^{1,\alpha}$ regularity for the reduced boundary of E.

4 Formation of a tree leaf

In [28], we build a mathematical model to understand the formation of a plant leaf. Our model is based on the idea that a leaf tends to maximize internal efficiency by developing an efficient transport system for water and nutrients. From a single cell, a leaf with various cost functionals on transport systems will grow into leaves with various shapes and various venation patterns. The efficient transport system of a tree leaf built here is a modified version of the optimal transport path ([23][24][25]) as discussed earlier. In [28], the cost functional on transport systems is controlled by two meaningful parameters. Different parameters will provide tree leaves with different shapes and different venation patterns. Unlike other models, the shapes of tree leaves are not predetermined here. Also, the aggregation of cells here is not random. It depends on the relative transportation cost from the cells to the root. Based on this model, we also provide some computer visualization of tree leaves, which resemble many known leaves including the maple and mulberry leaf.

Figure 1: Some leaves



Further Research: Here, the aggregation of cells depends on the relative transport cost from the root to the cells. This phenomenon is universal and deserves a more general theoretic treatment. Also, with the consideration of further affects like the minimization of boundary as well as the environment effects, the simulation of leaves will be more realistic.

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