Complex Analysis Lecture Notes

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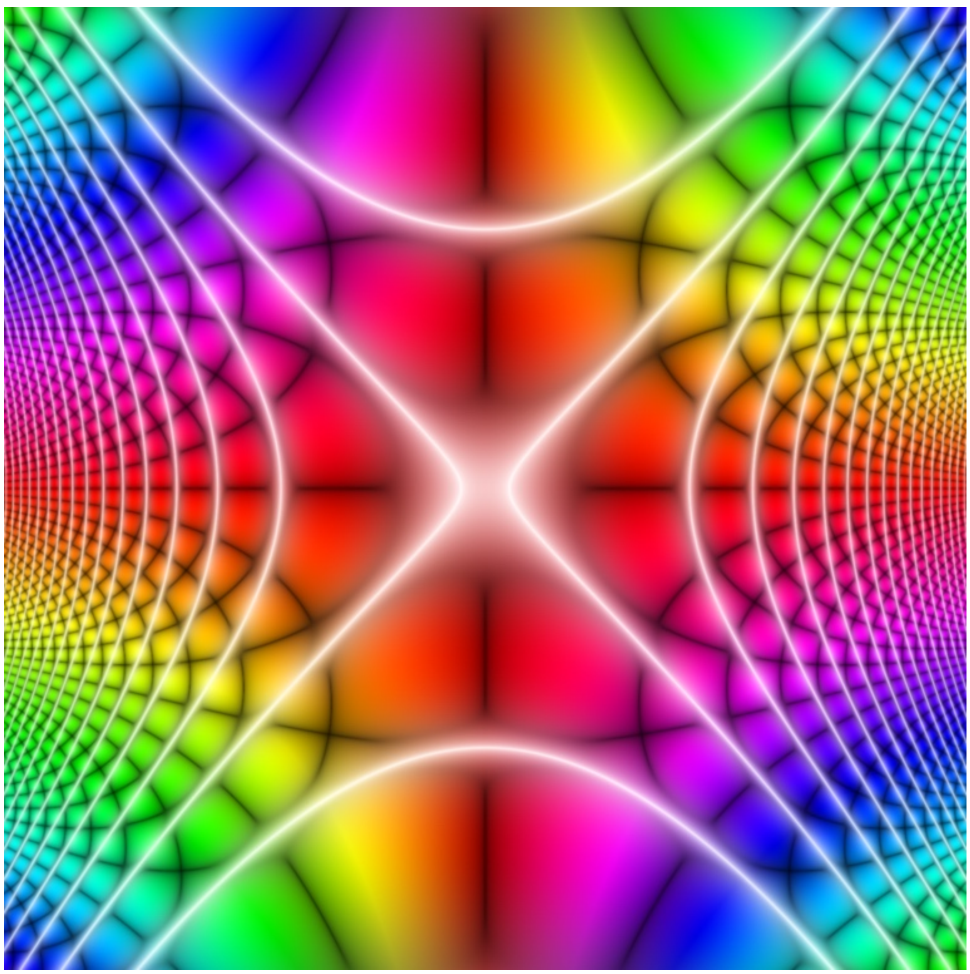

Note. I created these notes for the course *Math 205A: Complex Analysis* I taught at UC Davis in 2016 and 2018. With a few exceptions, the exposition follows the textbook *Complex Analysis* by E. M. Stein and R. Shakarchi (Princeton University Press, 2003). The notes were not heavily vetted for accuracy and may contain minor typos or errors. You can help me continue to improve them by emailing me with any comments or corrections you have.

Acknowledgements. I am grateful to Jianping Pan, Anthony Nguyen, Christopher Alexander, Brynn Caddel, Jennifer Brown, and Brad Velasquez for comments that helped me improved the notes. Figure 5 on page 23 was created by Jennifer Brown.

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Cover figure: a heat map plot of the entire function $z \mapsto z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$.

Created with `Mathematica 10` using code by Simon Woods, available at

`http://mathematica.stackexchange.com/questions/7275/how-can-i-generate-this-domain-coloring-plot`

Contents

1 Introduction

1. Complex analysis is in my opinion one of the most beautiful areas of mathematics. It has one of the highest ratios of theorems to definitions (i.e., a very low "entropy"), and lots of applications to things that seem unrelated to complex numbers, for example:

   - Solving cubic equations that have only real roots (historically, this was the motivation for introducing complex numbers by Cardano, who published the famous formula for solving cubic equations in 1543, after learning of the solution found earlier by Scipione del Ferro).

     Example. Using Cardano's formula, it can be found that the solutions to the cubic equation
     $$z^3 + 6z^2 + 9z + 3 = 0$$
     are
     $$\begin{aligned} z_1 &= 2\cos(2\pi/9) - 2, \\ z_2 &= 2\cos(8\pi/9) - 2, \\ z_3 &= 2\sin(\pi/18) - 2. \end{aligned}$$

   - Proving Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$.
   - Proving the prime number theorem: $\pi(n) \sim \frac{n}{\log n}$.
   - Proving many other asymptotic formulas in number theory and combinatorics, e.g., the Hardy-Ramanujan formula
     $$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}},$$
     where $p(n)$ is the number of integer partitions of n.
   - Evaluation of complicated definite integrals, for example
     $$\int_0^\infty \sin(t^2)\, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$
   - Solving physics problems in hydrodynamics, heat conduction, electrostatics and more.
   - Analyzing alternating current electrical networks by extending Ohm's law to electrical impedance.
   - Probability and combinatorics, e.g., the Cardy-Smirnov formula in percolation theory and the connective constant for self-avoiding walks on the hexagonal lattice.
   - It was proved in 2016 that the optimal densities for sphere packing in 8 and 24 dimensions are $\pi^4/384$ and $\pi^{12}/12!$, respectively. The proofs make spectacular use of complex analysis (and more specifically, modular forms).

Figure 1: *Print Gallery*, a lithograph by M.C. Escher which was discovered to be based on a mathematical structure related to a complex function $z \mapsto z^\alpha$ for a certain complex number α, although it was constructed by Escher purely using geometric intuition. See the paper *Artful mathematics: the heritage of M.C. Escher*, by B. de Smit and H.W. Lenstra Jr. (*Notices Amer. Math. Soc.* **50** (2003), 446–457).

- Nature uses complex numbers in Schrödinger's equation and quantum field theory. Why? No one knows.
- Conformal maps, which were used by M.C. Escher (though he had no mathematical training) to create amazing art, and used by others to better understand and even to improve Escher's work. See Fig. 1.
- Complex dynamics, e.g., the iconic Mandelbrot set. See Fig. 2.

2. In the next section I will begin our journey into the subject by illustrating a few beautiful ideas and along the way begin to review the concepts from undergraduate complex analysis.

2 The fundamental theorem of algebra

3. **The Fundamental Theorem of Algebra.** Every nonconstant polynomial $p(z)$ over the complex numbers has a root.

   I will show three proofs. Let me know if you see any "algebra"...

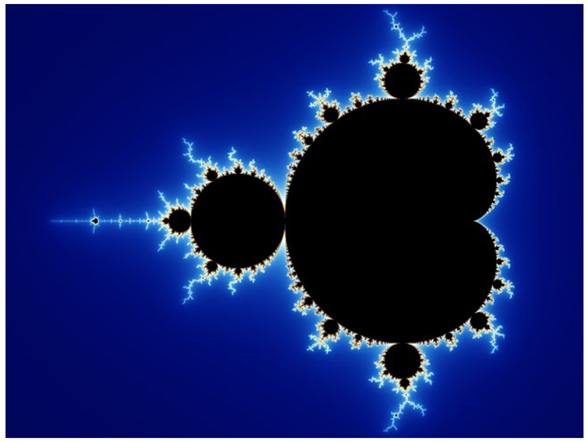


Figure 2: The Mandelbrot set. [Source: Wikipedia]

4. **Analytic proof.** Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

be a polynomial of degree n, and consider where $|p(z)|$ attains its infimum.

First, note that it can't happen as $|z| \to \infty$, since

$$|p(z)| = |z|^n \cdot (|a_n + a_{n-1} z^{-1} + a_{n-2} z^{-2} + \ldots + a_0 z^{-n}|),$$

and in particular $\lim_{|z|\to\infty} \frac{|p(z)|}{|z|^n} = |a_n|$, so for large $|z|$ it is guaranteed that $|p(z)| \geq |p(0)| = |a_0|$. Fixing some radius $R > 0$ for which $|z| > R$ implies $|p(z)| \geq |a_0|$, we therefore have that

$$m_0 := \inf_{z\in\mathbb{C}} |p(z)| = \inf_{|z|\leq R} |p(z)| = \min_{|z|\leq R} |p(z)| = |p(z_0)|$$

where $z_0 = \arg\min_{|z|\leq R} |p(z)|$, and the minimum exists because $p(z)$ is a continuous function on the disc $D_R(0)$.

Denote $w_0 = p(z_0)$, so that $m_0 = |w_0|$. We now claim that $m_0 = 0$. Assume by contradiction that it doesn't, and examine the local behavior of $p(z)$ around z_0; more precisely, expanding $p(z)$ in powers of $z - z_0$ we can write

$$p(z) = w_0 + \sum_{j=1}^{n} c_j (z - z_0)^j = w_0 + c_k (z - z_0)^k + \ldots + c_n (z - z_0)^n,$$

where k is the minimal positive index for which $c_j \neq 0$. (Exercise: why can we expand $p(z)$ in this way?) Now imagine starting with $z = z_0$ and traveling away from z_0 in some direction $e^{i\theta}$. What happens to $p(z)$? Well, the expansion gives

$$p(z_0 + re^{i\theta}) = w_0 + c_k r^k e^{ik\theta} + c_{k+1} r^{k+1} e^{i(k+1)\theta} + \ldots + c_n r^n e^{in\theta}.$$

When r is very small, the power r^k dominates the other terms r^j with $k < j \le n$, i.e.,

$$\begin{aligned} p(z_0 + re^{i\theta}) &= w_0 + r^k(c_k e^{ik\theta} + c_{k+1} r e^{i(k+1)\theta} + \ldots + c_n r^{n-k} e^{in\theta}) \\ &= w_0 + c_k r^k e^{ik\theta}(1 + g(r,\theta)), \end{aligned}$$

where $\lim_{r\to 0} |g(r,\theta)| = 0$. To reach a contradiction, it is now enough to choose θ so that the vector $c_k r^k e^{ik\theta}$ "points in the opposite direction" from w_0, that is, such that

$$\frac{c_k r^k e^{ik\theta}}{w_0} \in (-\infty, 0).$$

Obviously this is possible: take $\theta = \frac{1}{k}(\arg w_0 - \arg(c_k) + \pi)$. It follows that, for r small enough,

$$|w_0 + c_k r^k e^{ik\theta}| < |w_0|$$

and for r small enough (possibly even smaller)

$$|p(z_0 + re^{i\theta})| = |w_0 + c_k r^k e^{ik\theta}(1 + g(r,\theta))| < |w_0|,$$

a contradiction. This completes the proof. □

Exercise. Complete the last details of the proof (for which r are the inequalities valid, and why?) Note that "complex analysis" is part of "analysis" — you need to develop facility with such estimates until they become second nature.

5. **Topological proof.** Let $w_0 = p(0)$. If $w_0 = 0$, we are done. Otherwise consider the image under p of the circle $|z| = r$. Specifically:

   (a) For r very small the image is contained in a neighborhood of w_0, so it cannot "go around" the origin.

   (b) For r very large we have

   $$\begin{aligned} p(re^{i\theta}) &= a_n r^n e^{in\theta}\left(1 + \frac{a_{n-1}}{a_n} r^{-1} e^{-i\theta} + \ldots + \frac{a_0}{a_n} r^{-n} e^{-in\theta}\right) \\ &= a_n r^n e^{in\theta}(1 + h(r,\theta)) \end{aligned}$$

   where $\lim_{r\to\infty} h(r,\theta) = 0$ (uniformly in θ). As θ goes from 0 to 2π, this is a closed curve that goes around the origin n times (approximately in a circular path, that becomes closer and closer to a circle as $r \to \infty$).

   As we gradually increase r from 0 to a very large number, in order to transition from a curve that doesn't go around the origin to a curve that goes around the origin n times, there has to be a value of r for which the curve crosses 0. That means the circle $|z| = r$ contains a point such that $p(z) = 0$, which was the claim. □

6. **Remark.** The argument presented in the topological proof is imprecise. It can be made rigorous in a couple of ways — one way we will see a bit later is using Rouché's theorem and the argument principle. This already gives a hint as to the importance of subtle topological arguments in complex analysis.

7. **Remark.** The topological proof should be compared to the standard calculus proof that any odd-degree polynomial over the reals has a real root. That argument is also "topological," although much more trivial.

8. **Standard textbook proof using Liouville's theorem.** Recall:

   Liouville's theorem. A bounded entire function is constant.

   Assuming this result, if $p(z)$ is a polynomial with no root, then $1/p(z)$ is an entire function. Moreover, it is bounded, since as we noted before $\lim_{|z|\to\infty} \frac{|p(z)|}{|z|^n} = |a_n|$, so $\lim_{|z|\to\infty} 1/p(z) = 0$. It follows that $1/p(z)$ is a constant, which then has to be 0, which is a contradiction. $\square$

9. **Summary.** We saw three proofs of FTA. I like the first one best since it is elementary and doesn't use Cauchy's theorem or any of its consequences, or subtle topological concepts. Moreover, it is a "local" argument that is based on understanding how a polynomial behaves locally. The other two proofs can be characterized as "global." It is a general philosophical principle in analysis (that has analogies in other areas, such as number theory) that local arguments are easier than global ones.

3 Analyticity, conformality and the Cauchy-Riemann equations

10. **Definition.** A function $f(z)$ of a complex variable is holomorphic (a.k.a. complex-differentiable, analytic[1]) at z if

$$f'(z) := \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists.

11. Geometric meaning of holomorphicity in the case $f'(z) \neq 0$: f is locally a rotation and rescaling.

12. Interpretation: analytic functions are **conformal mappings** where $f'(z) \neq 0$: if γ_1 are two differentiable curves such that $\gamma_1(0) = \gamma_2(0) = z$, f is differentiable at z and $f'(z) \neq 0$, then, denoting $v_1 = \gamma_1'(0)$, $v_2 = \gamma_2'(0)$, $w_1 = (f\circ\gamma_1)'(0)$, $w_2 = (f\circ\gamma_2)'(0)$, we have

$$\begin{aligned}
\langle v_1, v_2\rangle &= \mathrm{Re}(v_1\overline{v_2}), \\
\langle w_1, w_2\rangle &= \langle (f'(\gamma_1(0))\gamma_1'(0)), (f'(\gamma_2(0))\gamma_2'(0))\rangle \\
&= f'(z)\overline{f'(z)}\langle v_1, v_2\rangle = |f'(z)|^2\langle v_1, v_2\rangle,
\end{aligned}$$

so, if we denote by θ (resp. φ the angle between v_1, v_2 (resp. w_1, w_2), we have

$$\cos\varphi = \frac{\langle w_1, w_2\rangle}{|w_1|\,|w_2|} = \frac{|f'(z)|^2\langle v_1, v_2\rangle}{|f'(z)v_1|\,|f'(z)v_2|} = \frac{\langle v_1, v_2\rangle}{|v_1|\,|v_2|} = \cos\theta.$$

[1]Note: some people use "analytic" and "holomorphic" with two a priori different definitions that are then proved to be equivalent; I find this needlessly confusing so I may use these two terms interchangeably.

13. Conversely, if f is conformal in a neighborhood of z then (under some additional mild assumptions) it is analytic — we will prove this below after discussing the Cauchy-Riemann equations.

14. Properties of derivatives: under appropriate assumptions (explain them precisely — see Proposition 2.2 on page 10 of [Stein-Shakarchi]),

$$\begin{aligned}(f+g)'(z) &= f'(z)+g'(z),\\ (fg)(z) &= f'(z)g(z)+f(z)g'(z),\\ \left(\frac{1}{f}\right)' &= -\frac{f'(z)}{f(z)^2},\\ \left(\frac{f}{g}\right)' &= \frac{f'(z)g(z)-f(z)g'(z)}{g(z)^2},\\ (f\circ g)'(z) &= f'(g(z))g'(z).\end{aligned}$$

15. Denote $z = x+iy$, $f = u+iv$. Note that if f is analytic at z then

$$\begin{aligned}f'(z) &= \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\\ &= \lim_{h\to 0,\ h\in\mathbb{R}}\frac{u(x+h+iy)-u(x+iy)}{h}+i\frac{v(x+h+iy)-v(x+iy)}{h}\\ &= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.\end{aligned}$$

On the other hand also

$$\begin{aligned}f'(z) &= \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\\ &= \lim_{h\to 0,\ h\in i\mathbb{R}}\frac{u(x+h+iy)-u(x+iy)}{h}+i\frac{v(x+h+iy)-v(x+iy)}{h}\\ &= \lim_{h\to 0,\ h\in\mathbb{R}}\frac{u(x+iy+ih)-u(x+iy)}{ih}+i\frac{v(x+iy+ih)-v(x+iy)}{ih}\\ &= -i\frac{\partial u}{\partial y}-i\cdot i\frac{\partial v}{\partial y}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}.\end{aligned}$$

Since these limits are equal, by equating their real and imaginary parts we get the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

16. Conversely, if $f = u+iv$ is continuously differentiable (in the real analysis sense) at $z = x+iy$ and satisfies the C-R equations there, f is analytic at z.

Proof. The assumption implies that f has a differential at z, i.e., in the notation of vector calculus, denoting $f = (u, v)$, $z = (x, y)^\top$, $\Delta z = (h_1, h_2)^\top$, we have

$$f(z+\Delta z) = \begin{pmatrix} u(z)\\ v(z)\end{pmatrix} + \begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix} + E(h_1, h_2),$$

where $E(h_1, h_2) = o(|\Delta z|)$ as $|\Delta z| \to 0$. Now, by the assumption that the C-R equations hold, we also have

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \\ -\frac{\partial u}{\partial y} h_1 + \frac{\partial u}{\partial x} h_2 \end{pmatrix},$$

which is the vector calculus notation for the complex number

$$\left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + i h_2) = \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) \Delta z.$$

So, we have shown that (again, in complex analysis notation)

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} + \frac{E(\Delta z)}{\Delta z} \right) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$

This proves that f is holomorphic at z with derivative given by $f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$. $\square$

17. Interesting consequence of C-R (1). Theorem: if $f = u + iv$ is conformal at z, continuously differentiable in the real analysis sense, and satisfies $\det J_f > 0$ (i.e., f preserves orientation as a planar map), then f is holomorphic at z.

    Proof. In the notation of the proof above, we have as before that

$$f(z + \Delta z) = \begin{pmatrix} u(z) \\ v(z) \end{pmatrix} + \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + E(h_1, h_2),$$

    where $E(h_1, h_2) = o(|\Delta z|)$ as $|\Delta z| \to 0$. The assumption is that the differential map

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

    preserves orientation and is conformal; the conclusion is that the Cauchy-Riemann equations are satisfied (which would imply that f is holomorphic at z by the result shown above. So the whole thing reduces to proving the following simple claim about 2×2 matrices:

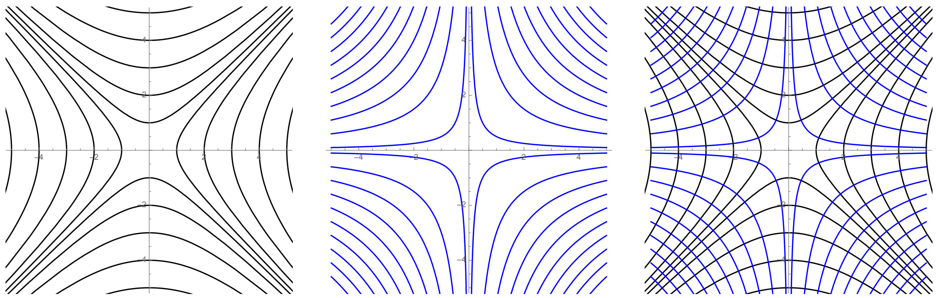
    Conformality lemma. Assume that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 real matrix. The following are equivalent:

    (a) A preserves orientation (that is, $\det A > 0$) and is conformal, that is

$$\frac{\langle Aw_1, Aw_2 \rangle}{|Aw_1|\,|Aw_2|} = \frac{\langle w_1, w_2 \rangle}{|w_1|\,|w_2|}$$

    for all $w_1, w_2 \in \mathbb{R}^2$.

    (b) A takes the form $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 > 0$.



(a) (b) (c)

Figure 3: The level curves for the (a) real and (b) imaginary parts of $z^2 = (x^2 - y^2) + i(2xy)$. (c) shows the superposition of both families of level curves.

(c) A takes the form $A = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $r > 0$ and $\theta \in \mathbb{R}$. (That is, geometrically A acts by a rotation followed by a scaling.)

Proof that (a) $\Longrightarrow$ (b). Note that both columns of A are nonzero vectors by the assumption that $\det A > 0$. Now applying the conformality assumption with $w_1 = (1,0)^\top$, $w_2 = (0,1)^\top$ yields that $(a,c) \perp (b,d)$, so that $(b,d) = \kappa(-c,a)$ for some $\kappa \in \mathbb{R} \setminus \{0\}$. On the other hand, applying the conformality assumption with $w_1 = (1,1)^\top$ and $w_2 = (1,-1)^\top$ yields that $(a+b, c+d) \perp (a-b, c-d)$, which is easily seen to be equivalent to $a^2 + c^2 = b^2 + d^2$. Together with the previous relation that implies that $\kappa = \pm 1$. So A is of one of the two forms $\begin{pmatrix} a & -c \\ c & a \end{pmatrix}$ or $\begin{pmatrix} a & c \\ c & -a \end{pmatrix}$. Finally, the assumption that $\det A > 0$ means it is the first of those two possibilities that must occur. $\square$

Exercise. Show also that (b) $\Longleftrightarrow$ (c) and that (b) $\Longrightarrow$ (a).

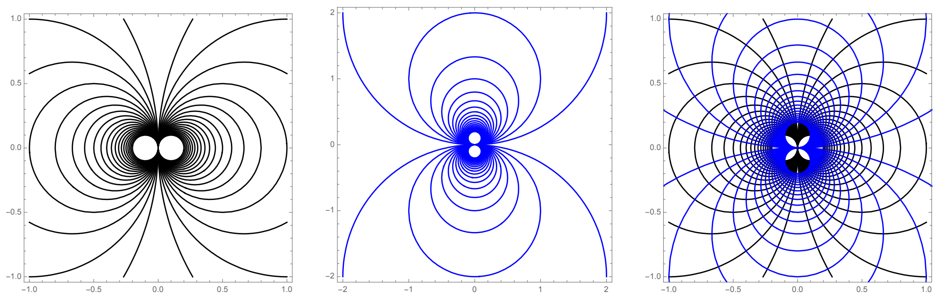
18. Interesting consequence of C-R (2): orthogonality of level curves of u and of v: if $f = u + iv$ is analytic then

$$\nabla u \cdot \nabla v = (u_x, u_y) \perp (v_x, v_y) = u_x v_x + u_y v_y = v_y v_x - v_x v_y = 0.$$

Since ∇u (resp. ∇v) is orthogonal to the level curve $\{u = c\}$ (resp. the level curve $\{v = d\}$, this proves that the level curves $\{u = c\}$, $\{v = d\}$ meet at right angles whenever they intersect.

19. Interesting consequence of C-R (3): Assume that f is analytic at z and *twice* continuously differentiable there. Then

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0. \end{aligned}$$



(a) (b) (c)

Figure 4: The level curves for the real and imaginary parts of $z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$.

Similarly (check), v also satisfies

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

That is, we have shown that u and v are **harmonic functions**. This is an extremely important connection between complex analysis and the theory of partial differential equations, which also relates to many other areas of real analysis.

20. We will later see that the assumption of twice continuous differentiability is unnecessary.

21. The Jacobian of an analytic function considered as a two-dimensional map: if $f = u + iv$ then

$$J_f = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x = u_x^2 + v_x^2 = |u_x + iv_x| = |f'(z)|^2.$$

This can also be understood geometrically (exercise: how?).

4 Power series

22. Power series are functions of a complex variable, defined by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where $(a_n)_{n=0}^{\infty}$ is a sequence of complex numbers, or more generally by

$$g(z) = f(z - z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

23. Where does this formula make sense? It is not hard to see that it converges absolutely precisely for $0 \le |z| < R$ where

$$R = \left(\limsup_{n\to\infty} |a_n|^{1/n} \right)^{-1}.$$

R is called the radius of convergence of the power series.

Proof. Assume $0 < R < \infty$ (the edge cases $R = 0$ and $R = \infty$ are left as an exercise). The defining property of R is that for all $\epsilon > 0$, we have that $|a_n| < \left(\frac{1}{R} + \epsilon\right)^n$ if n is large enough, and R is the minimal number with that property. Let $z \in D_R(0)$. Since $|z| < R$, we have $|z| \left(\frac{1}{R} + \epsilon\right) < 1$ for some fixed $\epsilon > 0$ chosen small enough. That implies that for $n > N$ (for some large enough N as a function of ϵ),

$$\sum_{n=N}^{\infty} |a_n z^n| < \sum_{n=N}^{\infty} \left[\left(\frac{1}{R} + \epsilon \right) |z| \right]^n,$$

so the series is dominated by a convergent geometric series, and hence converges. Conversely, if $|z| > R$, then, $|z| \left(\frac{1}{R} - \epsilon\right) > 1$ for some small enough fixed $\epsilon > 0$. Taking a subsequence $(a_{n_k})_{k=1}^{\infty}$ for which $|a_{n_k}| > \left(\frac{1}{R} - \epsilon\right)^{n_k}$ (guaranteed to exist by the definition of R), we see that

$$\sum_{n=0}^{\infty} |a_n z^n| \ge \sum_{k=1}^{\infty} \left[|z| \left(\frac{1}{R} - \epsilon \right) \right]^{n_k} = \infty,$$

so the power series diverges. $\square$

Exercise. Complete the argument in the extreme cases $R = 0, \infty$.

24. Another important theorem is: power series are holomorphic functions and can be differentiated termwise in the disc of convergence.

Proof. Denote

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n = S_N(z) + E_N(z), \\ S_N(z) &= \sum_{n=0}^{N} a_n z^n, \\ E_N(z) &= \sum_{n=N+1}^{\infty} a_n z^n, \\ g(z) &= \sum_{n=0}^{\infty} n a_n z^n. \end{aligned}$$

The claim is that f is differentiable on the disc of convergence and its derivative is the power series g. Since $n^{1/n} \to 1$ as $n \to \infty$, it is easy to see that $f(z)$ and $g(z)$ have the same radius of convergence. Fix z_0 with $|z| < r < R$. We wish to show that $\frac{f(z_0+h)-f(z_0)}{h}$ converges to $g(z_0)$ as $h \to 0$. Observe that

$$\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right) + \frac{E_N(z_0+h)-E_N(z_0)}{h} + (S_N'(z_0) - g(z_0))$$

The first term converges to 0 as $h \to 0$ for any fixed N. To bound the second term, fix some $\epsilon > 0$, and note that, if we assume that not only $|z_0| < r$ but also $|z_0 + h| < r$ (an assumption that's clearly satisfied for h close enough to 0) then

$$\begin{aligned}\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| &\le \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \\ &= \sum_{n=N+1}^{\infty} |a_n| \left|\frac{h\sum_{k=0}^{n-1} h^k (z_0+h)^{n-1-k}}{h}\right| \\ &\le \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},\end{aligned}$$

where we use the algebraic identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}).$$

The last expression in this chain of inequalities is the tail of an absolutely convergent series, so can be made $< \epsilon$ be taking N large enough (before taking the limit as $h \to 0$).

Third, when choosing N also make sure it is chosen so that $|S_N'(z_0) - g(z_0)| < \epsilon$, which of course is possible since $S_N'(z_0) \to g(z_0)$ as $N \to \infty$. Finally, having thus chosen N, we get that

$$\limsup_{h \to 0} \left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| \le 0 + \epsilon + \epsilon = 2\epsilon.$$

Since ϵ was an arbitrary positive number, this shows that $\frac{f(z_0+h)-f(z_0)}{h} \to g(z_0)$ as $h \to 0$, as claimed. $\square$

25. The proof above can be thought of as a special case of the following more conceptual result: if g_n is a sequence of holomorphic functions on a region Ω, and $g_n \to g$ *uniformly on closed discs* in Ω, $g_n' \to h$ uniformly on closed discs on Ω, and h is continuous, then g is holomorphic and $g' = h$ on Ω. (Exercise: prove this, and explain the connection to the previous result.)

26. **Corollary.** Analytic functions defined as power series are (complex-)differentiable infinitely many times in the disc of convergence.

27. **Corollary.** For a power series $g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ with a positive radius of convergence, we have

$$a_n = \frac{g^{(n)}(z_0)}{n!}.$$

In other words $g(z)$ satisfies Taylor's formula

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n.$$

5 Contour integrals

28. Parametrized curves: $\gamma : [a, b] \to \mathbb{C}$. Two curves $\gamma_1 : [a, b] \to \mathbb{C}$, $\gamma_2 : [c, d] \to \mathbb{C}$ are equivalent, denoted $\gamma_1 \sim \gamma_2$, if $\gamma_2(t) = \gamma_1(I(t))$ where $I : [c, d] \to [a, b]$ is a continuous, one-to-one, onto, increasing function. A "curve" is an equivalence class of parametrized curves.

    In practice, we will usually refer to parametrized curves as "curves", which is the usual abuse of terminology (that one sees in various places in mathematics), in which one blurs the distinction between equivalence classes and their members, remembering that various arguments need to "respect the equivalence" in the sense that they do not depend of the choice of member. (Meta-exercise: think of 2–3 other examples of this phenomenon.)

29. We shall assume all our curves are piecewise continuously differentiable. (More generally, one can assume them to be rectifiable, but we will not bother to develop that theory).

30. Reminder from vector calculus: line integrals of the first and second kind:

$$\int_\gamma u(z)ds = \lim_{\max_j \Delta s_j \to 0} \sum_{j=1}^n u(z_j)\Delta s_j \qquad \text{(line integral of the first kind)},$$

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\gamma P\,dx + Q\,dy = \lim_{\max_j \Delta s_j \to 0} \sum_{j=1}^n P(z_j)\Delta x_j + Q(z_j)\Delta y_j$$

$$(\mathbf{F} = (P, Q);\ \text{line integral of the second kind}).$$

31. Standard fact from calculus: the line integrals can be computed as

$$\int_\gamma u(z)\,ds = \int_a^b u(\gamma(t))|\gamma'(t)|\,dt,$$

$$\int_\gamma P\,dx + Q\,dy = \int_a^b \mathbf{F}(\gamma(\mathbf{t})) \cdot \gamma'(t)\,dt.$$

32. Reminder: the fundamental theorem of calculus for line integrals: if $F = \nabla u$ then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = u(\gamma(b)) - u(\gamma(a)).$$

33. **Important definition: contour integrals.** For a function $f = u + iv$ of a complex variable z and a curve γ, define

$$\begin{aligned}\int_\gamma f(z)\,dz &= \text{“}\int_\gamma (u + iv)(dx + idy)\text{”} \\ &= \left(\int_\gamma u\,dx - v\,dy\right) + i\left(\int_\gamma v\,dx + u\,dy\right) \\ &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \qquad \text{(contour integral)},\end{aligned}$$

$$\int_\gamma f(z)\,|dz| = \int_\gamma f(z)\,ds = \int_\gamma u\,ds + i\int_\gamma v\,ds \qquad \text{(arc length integral)}.$$

If γ is a *closed* curve (the two endpoints are the same, i.e., it satisfies $\gamma(a) = \gamma(b)$), we denote the contour integral as $\oint_\gamma f(z)\,dz$.

34. A special case of an arc length integral is the length of the curve, defined by the integral of the constant function 1:

$$\operatorname{len}(\gamma) = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\,dt.$$

35. These definitions do not depend on the parametrization of the curve. Indeed, if $\gamma_2(t) = \gamma_1(I(t))$, then (using the change of variables formula for real integrals) we have that

$$\begin{aligned}\int_{\gamma_2} f(z)\,dz &= \int_c^d f(\gamma_2(t))\gamma_2'(t)dt = \int_c^d f(\gamma_1(I(t)))(\gamma_1 \circ I)'(t)\,dt \\ &= \int_c^d f(\gamma_1(I(t)))\gamma_1'(I(t))I'(t)\,dt = \int_a^b f(\gamma_1(\tau))\gamma_1'(\tau)\,d\tau \\ &= \int_{\gamma_1} f(z)\,dz.\end{aligned}$$

Exercise. Show that the integral with respect to arc length also does not depend on the parametrization.

36. **Proposition (properties of contour integrals).** Contour integrals satisfy the following properties:

   (a) Linearity (as an operator on functions): $\int_\gamma(\alpha f(z)+\beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz$.

   (b) Linearity (as an operator on curves): if a contour Γ is a "composition" of two contours γ_1 and γ_2 (in a sense that is easy to define graphically, but tedious to write down precisely), then

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

   Similarly, if γ_2 is the "reverse" contour of γ_1, then

$$\int_{\gamma_2} f(z)\,dz = -\int_{\gamma_1} f(z)\,dz.$$

   (c) Triangle inequality:

$$\left|\int_\gamma f(z)\,dz\right| \le \int |f(z)|\,|dz| \le \operatorname{len}(\gamma) \cdot \sup_{z\in\gamma} |f(z)|.$$

Exercise. Prove this claim (part of the exercise is to define precisely the notions of "composition of curves" and "reverse curve").

37. **The fundamental theorem of calculus for contour integrals.** If γ is a curve connecting two points w_1, w_2 in a region Ω on which a function F is holomorphic, then

$$\int_\gamma F'(z)\,dz = F(w_2) - F(w_1).$$

Equivalently, to compute the contour integral $\int_\gamma f(z)\,dz$, try to find a **primitive** to f, that is, a function F such that $F'(z) = f(z)$ on all of Ω. Then $\int_\gamma f(z)\,dz$ is given by $F(w_2) - F(w_1)$.

Proof. For smooth curves,

$$\begin{aligned}\int_\gamma F'(z)\,dz &= \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b (F\circ\gamma)'(t)\,dt = (F\circ\gamma)(t)|_{t=a}^{t=b} \\ &= F(\gamma(b)) - F(\gamma(a)) = F(w_2) - F(w_1).\end{aligned}$$

For piecewise smooth curves, this is a trivial extension that is left as an exercise.

38. **Corollary.** If $f = F'$ where F is holomorphic on a region Ω (in that case we say that f **has a primitive**), γ is a closed curve in Ω, then

$$\oint_\gamma f(z)\,dz = 0.$$

39. A converse to the last claim: if $f : \Omega \to \mathbb{C}$ is a continuous function on a region Ω such that

$$\oint_\gamma f(z)\,dz = 0$$

holds for any closed contour in Ω, then f has a primitive.

Proof. Fix some $z_0 \in \Omega$. For any $z \in \Omega$, there is some path $\gamma(z_0, z)$ connecting z_0 and z (since Ω is connected and open, hence pathwise-connected — a standard exercise in topology, see the exercises in Chapter 1 of [Stein-Shakarchi]). Define

$$F(z) = \int_{\gamma(z_0,z)} f(w)\,dw.$$

By the assumption, this integral does not depend on which contour $\gamma(z_0, z)$ connecting z_0 and z was chosen, so $F(z)$ is well-defined. We now claim that F is holomorphic and its derivative is equal to f. To see this, note that

$$\begin{aligned}&\frac{F(z+h) - F(z)}{h} - f(z) \\ &\quad= \frac{1}{h}\left(\int_{\gamma(z_0,z+h)} f(w)\,dw - \int_{\gamma(z_0,z)} f(w)\,dw\right) - f(z) \\ &\quad= \frac{1}{h}\int_{\gamma(z,z+h)} f(w)\,dw - f(z) = \frac{1}{h}\int_{\gamma(z,z+h)} (f(w) - f(z))\,dw\end{aligned}$$

where $\gamma(z, z+h)$ denotes a contour connecting z and $z + h$. When $|h|$ is sufficiently small so that the disc $D(z, h)$ is contained in Ω, one can take $\gamma(z, z+h)$

as the straight line segment connecting z and $z+h$. For such h we get that

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \frac{1}{h}\operatorname{len}(\gamma(z,z+h)) \sup_{w\in D(z,h)} |f(w)-f(z)|$$
$$= \sup_{w\in D(z,h)} |f(w)-f(z)| \xrightarrow[h\to 0]{} 0,$$

by continuity of f. □

40. **Remark.** Note that with the last result, if we knew that holomorphic functions are differentiable infinitely many times (the so-called regularity theorem), we could conclude that a function that satisfies the assumption that all its contour integrals on closed contours were 0 is holomorphic. This is in fact true, and is called **Morera's theorem** (and is an important fact in complex analysis), but we won't be able to prove it until we've proved Cauchy's theorem.

41. **Example.** Compute $\oint_{|z|=1} z^n\,dz$ for $n \in \mathbb{Z}$. What do we learn from the fact that the integral is not zero for $n=-1$? (Hint: something; but what?) And what do we learn from the fact that it's 0 when $n \neq -1$? (Hint: nothing; but why?)

42. If f is holomorphic on Ω and $f' \equiv 0$ then f is a constant.

    Proof. Fix some $z_0 \in \Omega$. For any $z \in \Omega$, as we discussed above there is a path $\gamma(z_0,z)$ connecting z_0 and z. Then

    $$f(z) - f(z_0) = \int_{\gamma(z_0,z)} f'(w)\,dw = 0,$$

    hence $f(z) \equiv f(z_0)$, so f is constant. □

6 Cauchy's theorem

43. One of the central results in complex analysis is **Cauchy's theorem**:

    Cauchy's theorem. If f is holomorphic on a **simply-connected** region Ω, then for any closed curve in Ω we have

    $$\oint_\gamma f(z)\,dz = 0.$$

    The challenges are: first, to prove Cauchy's theorem for curves and regions that are relatively simple (where we do not have to deal with subtle topological considerations); second, to define what simply-connected means; third, which will take a bit longer and we won't do immediately, to extend the theorem to the most general setting.

44. Two other theorems that are closely related to Cauchy's theorem are Goursat's theorem and Morera's theorem.

45. **Goursat's theorem (a relatively easy special case of Cauchy's theorem).** If f is holomorphic on a region Ω, and T is a triangle contained in Ω, then $\oint_{\partial T} f(z)\,dz = 0$ (where T refers to the "full" triangle, and ∂T refers to its boundary considered as a curve oriented in the usual positive direction).

46. **Morera's theorem ("the converse of Cauchy's theorem").** If $f : \Omega \to \mathbb{C}$ is a continuous function on a region Ω such that

$$\oint_\gamma f(z)\,dz = 0$$

holds for any closed contour in Ω, then f is holomorphic on Ω.

47. **Proof of Goursat's theorem.** The proof idea: "localize the damage". Namely, try to translate a global statement about the integral around the triangle to a local statement about behavior near a specific point inside the triangle, which would become manageable since we have a good understanding of the local behavior of a holomorphic function near a point. If something goes wrong with the global integral, something has to go wrong at the local level, and we will show that can't happen (although technically the proof is not a proof by contradiction, conceptually I find this a helpful way to think about it).

The idea can be made more precise using *triangle subdivision*. Specifically, let $T^{(0)} = T$, and define a hierarchy of subdivided triangles

$$\begin{array}{rl}
\text{order 0 triangle:} & T^{(0)}, \\
\text{order 1 triangles:} & T^{(1)}_j, 1 \le j \le 4, \\
\text{order 2 triangles:} & T^{(2)}_{j,k}, 1 \le j,k \le 4 \\
\text{order 3 triangles:} & T^{(3)}_{j,k,\ell}, 1 \le j,k,\ell \le 4, \\
\vdots & \\
\text{order } n \text{ triangles:} & T^{(n)}_{j_1,\ldots,j_n}, 1 \le j_1,\ldots,j_n \le 4. \\
\vdots &
\end{array}$$

Here, the triangles $T^{(n)}_{j_1,\ldots,j_n}$ for $j_n = 1,2,3,4$ are obtained by subdividing the order $n-1$ triangle $T^{(n-1)}_{j_1,\ldots,j_{n-1}}$ into 4 subtriangles whose vertices are the vertices and/or edge bisectors of $T^{(n-1)}_{j_1,\ldots,j_{n-1}}$ (see Figure 1 on page 35 of [Stein-Shakarchi]). Now, given the way this subdivision was done, it is clear that we have the equality

$$\oint_{\partial T^{(n-1)}_{j_1,\ldots,j_{n-1}}} f(z)\,dz = \sum_{j_n=1}^{4} \oint_{\partial T^{(n)}_{j_1,\ldots,j_n}} f(z)\,dz$$

due to cancellation along the internal edges, and hence

$$\oint_{\partial T^{(0)}} f(z)\,dz = \sum_{j_1,\ldots,j_n=1}^{4} \oint_{\partial T^{(n)}_{j_1,\ldots,j_n}} f(z)\,dz.$$

That is, the integral along the boundary of the original triangle is equal to the sum of the integrals over all 4^n triangles of order n. Now, the crucial observation is that *one of these integrals has to have a modulus that is at least as big as the average*. That is, we have

$$\left| \oint_{\partial T^{(0)}} f(z)\,dz \right| \le \sum_{j_1,\ldots,j_n=1}^{4} \left| \oint_{\partial T^{(n)}_{j_1,\ldots,j_n}} f(z)\,dz \right| \le 4^n \left| \oint_{\partial T^{(n)}_{\mathbf{j}(n)}} f(z)\,dz \right|$$

where $\mathbf{j}(n) = (j_1^{(n)}, \ldots, j_n^{(n)})$ is some n-tuple chosen such that the second inequality holds. Moreover, we can choose $\mathbf{j}(n)$ inductively in such a way that the triangles $T^{(n)}_{\mathbf{j}(n)}$ are nested — that is, $T^{(n)}_{\mathbf{j}(n)} \subset T^{(n-1)}_{\mathbf{j}(n-1)}$ for $n \ge 1$, or equivalently $\mathbf{j}(n) = (j_1^{(n-1)}, \ldots, j_{n-1}^{(n-1)}, k)$ for some $1 \le k \le 4$ — to make this happen, choose k to be such that $\left| \oint_{\partial T^{(n)}_{(\mathbf{j}(n-1),k)}} f(z)\,dz \right|$ is greater than (or equal to) the average

$$\frac{1}{4} \sum_{d=1}^{4} \left| \oint_{\partial T^{(n)}_{(\mathbf{j}(n-1),d)}} f(z)\,dz \right|,$$

which in turn is (by induction) greater than or equal to

$$\left| \frac{1}{4} \sum_{d=1}^{4} \oint_{\partial T^{(n)}_{(\mathbf{j}(n-1),d)}} f(z)\,dz \right| = \left| \oint_{\partial T^{(n-1)}_{\mathbf{j}(n-1)}} f(z)\,dz \right| \ge 4^{-(n-1)} \oint_{\partial T} f(z)\,dz.$$

Now observe that the sequence of nested triangles shrinks to a single point. That is, we have

$$\bigcap_{n=0}^{\infty} T^{(n)}_{\mathbf{j}(n)} = \{z_0\}$$

for some point $z_0 \in T$. This is true because the diameter of the triangles goes to 0 as $n \to \infty$, so certainly there can't be two distinct points in the intersection; whereas, on the other hand, the intersection cannot be empty, since the sequence $(z_n)_{n=0}^{\infty}$ of centers (in some obvious sense, e.g., intersection of the angle bisectors) of each of the triangles is easily seen to be a Cauchy sequence (and hence a convergent sequence, by the completeness property of the complex numbers), whose limit must be an element of the intersection.

Having defined z_0, write $f(z)$ for z near z_0 as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

where

$$\psi(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0).$$

The holomorphicity of f at z_0 implies that $\psi(z) \to 0$ as $z \to z_0$. Denote by $d^{(n)}$ the diameter of $T^{(n)}_{\mathbf{j}(n)}$ and by $p^{(n)}$ its perimeter. Each subdivision shrinks both the diameter and perimeter by a factor of 2, so we have

$$d^{(n)} = 2^{-n} d^{(0)}, \qquad p^{(n)} = 2^{-n} p^{(0)}.$$

It follows that

$$\left| \int_{\partial T^{(n)}_{\mathbf{j}(n)}} f(z)\,dz \right| = \left| \int_{\partial T^{(n)}_{\mathbf{j}(n)}} f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0)\,dz \right|$$
$$= \left| \int_{\partial T^{(n)}_{\mathbf{j}(n)}} \psi(z)(z-z_0)\,dz \right| \le p^{(n)} d^{(n)} \sup_{z \in T^{(n)}_{\mathbf{j}(n)}} |\psi(z)|$$

Finally, combining this with the relationship between $\left|\oint_{\partial T^{(0)}} f(z)\,dz\right|$ and $|\int_{\partial T^{(n)}_{\mathbf{j}(n)}} f(z)\,dz|$, we get that

$$\left| \int_{\partial T^{(0)}} f(z)\,dz \right| \le p^{(0)} d^{(0)} \sup_{z \in T^{(n)}_{\mathbf{j}(n)}} |\psi(z)| \xrightarrow[n\to\infty]{} 0,$$

which finishes the proof. ☐

48. **Goursat's theorem for rectangles.** The theorem is also true when we replace the word "triangle" with "rectangle", since a rectangle can be decomposed as the union of two triangles, with the contour integral around the rectangle being the sum of the integrals around the two triangles.

49. **Corollary: existence of a primitive for a holomorphic function on a disc.** If f is holomorphic on a disc D, then $f = F'$ for some holomorphic function F on D.

    Proof. The idea is similar to the proof of the result in 39 above. If we knew that all contour integrals of f around closed contours vanished, that result would give us what we want. As it is, we know this is true but only for triangular contours. How can we make use of that information? [Stein-Shakarchi] gives a clever approach in which the contour $\gamma(z_0, z)$ is comprised of a horizontal line segment followed by a vertical line segment. Then one shows in three steps, each involving a use of Goursat's theorem (see Figure 4 on page 38 of [Stein-Shakarchi]), that $F(z_0+h) - F(z_0)$ is precisely the contour integral over the line segment connecting z_0 and $z_0 + h$. From there the theorem proceeds in exactly the same way as before. ☐

50. **Corollary: Cauchy's theorem for a disc.** If f is holomorphic on a disc, then $\oint_\gamma f\,dz = 0$ for any closed contour γ in the disc.

    Proof. f has a primitive, and we saw that that implies the claimed consequence. ☐

51. **Cauchy's theorem for a region enclosed by a "toy contour".** Repeat the same ideas, going from Goursat's theorem, to the fact that the function has a primitive, to the fact that its contour integrals along closed curves vanish. The difficulty as the toy contour gets more complicated is to make sure that the geometry works out when proving the existence of the primitive — see for example the (incomplete) discussion of the case of "keyhole contours" on pages 40–41 of [Stein-Shakarchi].

7 Consequences of Cauchy's theorem

52. **Cauchy's integral formula.** If f is holomorphic on a region Ω, and $C = \partial D$ is a circular contour contained in Ω, then

$$\frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw = \begin{cases} f(z) & \text{if } z \in D, \\ 0 & \text{if } z \in \Omega\setminus\overline{D}, \\ \text{undefined} & \text{if } z \in C. \end{cases}$$

Proof. The case when $z \notin \overline{D}$ is covered by Cauchy's theorem in a disc, since in that case the function $w \mapsto f(w)/(w-z)$ is holomorphic in an open set containing $\overline{D}$. It remains to deal with the case $z \in D$. In this case, denote by z_0 the center of the circle C. The idea is now to consider instead the integral

$$\oint_{\Gamma_{\epsilon,\delta}} F_z(w)\,dw = \oint_{\Gamma_{\epsilon,\delta}} \frac{f(w)}{w-z}\,dw,$$

where $\Gamma_{\epsilon,\delta}$ is a so-called **keyhole contour**, namely a contour comprised of a large circular arc around z_0 that is a subset of the circle C, and another smaller circular arc of radius ϵ centered at z, with two straight line segments connecting the two circular arcs to form a closed curve, such that the width of the "neck" of the keyhole is δ (think of δ as being much smaller than ϵ); see Fig. 5. Note that the function $F_z(w)$ is holomorphic inside the region enclosed by $\Gamma_{\epsilon,\delta}$, so Cauchy's theorem for toy contours gives that

$$\oint_{\Gamma_{\epsilon,\delta}} F_z(w)\,dw = 0.$$

As $\delta \to 0$, the two parts of the integral along the "neck" of the contour $\Gamma_{\epsilon,\delta}$ cancel out in the limit because F_z is continuous, and hence uniformly continuous, on the compact set $\overline{D}\setminus D(z,\epsilon)$. So we can conclude that

$$\oint_C F_z(w)\,dw = \oint_{|w-z|=\epsilon} F_z(w)\,dw.$$

The next, and final, step, is to take the limit as $\epsilon \to 0$ of the right-hand side of this equation, after first decomposing $F_z(w)$ as

$$F_z(w) = \frac{f(w)-f(z)}{w-z} + f(z)\cdot\frac{1}{w-z},$$

Integrating each term separately, we have for the first term

$$\begin{aligned} \left|\oint_C \frac{f(w)-f(z)}{w-z}\,dw\right| &\le 2\pi\epsilon\cdot \sup_{|w-z|=\epsilon} \frac{|f(w)-f(z)|}{\epsilon} \\ &= 2\pi \sup_{|w-z|=\epsilon} |f(w)-f(z)| \xrightarrow[\epsilon\to 0]{} 0, \end{aligned}$$

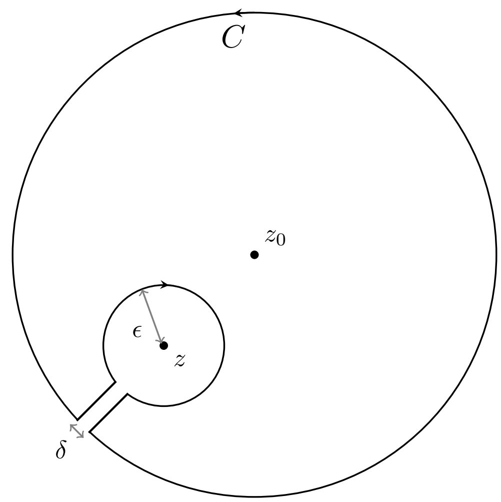


Figure 5: The keyhole contour used in the proof of Cauchy's integral formula.

by continuity of f; and for the second term,

$$\oint_{|w-z|=\epsilon} f(z)\cdot\frac{1}{w-z}\,dw = f(z)\oint_{|w-z|=\epsilon}\frac{1}{w-z}\,dw = 2\pi i f(z)$$

(by a standard calculation, see 41 above). Putting everything together gives $\oint_C \frac{1}{2\pi i}F_z(w)\,dw = f(z)$, which was the formula to be proved. $\square$

53. **Example:** in the case when z is the *center* of the circle $C = \{w : |w-z| = r\}$, Cauchy's formula gives that

$$f(z) = \frac{1}{2\pi}\oint_{|w-z|=r} f(w)\,\frac{dw}{i(w-z)} = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})dt.$$

In other words, we have proved:

Theorem (the mean value property for holomorphic functions). The value of a holomorphic function f at z is equal to the average of its values around a circle $|w-z| = r$ (assuming it is holomorphic on an open set containing the disc $|w-z| \le r$).

54. Considering what the mean value property means for the real and imaginary parts of $f = u + iv$, which are harmonic functions, we see that they in turn also satisfy a similar mean value property:

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + r\cos t, y + r\sin t)\,dt.$$

This is in fact true for all harmonic functions — a fact, known as the **mean value property for harmonic functions**, that can be proved separately using PDE/real analysis methods, or derived from the above considerations by proving that every harmonic function in a disc is the real part of a holomorphic function.

55. **Cauchy's integral formula, extended version.** Under the same assumptions, f is infinitely differentiable, and for $z \in D$ its derivatives $f^{(n)}(z)$ are given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}}\,dw.$$

The fact that holomorphic functions are differentiable infinitely many times is referred to by [Stein-Shakarchi] as the **regularity theorem**.

Proof. The key observation is that the expression on the right-hand side of Cauchy's integral formula for $f(z)$ (which is the case $n = 0$ of the "extended" version) can be differentiated under the integral sign. To make this precise, let $n \geq 1$, and assume inductively that we proved

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^n}\,dw.$$

Then

$$\begin{aligned}\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h}& \\ &= \frac{(n-1)!}{2\pi i} \oint_C f(w) \cdot \frac{1}{h}\left(\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n}\right) dw.\end{aligned}$$

It is easily seen that as $h \to 0$, the divided difference $\frac{(w-z-h)^{-n}-(w-z)^{-n}}{h}$ converges to $n(w-z)^{-n-1}$, *uniformly over* $w \in C$. (The same claim without the uniformity is just the rule for differentiation of a power function; to get the uniformity one needs to "go back to basics" and repeat the elementary algebraic calculation that was originally used to derive this power rule — an illustration of the idea that in mathematics it is important not just to understand results but also the techniques used to derive them.) It follows that we can go to the limit $h \to 0$ in the above integral representation, to get

$$f^{(n)}(z) = \frac{(n-1)!}{2\pi i} \oint_C f(w) n (w-z)^{-n-1}\,dz,$$

which is precisely the nth case of Cauchy's integral formula. $\square$

56. **Proof of Morera's theorem.** We already proved that if f is a function all of whose contour integrals over closed curves vanish, then f has a primitive F. The regularity property now implies that the derivative $F' = f$ is also holomorphic, hence f is holomorphic, which was the claim of Morera's theorem. $\square$

57. As another immediate corollary to Cauchy's integral formula, we now get an extremely useful family of inequalities that bounds a function $f(z)$ and its derivatives at some specific point $z \in C$ in terms of the values of the function on the boundary of a circle centered at z.

Cauchy inequalities. For f holomorphic in a region Ω that contains the closed disc $\overline{D_R(z)}$, we have

$$|f^{(n)}(z)| \leq n! R^{-n} \sup_{z \in \partial D_R(z)} |f(z)|$$

(where $\partial D_R(z)$ refers to the circle of radius R around z).

58. **Analyticity of holomorphic functions.** If f is holomorphic in a region Ω that contains a closed disc $\overline{D_R(z_0)}$, then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

that is convergent for all $z \in D_R(z_0)$, where (of course) $a_n = f^{(n)}(z_0)/n!$.

Proof. The idea is that Cauchy's integral formula gives us a representation of $f(z)$ as a weighted "sum" (=an integral, which is a limit of sums) of functions of the form $z \mapsto (w-z)^{-1}$. Each such function has a power series expansion since it is, more or less, a geometric series, so the sum also has a power series expansion.

To make this precise, write

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0} \cdot \frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} \\ &= \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty} (w-z_0)^{-n-1} (z-z_0)^n.\end{aligned}$$

This is a power series in $z-z_0$ that, assuming $w \in C_R(z_0)$, converges absolutely for all z such that $|z-z_0| < R$ (that is, for all $z \in D_R(z_0)$). Moreover the convergence is clearly uniform in $w \in C_R(z_0)$. Since infinite summations that are absolutely and uniformly convergent can be interchanged with integration operations, we then get, using the extended version of Cauchy's integral formula, that

$$\begin{aligned}f(z) &= \frac{1}{2\pi i} \oint_{C_R(z_0)} \frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i} \oint_{C_R(z_0)} f(w) \sum_{n=0}^{\infty} (w-z_0)^{-n-1} (z-z_0)^n\,dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_R(z_0)} f(w)(w-z_0)^{n-1}\,dw \right) (z-z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n,\end{aligned}$$

which is precisely the expansion we were after. $\square$

59. **Remark.** In the above proof, if we only knew the simple ($n=0$) case of Cauchy's integral formula (and in particular didn't know the regularity theorem that follows from the extended case of this formula), we would still conclude from the penultimate expression in the above chain of equalities that $f(z)$ has a power series expansion of the form $\sum_n a_n (z-z_0)^n$, with $a_n = (2\pi i)^{-1} \int_{C_R(z_0)} f(w)(w-z)^{-n-1}$. It would then follow from earlier results we proved that $f(z)$ is differentiable infinitely many times, and that $a_n = f^{(n)}(z_0)/n!$, which would again give the extended version of Cauchy's integral formula.

60. **Liouville's theorem.** A bounded entire function is constant.

Proof. An easy application of the (case $n = 1$ of the) Cauchy inequalities gives upon taking the limit $R \to \infty$ that $f'(z) = 0$ for all z, hence, as we already proved, f must be constant.

61. **Theorem.** If f is holomorphic on a region Ω, and $f = 0$ for z in a set containing a limit point in Ω, then f is identically zero on Ω.

Proof. If the limit point is $z_0 \in \Omega$, that means there is a sequence $(w_k)_{k=0}^{\infty}$ of points in Ω such that $f(w_k) = 0$ for all n, $w_k \to z$, and $w_k \neq z_0$ for all k. We know that in a neighborhood of z_0, f has a convergent power series expansion. If we assume that f is not identically zero in a neighborhood of z_0, then we can write the power series expansion as

$$\begin{aligned} f(z) &= \sum_{n=0} a_n (z-z_0)^k = \sum_{n=m}^{\infty} a_n (z-z_0)^{\infty} \\ &= a_m (z-z_0)^m \sum_{n=0}^{\infty} \frac{a_{n+m}}{a_m} (z-z_0)^n = a_m (z-z_0)^m (1 + g(z)), \end{aligned}$$

where m is the smallest index such that $a_m \neq 0$, and where $g(z) = \sum_{n=1}^{\infty} \frac{a_{n+m}}{a_m} (z - z_0)^n$ is a holomorphic function in the neighborhood of z_0 that satisfies $g(z_0) = 0$. It follows that for all k,

$$a_m (w_k - z_0)^m (1 + g(w_k)) = f(w_k) = 0,$$

but for large enough k this is impossible, since $w_k - z_0 \neq 0$ for all k and $g(w_k) \to g(z_0) = 0$ as $k \to \infty$.

The conclusion is that f is identically zero at least in a neighborhood of z_0. But now we claim that that also implies that f is identically zero on all of Ω, because Ω is a region (open and connected). More precisely, denote by U the set of points $z \in \Omega$ such that f is equal to 0 in a neighborhood of z. It is obvious that U is open, hence also relatively open in Ω since Ω itself is open; U is also closed, by the argument above; and U is nonempty (it contains z_0, again by what we showed above). It follows that $U = \Omega$ by the well-known characterization of a connected topological space as a topological space that has no "'clopen" (closed and open) sets other than the empty set and the entire space.

An alternative way to finish the proof is the following. For every point $z \in \Omega$, let $r(z)$ be the radius of convergence of the power series expansion of f around z. Thus the discs $\{D_{r(z)}(z) : z \in \Omega\}$ form an open covering of Ω. Take $w \in \Omega$ (with z_0 being as above), and take a path $\gamma : [a, b] \to \Omega$ connecting z_0 and w (it exists because Ω is open and connected, hence pathwise-connected). The open covering of Ω by discs is also an open covering of the compact set $\gamma[a, b]$ (the range of γ). By the Heine-Borel property of compact sets, it has a finite subcovering $\{D_{r(z_j)}(z_j) : j = 0, \ldots, m\}$ (where we take $w = z_{m+1}$. The proof above shows that f is identically zero on $D_{r(z_0)}(z_0)$, and also shows that if we know f is zero on $D_{r(z_j)}(z_j)$ then we can conclude that it is zero on the next disc

$D_{r(z_{j+1})}(z_{j+1})$. It follows that we can get all the way to the last disc $D_{r(w)}(w)$. In particular, $f(w) = 0$, as claimed. $\square$

62. **Remark.** The above result is also sometimes described under the heading **zeros of holomorphic functions are isolated**, since it can be formulated as the following statement: if f is holomorphic on Ω, is not identically zero on Ω, and $f(z_0) = 0$ for $z_0 \in \Omega$, then for some $\epsilon > 0$, the punctured neighborhood $D_\epsilon(z_0) \setminus \{z_0\}$ of z_0 contains no zeros of f. In other words, the set of zeros of f contains only isolated points.

63. **Remark 2.** The condition that the limit point z_0 be in Ω is needed. Note that it *is* possible to have a sequence $z_n \to z_0$ of points in Ω such that $f(z_n) = 0$ for all n. For example, consider the function $e^{1/z} - 1$ — it has zeros in every neighborhood of $z_0 = 0$.

64. **Corollary.** If f, g are holomorphic on a region Ω, and $f(z) = g(z)$ for z in a set with limit point in Ω (e.g., an open disc, or even a sequence of points z_n converging to some $z \in \Omega$), then $f \equiv g$ everywhere in Ω.

    Proof. Apply the previous result to $f - g$. $\square$

65. The previous result is usually reformulated slightly as the following conceptually important result:

    Principle of analytic continuation. If f is holomorphic on a region Ω, and f_+ is holomorphic on a bigger region $\Omega_+ \supset \Omega$ and satisfies $f(z) = f_+(z)$ for all $z \in \Omega$, then f_+ is the *unique* such extension, in the sense that if $\tilde{f_+}$ is another function with the same properties then $f_+(z) = \tilde{f_+}(z)$ for all $z \in \Omega_+$.

66. **Example.** In real analysis, we learn that "formulas" such as

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots = \frac{1}{2},$$
$$1 + 2 + 4 + 8 + 16 + 32 + \ldots = -1$$

    don't have any meaning. However, in the context of complex analysis one can in fact make perfect sense of such identities, using the principle of analytic continuation! Do you see how? We will also learn later in the course about additional such amusing identities, the most famous of which being

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12},$$
$$1 - 2 + 3 - 4 + \ldots = \frac{1}{4}.$$

    Such supposedly "astounding" formulas have attracted a lot of attention recently, being the subject of a popular Numberphile video, a New York Times article, a discussion on the popular math blog by Terry Tao, a Wikipedia article, a discussion on Mathematics StackExchange, and more.

67. **A "toy" (but stil very interesting) example of analytic continuation: removable singularities**. A point $z_0 \in \Omega$ is called a **removable singularity**

of a function $f : \Omega \to \mathbb{C} \cup \{\text{undefined}\}$ if f is holomorphic in a punctured neighborhood of Ω, is not holomorphic at z_0, but its value at z_0 can be redefined so as to make it holomorphic at z_0.

Riemann's removable singularities theorem. If f is holomorphic in Ω except at a point $z_0 \in \Omega$ (where it may be undefined, or be defined but not known to be holomorphic or even continuous). Assume that f is bounded in a punctured neighborhood $D_r(z_0) \setminus \{z_0\}$ of z_0. Then f can be extended to a holomorphic function $\tilde{f}$ on all of Ω by defining (or redefining) its value at z_0 appropriately.

Proof. Fix some disc $D = D_R(z_0)$ around z_0 whose closure is contained in Ω. The idea is to prove that the Cauchy integral representation formula

$$f(z) = \frac{1}{2\pi i} \oint_{C_R(z_0)} \frac{f(w)}{w-z}\, dw =: \tilde{f}(z)$$

is satisfied for all $z \in D \setminus \{z_0\}$. Once we show this, we will set $\tilde{f}(z_0)$ to be defined by the same integral representation, and it will be easy to see that that gives the desired extension.

To prove that the representation above holds, consider a "double keyhole" contour $\Gamma_{\epsilon,\delta}$ that surrounds most of circle $C = \partial D$ but makes diversions to avoid the points z_0 and z, circling them in the negative direction around most of a circle of radius ϵ. After applying a limiting argument similar to the one used in the proof of Cauchy's integral formula, we get that

$$\frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} = \frac{1}{2\pi i} \oint_{C_\epsilon(z)} \frac{f(w)}{w-z} + \frac{1}{2\pi i} \oint_{C_\epsilon(z_0)} \frac{f(w)}{w-z}.$$

On the right-hand side, the first term is $f(z)$ by Cauchy's integral formula (since f is known to be holomorphic on an open set containing $\overline{D_\epsilon(z)}$). The second term can be bounded in magnitude using the assumption that f is bounded in a neighborhood of z_0; more precisely, we have

$$\left| \oint_{C_\epsilon(z_0)} \frac{f(w)}{w-z} \right| \le 2\pi\epsilon \sup_{w \in C_\epsilon(z_0)} |f(w)| \cdot \frac{1}{|z-z_0|-\epsilon} \xrightarrow[\epsilon \to 0]{} 0.$$

Thus by taking the limit as $\epsilon \to 0$ we obtain precisely the desired representation for f.

Finally, once we have the integral representation $\tilde{f}$ (defined only in terms of the values of $f(w)$ for $w \in C_R(z_0)$), the fact that this defines a holomorphic function on all of $D_R(z_0)$ is easy to see, and is something we implicitly were aware of already. For example, the relevant argument (involving a direct manipulation of the divided differences $\frac{1}{h}(\tilde{f}(z+h) - \tilde{f}(z))$) appeared in the proof of the extended version of Cauchy's integral formula. Another approach is to show that integrating $\tilde{f}$ over closed contours gives 0 (which requires interchanging the order of two integration operations, which will not be hard to justify) and then use Morera's theorem. The details are left as an exercise. $\square$

68. **Definition: Uniform convergence on compact subsets.** If $f, (f_n)_{n=0}^{\infty}$ are holomorphic functions on a region Ω, we say that the sequence f_n converges to f **uniformly on compact subsets** if for any compact set $K \subset \Omega$, $f_n(z) \to f_n(z)$ uniformly on K.

69. **Theorem.** If $f_n \to f$ uniformly on compact subsets in Ω and f_n are holomorphic, then f is holomorphic, and $f_n' \to f'$ uniformly on compact subsets in Ω.

    Proof. The fact that f is holomorphic is an easy consequence of a combination of Cauchy's theorem and Morera's theorem. More precisely, note that for each closed disc $\overline{D} = \overline{D_r(z_0)} \subset \Omega$ we have $f_n(z) \to f$ uniformly on $\overline{D}$. In particular, for each curve γ whose image is contained in the open disc $D = D_r(z_0)$,

    $$\int_\gamma f_n(z)\,dz \xrightarrow[n\to\infty]{} \int_\gamma f(z)\,dz.$$

    By Cauchy's theorem, the integrals in this sequence are all 0, so $\int_\gamma f(z)\,dz$ is also zero. Since this is true for all such γ, by Morera's theorem f is holomorphic on D. This was true for any disc in Ω, and holomorphicity is a local property, so in other words f is holomorphic on all of Ω.

    Next, let $D = D_r(z_0)$ be a disc whose closure $\overline{D}$ satisfies $\overline{D} \subset \Omega$. for $z \in D$ we have by Cauchy's integral formula that

    $$\begin{aligned} f_n'(z) - f'(z) &= \frac{1}{2\pi i}\oint_{\partial D} \frac{f_n(w)}{(w-z)^2}\,dw - \frac{1}{2\pi i}\oint_{\partial D} \frac{f(w)}{(w-z)^2}\,dw \\ &= \frac{1}{2\pi i}\oint_{\partial D} \frac{f_n(w)-f(w)}{(w-z)^2}\,dw. \end{aligned}$$

    It is easy to see therefore that $f_n'(z) \to f'(z)$ as $n \to \infty$, *uniformly as z ranges on the disc $D_{r/2}(z_0)$*, since $f_n(w) \to f(w)$ uniformly for $w \in \partial D \subset \overline{D}$, and $|w-z|^{-2} \le (r/2)^{-2}$ for $z \in D_{r/2}(z_0)$, $w \in \partial D$.

    Finally, let $K \subset D$ be compact. For each $z \in K$ let $r(z)$ be the radius of a disc $D_{r(z)}(z)$ around z whose closure is contained in Ω. The family of discs $\{D_z = D_{r(z)/2}(z) : z \in \Omega\}$ is an open covering of K, so by the Heine-Borel property of compact sets it has a finite subcovering $D_{z_1}, \ldots, D_{z_n}$. We showed that $f_n'(z) \to f'(z)$ uniformly on every D_{z_j}, so we also have uniform convergence on their union, which contains K, so we get that $f_n' \to f'$ uniformly on K, as claimed. $\square$

8 Zeros, poles, and the residue theorem

70. **Definition (zeros).** z_0 is a zero of a holomorphic function f if $f(z_0) = 0$.

71. **Lemma/Definition.** If f is a holomorphic function on a region Ω that is not identically zero and z_0 is a zero of f, then f can be represented in the form

    $$f(z) = (z - z_0)^m g(z)$$

in some neighborhood of z_0, where $m \geq 1$ and g is a holomorphic function in that neighborhood such that $g(z) \neq 0$. The number m is determined uniquely and is called the **order** of the zero z_0, i.e, z_0 will be described as "a zero of order m."

Remark 1. In the case when z_0 is *not* a zero of f, the same representation holds with $m = 0$ (and $g = f$), so in certain contexts one may occasionally say that z_0 is a zero of order 0.

Remark 2. A zero of order 1 is called a **simple zero**.

Proof. Power series expansions – this is similar to the argument used in the proof that zeros of holomorphic functions are isolated. That is, write the power series expansion (known to converge in a neighborhood of z_0)

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=m}^{\infty} a_n (z - z_0)^n \\ &= (z - z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^n =: (z - z_0)^m g(z), \end{aligned}$$

where m is the smallest index ≥ 0 such that $a_m \neq 0$. This gives the desired representation. On the other hand, given a representation of this form, expanding $g(z)$ as a power series shows that m has to be the smallest index of a nonzero coefficient in the power series expansion of $f(z)$, which proves the uniqueness claim. $\square$

72. **Definition (poles).** If f is defined and holomorphic in a punctured neighborhood of a point z_0, we say that it has a **pole of order** m at z_0 if the function $h(z) = 1/f(z)$ (defined to be 0 at z_0) has a zero of order m at z_0. A pole of order 1 is called a **simple pole**.

    Remark. As with the case of zeros, one can extend this definition in an obvious way to define a notion of a "pole of order 0". If $f(z)$ is actually holomorphic and nonzero at z_0 (or has a removable singularity at z_0 and can be made holomorphic and nonzero by defining its value at z_0 appropriately), we define the order of the pole as 0 and consider f to have a pole of order 0 at z_0.

73. **Lemma.** f has a pole of order m at z_0 if and only if it can be represented in the form

    $$f(z) = (z - z_0)^{-m} g(z)$$

    in a punctured neighborhood of z_0, where g is holomorphic in a neighborhood of z_0 and satisfies $g(z_0) \neq 0$.

    Proof. Apply the previous lemma to $1/f(z)$. $\square$

74. **Theorem.** If f has a pole of order m at z_0, then it can be represented uniquely as

    $$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z - z_0} + G(z)$$

    where G is holomorphic in a neighborhood of z_0.

Proof. This follows immediately on expressing $f(z)$ as $(z - z_0)^{-m} g(z)$ as in the previous lemma and separating the power series expansion of $g(z)$ into the powers $(z - z_0)^k$ with $0 \leq k \leq m - 1$ and the powers with $k \geq m$. $\square$

75. **Definition.** The expansion $\frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z-z_0}$ in the above representation is called the **principal part** of f at the pole z_0. The coefficient a_{-1} is called the **residue of f at** z_0 and denoted $\mathrm{Res}_{z_0}(f)$.

76. **Exercise.** The definitions of the order of a zero and a pole can be consistently unified into a single definition of the **(generalized) order of a zero**, where if f has a pole of order m at z_0 then we say instead that f has a zero of order $-m$. Denote the order of a zero of f at z_0 by $\mathrm{ord}_{z_0}(f)$. With these definitions, prove that

$$\mathrm{ord}_{z_0}(f + g) \geq \min\left(\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\right)$$

(can you give a useful condition when equality holds?), and that

$$\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g).$$

77. **The residue theorem (simple version).** Assume that f is holomorphic in a region containing a closed disc $\overline{D}$, except for a pole at $z_0 \in D$. Then

$$\oint_{\partial D} f(z)\, dz = 2\pi i \,\mathrm{Res}_{z_0}(f).$$

Proof. By the standard argument involving a keyhole contour, we see that the circle $C = \partial D$ in the integral can be replaced with a circle $C_\epsilon = C_\epsilon(z_0)$ of a small radius $\epsilon > 0$ around z_0, that is , we have

$$\oint_{\partial D} f(z)\, dz = \oint_{C_\epsilon} f(z)\, dz.$$

When ϵ is small enough, inside C_ϵ we can use the decomposition

$$f(z) = \sum_{k=-m}^{-1} a_k (z - z_0)^k + G(z)$$

for f into its principal part and a remaining holomorphic part. Integrating termwise gives 0 for the integral of $G(z)$, by Cauchy's theorem; 0 for the integral powers $(z - z_0)^k$ with $-m \leq k \leq -2$, by a standard computation; and $2\pi i a_{-1} = 2\pi i \,\mathrm{Res}_{z_0}(f)$ for the integral of $r(z - z_0)^{-1}$, by the same standard computation. This gives the result. $\square$

78. **The residue theorem (extended version).** Assume that f is holomorphic in a region containing a closed disc $\overline{D}$, except for a finite number of poles at $z_1, \ldots, z_N \in D$. Then

$$\oint_{\partial D} f(z)\, dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}_{z_k}(f).$$

Proof. The idea is the same, except one now uses a contour with multiple keyholes to deduce after a limiting argument that

$$\oint_{\partial D} f(z)\, dz = \sum_{k=1}^{N} \oint_{C_\epsilon(z_k)} f(z)\, dz$$

for a small enough ϵ, and then proceeds as before.

(Note: The above argument seems slightly dishonest to me, since it relies on the assertion that a multiple keyhole contour with arbitrary many keyholes is a "toy contour"; while this is intuitively plausible, it will be undoubtedly quite difficult to think of, and write, a detailed proof of this argument.) $\square$

79. **The residue theorem (version for general toy contours).** Assume that f is holomorphic in a region containing a toy contour γ (oriented in the positive direction) and the region R_γ enclosed by it, except for poles at the points $z_1, \ldots, z_N \in R_\gamma$. Then

$$\oint_{\gamma} f(z)\, dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}_{z_k}(f).$$

Proof. Again, construct a multiple keyhole version of the same contour γ (assuming that one can believably argue that the resulting contour is still a toy contour), and then use a limiting argument to conclude that

$$\oint_{\gamma} f(z)\, dz = \sum_{k=1}^{N} \oint_{C_\epsilon(z_k)} f(z)\, dz,$$

for a small enough ϵ. Then proceed as before. $\square$

9 Meromorphic functions, holomorphicity at ∞ and the Riemann sphere

80. **Definition (meromorphic functions).** A meromorphic function on a region Ω is a function $f : \Omega \to \mathbb{C} \cup \{\text{undefined}\}$ such that f is holomorphic except for an isolated set of poles.

81. **Definition (holomorphicity at ∞).** Let $U \subset \mathbb{C}$ be an open set containing the complement $\mathbb{C} \setminus \overline{D_R(0)}$ of a closed disc around 0. A function $f : U \to \mathbb{C}$ is **holomorphic at** ∞ if $g(z) = f(1/z)$ (defined on a neighborhood $D_{1/R}(0)$ of 0) has a removable singularity at 0. In that case we define $f(\infty) = g(0)$ (the value that makes g holomorphic at 0).

82. **Definition (order of a zero/pole at ∞).** Let $U \subset \mathbb{C}$ be an open set containing the complement $\mathbb{C} \setminus \overline{D_R(0)}$ of a closed disc around 0. We say that a function $f : U \to \mathbb{C}$ has a zero (resp. pole) of order m at ∞ if $g(z) = f(1/z)$ has a zero (resp. pole) at $z = 0$, after appropriately defining the value of g at 0.

83. Conceptually, it is useful to think of meromorphic functions as holomorphic functions with range in the Riemann sphere $\hat{\mathbb{C}}$. Let's define what that means.

84. **Definition.** The Riemann sphere (a.k.a. the **extended complex numbers**) is the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, equipped with the following additional structure:

    - *Topologically*, we think of $\hat{\mathbb{C}}$ as the 1**-point compactification** of $\mathbb{C}$; that is, we add to $\mathbb{C}$ an additional element ∞ (called "the point at infinity") and say that the neighborhoods of ∞ are the complements of compact sets in $\mathbb{C}$. This turns $\hat{\mathbb{C}}$ into a topological space in a simple way.

    - *Geometrically*, we can identify $\hat{\mathbb{C}}$ with an actual sphere embedded in $\mathbb{R}^3$, namely
    $$S^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + \left(z - \tfrac{1}{2}\right)^2 = \tfrac{1}{2} \right\}.$$
    The identification is via **stereographic projection**, given explicitly by the formula
    $$(X,Y,Z) \in S^2 \longmapsto \begin{cases} x + iy = \frac{X}{1-Z} + i\frac{Y}{1-Z} & \text{if } (X,Y,Z) \neq (0,0,1), \\ \infty & \text{if } (X,Y,Z) = (0,0,1). \end{cases}$$
    See page 88 in [Stein-Shakarchi] for a more detailed explanation. One can check that this geometric identification is a homeomorphism between S^2 (equipped with the obvious topology inherited from $\mathbb{R}^3$) and $\hat{\mathbb{C}}$ (with the 1-point compactification topology defined above).

    - *Analytically*, the above definition of what it means for a function on a neighborhood of ∞ to be holomorphic at ∞ provides a way of giving $\hat{\mathbb{C}}$ the structure of a **Riemann surface** (the simplest case of a manifold with a complex-analytic structure). The details can be found in many textbooks and online resources, and we will not discuss them in this course.

85. With the above definitions, the concept of a meromorphic function $f : \Omega \to \mathbb{C} \cup \{\text{undefined}\}$ can be seen to coincide with the notion of a holomorphic function $f : \Omega \to \hat{\mathbb{C}}$ — that is, the underlying concept of the definition is still holomorphicity, but it concerns functions taking values in $\hat{\mathbb{C}}$, a different Riemann surface, instead of $\mathbb{C}$.

86. **Definition (classification of singularities).** If a function $f : \Omega \to \hat{\mathbb{C}} \cup \{\text{undefined}\}$ is holomorphic in a punctured neighborhood $D_r(z_0) \setminus \{z_0\}$ of z_0, we say that f has a **singularity** at z_0 if f is not holomorphic at z_0. We classify singularities into three types, two of which we already defined:

    - **Removable singularities**: when f can be made holomorphic at z_0 by defining or redefining its value at z_0.

    - **Poles**.

    - Any singularity that is not removable or a pole is called an **essential singularity**.

For a function defined on a neighborhood of ∞ that is not holomorphic at ∞, we say that f has a singularity at ∞, and classify the singularity as a removable singularity, a pole, or an essential singularity, according to the type of singularity that $z \mapsto f(1/z)$ has at $z = 0$.

87. **Theorem (Casorati-Weierstrass theorem on essential singularities).** If f is holomorphic in a punctured neighborhood $D_r(z_0) \setminus \{z_0\}$ of z_0 and has an essential singularity at z_0, the image $f(D_r(z_0) \setminus \{z_0\})$ of the punctured neighborhood under f is dense in $\mathbb{C}$.

    Proof. Assume that the closure $\overline{f(D_r(z_0) \setminus \{z_0\})}$ does not contain a point $w \in \mathbb{C}$. Then $g(z) = \frac{1}{f(z)-w}$ is a function that's holomorphic and bounded in $D_r(z_0) \setminus \{z_0\}$. By Riemann's removable singularity theorem, its singularity at z_0 is removable, so we can assume it is holomorphic at z_0 after defining its value there. It then follows that
$$f(z) = w + \frac{1}{g(z)}$$
    has either a pole or a removable singularity at z_0, depending on whether $g(z_0) = 0$ or not. □

10 The argument principle

88. **Definition.** The **logarithmic derivative** of a holomorphic function $f(z)$ is $f'(z)/f(z)$.

89. **Lemma.** The logarithmic derivative of a product is the sum of the logarithmic derivatives. That is,
$$\frac{\left(\prod_{k=1}^n f_k\right)'}{\prod_{k=1}^n f_k} = \sum_{k=1}^n \frac{f_k'(z)}{f_k(z)}.$$
    Proof. Show this for $n = 2$ and proceed by induction. □

90. **Theorem (the argument principle).** Assume that f is meromorphic in a region Ω containing a closed disc $\overline{D}$, such that f has no poles on the circle ∂D. Denote its zeros and poles inside D by $z_1, \ldots, z_n$, where z_k is a zero of order $m_k = \operatorname{ord}_{z_k}(f)$ (in the sense mentioned above, where $m_k = m$ is a positive integer if z_k is a zero of order m, and $m_k = -m$ is a negative integer if z_k is a pole of order m). Then
$$\begin{aligned}\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^n m_k \\ &= [\text{total number of zeros of } f \text{ inside } D] \\ &\quad - [\text{total number of poles of } f \text{ inside } D].\end{aligned}$$
    Proof. Define
$$g(z) = \prod_{k=1}^n (z - z_k)^{-m_k} f(z).$$

Then $g(z)$ is meromorphic on Ω, has no singularities zeros on ∂D, and inside D it has no poles or zeros, only removable singularities at $z_1, \ldots, z_n$ (so after redefining its values at these points we can assume it is holomorphic on D). It follows that

$$f(z) = \prod_{k=1}^{n} (z - z_k)^{m_k} g(z).$$

Taking the logarithmic derivative of this equation gives that

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{m_k}{z - z_k} + \frac{g'(z)}{g(z)}.$$

The result now follows by integrating this equation and using the residue theorem (the term $g'(z)/g(z)$ is holomorpic on D so does not contribute anything to the integral). $\square$

91. By similar reasoning, the theorem also holds when the circle is replaced by a toy contour γ.

92. **Intuitive explanation for the argument principle.** Note that the integral in the argument principle can be represented as

$$\begin{aligned}\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz &= \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t)}\, dt = \frac{1}{2\pi i} \int_a^b \frac{(f \circ \gamma)'(t)}{(f \circ \gamma)(t)}\, dt \\ &= \frac{1}{2\pi i} \int_{f\circ\gamma} \frac{1}{w}\, dw,\end{aligned}$$

that is, an integral of dw/w over the contour $f \circ \gamma$ — the image of γ under f. Now note that the differential form dw/w has a special geometric meaning in complex analysis, namely we have

$$\frac{dw}{w} = \text{“}d\,(\log w)\text{''} = \text{“}d\,(\log|w| + i \arg w)\text{''}.$$

We put these expressions in quotes since the logarithm and argument are not single-valued functions so it needs to be explained what such formulas mean. However, at least $\log|w|$ is well-defined for a curve that does not cross 0, so when integrating over the closed curve $f \circ \gamma$, the real part is zero by the fundamental theorem of calculus. The imaginary part (which becomes real after dividing by $2\pi i$) can be interpreted intuitively as the **change in the argument over the curve** — that is, initially at time $t = a$ one fixes a specific value of $\arg w = \arg \gamma(a)$; then as t increases from $t = a$ to $t = b$, one tracks the increase or decrease in the argument as one travels along the curve $\gamma(t)$; if this is done correctly (i.e., in a continuous fashion), at the end the argument must have a well-defined value. Since the curve is closed, the total change in the argument must be an integer multiple of 2π, so the division by $2\pi i$ turns it into an integer.

Of course, this explanation also explains the name “the argument principle,” which may sound arbitrary and uninformative when you first hear it.

93. **Connection to winding numbers.** What the above reasoning shows is that in general, an integral of the form

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w}\,dw$$

over a closed curve γ that does not cross 0 carries the meaning of "the total number of times the curve γ goes around the origin," with the number being positive if the curve goes in the positive direction around the origin; negative if the curve goes in the negative direction around the origin; or zero if there is no net change in the argument. This number is more properly called the **winding number** of f around $w = 0$ (also sometimes referred to as the **index** of the curve around 0), and denoted

$$\mathrm{Ind}_0(f) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z}\,dz.$$

More generally, one can define the winding number at $z = z_0$ as the number of times a curve γ winds around an arbitrary point z_0, which (it is easy to see) will be given by

$$\mathrm{Ind}_{z_0}(f) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}\,dz,$$

assuming that γ does not cross z_0.

Note that winding number is a *topological* concept of planar geometry that can be considered and studied without any reference to complex analysis; indeed, in my opinion that is the correct approach. It is possible, and not especially difficult, to define it in purely topological terms without mentioning contour integrals, and then show that the complex analytic and topological definitions coincide. Try to think what such a definition might look like.

94. **Rouché's theorem.** Assume that f, g are holomorphic on a region Ω containing a circle $\gamma = C$ and its interior (or, more generally, a toy contour γ and the region U enclosed by it). If $|f(z)| > |g(z)|$ for all $z \in \gamma$ then f and $f+g$ have the same number of zeros inside the region U.

Proof. Define $f_t(z) = f(z) + tg(z)$ for $t \in [0,1]$, and note that $f_0 = f$ and $f_1 = f + g$, and that the condition $|f(z)| > |g(z)|$ on γ implies that f_t has no zeros on γ for any $t \in [0,1]$. Denote

$$n_t = \frac{1}{2\pi i}\oint_\gamma \frac{f_t'(z)}{f_t(z)}\,dz,$$

which by the argument principle is the number of "generalized zeros" (zeros or poles, counting multiplicities) of f_t in U. In particular, the function $t \mapsto n_t$ is integer-valued. If we also knew that it was continuous, then it would have to be constant (by the easy exercise: any integer-valued continuous function on an interval $[a,b]$ is constant), so in particular we would get the desired conclusion that $n_1 = n_0$.

To prove continuity of n_t, note that the function $g(t,z) = f_t'(z)/f_t(z)$ is continuous, hence also uniformly continuous, on the compact set $[0,1] \times \gamma$. For $s,t \in [0,1]$ satisfying $|t-s| < \delta$, we can write

$$\begin{aligned} |n_t - n_s| &\le \frac{1}{2\pi i} \oint_\gamma |g(t,z) - g(s,z)| \cdot |dz| \\ &\le \frac{1}{2\pi i} \operatorname{len}(\gamma) \sup\{|g(u,z) - g(v,z)| \,:\, z \in \gamma,\ u,v \in [0,1],\ |u-v| < \delta\}. \end{aligned}$$

Given $\epsilon > 0$, we can choose δ that ensures that this expression is $< \epsilon$ if $|t-s| < \delta$, by the uniform continuity. This is precisely what is needed to show that $t \mapsto n_t$ is continuous. $\square$

95. **Intuitive explanation for Rouché's theorem: "walking the dog".** The following intuitive explanation for Rouché's theorem appears in the book *Visual Complex Analysis* by Tristan Needham. Imagine that you are walking in a large empty park containing at some "origin" point 0 a large pole (in the English sense of a metal post sticking out of the ground, not the complex analysis sense). You start at some point X and go for a walk along some curve, ending back at the same starting point X. Let N denote your winding number around the pole at the origin — that is, the total number of times you went around the pole, with the appropriate sign.

Now imagine that you also have a dog that is walking alongside you in some erratic path that is sometimes close to you, sometimes less close. As you traverse your curve C_1, the dog walks along on its own curve C_2, which also begins and ends in the same place. Let M denote the *dog's* winding number around the pole at the origin. Can we say that $N = M$? The answer is: *yes, we can, provided that we know the dog's distance to you was always less than your distance to the pole.* To see this, imagine that you had the dog on a leash of variable length; if the distance condition was not satisfied, it would be possible for the dog to reach the pole and go in a short tour around it while you were still far away and not turning around the pole, causing an entanglement of the leash with the pole.

Amazingly, the above scenario maps in a precise way to Rouché's theorem, using the following dictionary: the curve $f \circ \gamma$ represents your path; the curve $(f+g)\circ\gamma$ represents the dog's path; $g \circ \gamma$ represents the vector pointing from you to the dog; the condition $|f| > |g|$ along γ is precisely the correct condition that the dog stays closer to you than your distance to the pole; and the conclusion that the two winding numbers are the same is precisely the theorem's assertion that f and $f+g$ have the same number of generalized zeros in the region U enclosed by γ (see the discussion above regarding the connection between the integral $(2\pi i)^{-1} \oint_\gamma f'/f\,dz$ and the winding number of $f \circ \gamma$ around 0).

Exercise. Spend a few minutes thinking about the above correspondence and make sure you understand it. You will probably forget the technical details of the proof of Rouché's theorem in a few weeks or months, but I hope you will remember this intuitive explanation for a long time.

96. As another small cryptic remark to think about, the proof of Rouché's theorem given above can be thought of as an argument about the invariance of a certain integral under the homotopy between two curves. Can you see how?

11 Applications of Rouché's theorem

97. **Topological proof of the fundamental theorem of algebra.** At the beginning of the course we discussed the topological proof of FTA. We can now make that argument precise using Rouché's theorem. The details will be assigned as a homework exercise.

98. **The open mapping theorem.** Holomorphic functions are **open mappings**, that is, they map open sets to open sets.

    Proof. Let f be holomorphic in a region Ω, $z_0 \in \Omega$, and denote $w_0 = f(z_0)$. What we need to show is that the image of any neighborhood $D_\epsilon(z_0)$ for $\epsilon > 0$ contains a neighborhood $D_\delta(z_0)$ of w_0 for some $\delta > 0$. Fixing w (visualized as being near w_0), denote

    $$h(z) = f(z) - w = (f(z) - w_0) + (w_0 - w) =: F(z) + G(z).$$

    The idea is now to apply Rouché's theorem to $F(z)$ and $G(z)$. Fix $\epsilon > 0$ small enough so that the disc $D_\epsilon(z_0)$ is contained in Ω and does not contain solutions of the equation $f(z) = w_0$ other than z_0 (this is possible, by the property that zeros of holomorphic functions are isolated). Defining

    $$\delta = \inf\{|f(z) - w_0| \,:\, z \in \overline{D_\epsilon(z_0)}\},$$

    we therefore have that $\delta > 0$ and $|f(z) - w_0| \geq \delta$ for z on the circle $|z - z_0| = \epsilon$. That means that under the assumption that $|w - w_0| < \delta$ (i.e., if w is assumed to be close enough to w_0), the condition $|F(z)| > |G(z)|$ in Rouché's theorem will be satisfied for $z \in D_\epsilon(z_0)$. The conclusion is that the equation $h(z) = 0$ (or equivalently $f(z) = w$) has the same number in solutions (in particular, at least one solution) as the equation $f(z) = w_0$ in the disc $D_\epsilon(z_0)$. This was precisely the claim to be proved. □

99. **Corollary: the maximum modulus principle.** If f is a non-constant holomorphic function on a region Ω, then $|f|$ cannot attain a maximum on Ω.

    Proof. Trivial exercise. □

12 Simply-connected regions and the general version of Cauchy's theorem

100. **Definition (homotopy of curves).** Given a region $\Omega \subset \mathbb{C}$, two parametrized curves $\gamma_1, \gamma_2 : [0,1] \to \Omega$ (assumed for simplicity of notation to be defined on $[0,1]$) are said to be **homotopic (with fixed endpoints)** if $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and there exists a function $F : [0,1] \times [0,1] \to \Omega$ such that

(a) F is continuous.

(b) $F(0,t) = \gamma_1(t)$ for all $t \in [0,1]$.

(c) $F(1,t) = \gamma_2(t)$ for all $t \in [0,1]$.

(d) $F(s,0) = \gamma_1(0)$ for all $s \in [0,1]$.

(e) $F(s,1) = \gamma_1(1)$ for all $s \in [0,1]$.

The map F is called a homotopy between γ_1 and γ_2. Intuitively, for each $s \in [0,1]$ the function $t \mapsto F(s,t)$ defines a curve connecting the two endpoints $\gamma_1(0)$, $\gamma_1(1)$. As s grows from 0 to 1, this family of curves transitions in a continuous way between the curve γ_1 and γ_2, with the endpoints being fixed in place.

101. **Exercise.** Prove that the relation of being homotopic is an equivalence relation.

102. **Definition (simply-connected regions).** A region Ω is called **simply-connected** if any two curves γ_1, γ_2 in Ω with the same endpoints are homotopic.

103. **Remark.** A common alternative way to define the notion of homotopy of curves is for closed curves, where the endpoints are not fixed but the homotopy must keep the curves closed as it is deforming them. The definition of a simply-connected region then becomes a region in which any two closed curves are homotopic. It is not hard to show that those two definitions are equivalent.

104. **Theorem.** If f is a holomorphic function on a region Ω, and γ_1,γ_2 are two curves on Ω with the same endpoints that are homotopic, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

105. **Proof.** See pages 93–95 in [Stein-Shakarchi]. □

106. **Cauchy's theorem (general version).** If f is holomorphic on a simply-connected region Ω, then for any closed curve in Ω we have

$$\oint_\gamma f(z)\,dz = 0.$$

107. **Proof.** Assume for simplicity that γ is parametrized as a curve on $[0,1]$. Then it can be thought of as the concatenation of two curves γ_1 and $-\gamma_2$, where $\gamma_1 = \gamma_{|[0,1/2]}$ and γ_2 is the "reverse" of the curve $\gamma_{|[1/2,1]}$. Note that γ_1 and γ_2 have the same endpoints. By the invariance property of contour integrals under homotopy proved above, we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1-\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = 0.$$

108. **Corollary.** Any holomorphic function on a simply-connected region has a primitive.

13 The logarithm function

109. The logarithm function can be defined as

$$\log z = \log |z| + i \arg z$$

on any region Ω that does not contain 0 and where one can make a consistent, smoothly varying choice of $\arg z$ as z ranges over Ω. It is easy to see that this formula gives an inverse to the exponential function.

For example, if

$$\Omega = \mathbb{C} \setminus (-\infty, 0]$$

(the "slit complex plane" with the negative real axis removed), we can set

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z$$

where $\operatorname{Arg} z$ is set to take values in $(-\pi, \pi)$. This is called the **principal branch of the logarithm**. However, sometimes we may want to consider the logarithm function on more strange or complicated regions. When can this be made to work? The answer is: precisely when Ω is simply-connected.

110. **Theorem.** Assume that Ω is a simply-connected region with $0 \notin \Omega$, $1 \in \Omega$. Then there exists a function $F(z) = \log_\Omega(z)$ with the properties:

(a) F is holomorphic in Ω.

(b) $e^{F(z)} = z$ for all $z \in \Omega$.

(c) $F(r) = \log r$ (the usual logarithm for real numbers) for all real numbers $r \in \Omega$ sufficiently close to 1.

Proof. We define F as a primitive function of the function $z \mapsto 1/z$, that is, as

$$F(z) = \int_1^z \frac{dw}{w},$$

where the integral is computed along a curve γ connecting 1 to z. By the general version of Cauchy's theorem for simply-connected regions, this integral is independent of the choice of curve. As we have already seen, this function is holomorphic and satisfies $F'(z) = 1/z$ for all $z \in \Omega$. It follows that

$$\frac{d}{dz}\left(ze^{-F(z)}\right) = e^{-F(z)} - zF'(z)e^{-F(z)} = e^{-F(z)}(1 - z/z) = 0,$$

so $ze^{-F(z)}$ is a constant function. Since its value at $z = 1$ is 1, we see that $e^{F(z)} = z$, as required. Finally, for real r close to 1 we have that $F(z) = \int_1^r \frac{dw}{w} = \log r$, which can be seen by taking the integral to be along the straight line segment connecting 1 and r. $\square$

111. **Exercise.** Prove that the principal branch of the logarithm has the Taylor series expansion

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n \qquad (|z| < 1).$$

112. **Exercise.** Modify the proof above to prove the existence of a branch of the logarithm function in any simply-connected region Ω not containing 0, without the assumption that $1 \in \Omega$. In what way is the conclusion weakened in that case?

113. **Exercise.** Explain in what sense the logarithm functions $F(z) = \log_\Omega(z)$ satisfying the properties proved in the theorem above (and its generalization described in the previous exercise) are unique.

114. **Exercise.** Prove the following generalization of the logarithm construction above: if f is a holomorphic function on a simply-connected region Ω, and $f \neq 0$ on Ω, then there exists a holomorphic function g on Ω, referred to as a branch of the logarithm of f, satisfying

$$e^{g(z)} = f(z).$$

115. **Definition (power functions and nth roots).** On a simply-connected region Ω we can now define the power function $z \mapsto z^\alpha$ for an arbitrary $\alpha \in \mathbb{C}$ by setting

$$z^\alpha = e^{\alpha \log z}.$$

In the special case $\alpha = 1/n$ this has the meaning of the nth root function $z \mapsto z^{1/n}$, which satisfies

$$(z^{1/n})^n = \left(e^{\frac{1}{n}\log z}\right)^n = e^{n\frac{1}{n}\log z} = e^{\log z} = z.$$

Note that if $f(z) = z^{1/n}$ is one choice of an nth root function, then for any $0 \le k \le n-1$, the function $g(z) = e^{2\pi i k/n} f(z)$ will be another function satisfying $g(z)^n = z$. Conversely, it is easy to see that those are precisely the possible choices for an nth root function.

14 The Euler gamma function

116. The **Euler gamma function** (often referred to simply as the gamma function) is one of the most important special functions in mathematics. It has applications to many areas, such as combinatorics, number theory, differential equations, probability, and more, and is probably the most ubiquitous transcendental function after the "elementary" transcendental functions (the exponential function, logarithms, trigonometric functions and their inverses) that one learns about in calculus. It is a natural meromorphic function of a complex variable that extends the factorial function to non-integer values. In complex analysis it is particularly important in connection with the theory of the Mellin transform (a version of the Fourier transform associated with the multiplicative group of positive real numbers).

117. Most textbooks define the gamma function in one way and proceed to prove several other equivalent representations of it. However, the truth is that none

of the representations of the gamma function is more fundamental or "natural" than the others. So, it seems more logical to start by simply listing the various formulas and properties associated with it, and then proving that the different representations are equivalent and that the claimed properties hold.

Theorem (the Euler gamma function). There exists a unique function $\Gamma(s)$ of a complex variable s that has the following properties:

(a) $\Gamma(s)$ is a meromorphic function on $\mathbb{C}$.

(b) **Connection to factorials:** $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.

(c) **Important special value:** $\Gamma(1/2) = \sqrt{\pi}$.

(d) **Integral representation:**

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \qquad (\operatorname{Re} s > 0).$$

(e) **Hybrid series-integral representation:**

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-x} x^{s-1}\, dx \qquad (s \in \mathbb{C}).$$

(f) **Infinite product representation:**

$$\Gamma(s)^{-1} = s e^{\gamma s} \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right) e^{-s/n} \qquad (s \in \mathbb{C}),$$

where $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right) \doteq 0.577215$ is the Euler-Mascheroni constant.

(g) **Limit of finite products representation:**

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\, n^s}{s(s+1)\cdots(s+n)} \qquad (s \in \mathbb{C}).$$

(h) **Zeros:** the gamma function has no zeros (so $\Gamma(s)^{-1}$ is an entire function).

(i) **Poles:** the gamma function has poles precisely at the non-positive integers $s = 0, -1, -2, \ldots$, and is holomorphic everywhere else. The pole at $s = -n$ is a simple pole with residue

$$\operatorname{Res}_{s=-n}(\Gamma) = \frac{(-1)^n}{n!} \qquad (n = 0, 1, 2, \ldots).$$

(j) **Functional equation:**

$$\Gamma(s+1) = s\,\Gamma(s) \qquad (s \in \mathbb{C}).$$

(k) **The reflection formula** (a surprising connection to trigonometry):

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad (s \in \mathbb{C}).$$

118. To begin the proofs, let's take the formula

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$$

as our working definition of $\Gamma(s)$. This improper integral is easily seen to converge absolutely for $\mathrm{Re}(s) > 0$, since

$$\left| \int_0^\infty e^{-x} x^{s-1}\, dx \right| \le \int_0^\infty e^{-x} |x^{s-1}|\, dx = \int_0^\infty e^{-x} x^{\mathrm{Re}(s)-1}\, dx.$$

I leave it as an exercise to check (or read the easy explanation in the book) that the function it defines is holomorphic in that region.

119. Next, perform an integration by parts, to get that, again for $\mathrm{Re}(s) > 0$, we have

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^s\, dx = -e^{-x} x^s \Big|_{x=0}^{x=\infty} + \int_0^\infty e^{-x} s x^{s-1}\, dx = s\,\Gamma(s),$$

which is the functional equation.

120. Combining the trivial evaluation $\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$ with the functional equation shows by induction that $\Gamma(n+1) = n!$.

121. The special value $\Gamma(1/2) = \sqrt{\pi}$ follows immediately by a change of variable $x = u^2$ in the integral and an appeal to the standard Gaussian integral $\int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{\pi}$:

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2}\, dx = \int_0^\infty e^{-u^2} 2\, du = \int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{\pi}.$$

122. The functional equation can now be used to perform an analytic continuation of $\Gamma(s)$ to a meromorphic function on $\mathbb{C}$: for example, we can define

$$\Gamma_1(s) = \frac{\Gamma(s+1)}{s},$$

which is a function that is holomorphic on $\mathrm{Re}(s) > -1, s \neq 0$ and coincides with $\gamma(s)$ for $\mathrm{Re}(s) > 0$. By the principle of analytic continuation this provides a unique extension of $\Gamma(s)$ to the region $\mathrm{Re}(s) > -1$. Because of the factor $1/s$ and the fact that $\Gamma(1) = 1$ we also see that $\Gamma_1(s)$ has a simple pole at $s = 0$ with residue 1.

Next, for $\mathrm{Re}(s) > -2$ we define

$$\Gamma_2(s) = \frac{\Gamma_1(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)},$$

a function that is holomorphic on $\mathrm{Re}(s) > -2$, $s \neq 0, -1$, and coincides with $\Gamma_1(s)$ for $\mathrm{Re}(s) > -1$, $s \neq 0$. Again, this provides an analytic continuation of $\Gamma(s)$ to that region. The factors $1/s(s+1)$ show that $\Gamma_2(s)$ has a simple pole at $s = -1$ with residue -1.

Continuing by induction, having defined an analytic continuation $\Gamma_{n-1}(s)$ of $\Gamma(s)$ to the region $\mathrm{Re}(s) > -n+1$, $s \neq 0, -1, -2, \ldots, -n+2$, we now define

$$\Gamma_n(s) = \frac{\Gamma_{n-1}(s+1)}{s} = \ldots = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)}.$$

By inspection we see that this gives a meromorphic function in $\mathrm{Re}(s) > -n$ whose poles are precisely at $s = -n+1, \ldots, 0$ and have the claimed residues.

123. An alternative way to perform the analytic continuation is to separate the integral defining $\Gamma(s)$ into

$$\Gamma(s) = \int_0^1 e^{-x} x^{s-1}\, dx + \int_1^\infty e^{-x} x^{s-1}\, dx$$

and to note that the integral over $[1, \infty)$ converges (and defines a holomorphic function of s) for *all* $s \in \mathbb{C}$, and the integral over $[0, 1]$ can be computed by expanding e^{-x} as a power series in x and integrating term by term. That is, for $\mathrm{Re}(s) > 0$ we have

$$\begin{aligned}\int_0^1 e^{-x} x^{s-1}\, dx &= \int_0^1 \sum_{n=0}^\infty \frac{(-1)^n}{n!} x^{n+s-1}\, dx = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 x^{n+s-1}\, dx \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}\end{aligned}$$

The justification for interchanging the summation and integration operations is easy and is left as an exercise. Thus, we have obtained not just an alternative proof for the meromorphic continuation of $\Gamma(s)$, but a proof of the hybrid series-integral representation of $\Gamma(s)$, which also clearly shows where the poles of $\Gamma(s)$ are and that they are simple poles with the correct residues.

124. **Lemma.** For $\mathrm{Re}(s) > 0$ we have

$$\Gamma(s) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1}\, dx.$$

Proof. As $n \to \infty$, the integrand converges to $e^{-x} x^{s-1}$ pointwise. Furthermore, the factor $\left(1 - \frac{x}{n}\right)^n$ is bounded from above by the function e^{-x} (because of the elementary inequality $1 - t \leq e^{-t}$ that holds for all real t). The claim therefore follows from the dominated convergence theorem. $\square$

125. **Lemma.** For $\mathrm{Re}(s) > 0$ we have

$$\int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1}\, dx = \frac{n!\, n^s}{s(s+1)\cdots(s+n)}.$$

Proof. For $n = 1$, the claim is that

$$\int_0^1 (1-x)\, x^{s-1}\, dx = \frac{1}{s(s+1)},$$

which is easy to verify directly. For the general claim, using a linear change of variables and an integration by parts we see that

$$\begin{aligned}\int_0^n \left(1-\frac{x}{n}\right)^n x^{s-1}\,dx &= n^s \int_0^1 (1-t)^n t^{s-1}\,dt \\ &= n^s \left[(1-t)^n \frac{t^s}{s}\Big|_{t=0}^{t=1} + \int_0^1 n(1-t)^{n-1}\frac{t^s}{s}\,dt \right] \\ &= n^s \cdot \frac{n}{s}\int_0^1 (1-t)^{n-1} t^{(s+1)-1}\,dt,\end{aligned}$$

so the claim follows by induction on n. □

126. **Corollary.** For $\mathrm{Re}(s) > 0$ we have

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\,n^s}{s(s+1)\cdots(s+n)}.$$

127. **Proof of the infinite product representation for $\Gamma(s)$.** For $\mathrm{Re}(s) > 0$ we have

$$\begin{aligned}\Gamma(s)^{-1} &= \lim_{n\to\infty} \frac{s(s+1)\cdots(s+n)}{n!\,n^s} \\ &= s \lim_{n\to\infty} e^{-s\log n}\left(1+\frac{s}{1}\right)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n}\right) \\ &= s \lim_{n\to\infty} e^{s(\sum_{k=1}^n \frac{1}{k} - \log n)} \prod_{k=1}^n \left(1+\frac{s}{k}\right)e^{-s/k} \\ &= s e^{\gamma s} \prod_{n=1}^\infty \left(1+\frac{s}{n}\right) e^{-s/n}.\end{aligned}$$

□

128. We now check that the infinite product actually converges absolutely and uniformly on compact subsets in all of $\mathbb{C}$, so defines an entire function. Let's start with some preliminary elementary observations on infinite products.

Lemma. For a sequence of complex numbers $(a_n)_{n=1}^\infty$, we have $\prod_{n=1}^\infty (1+|a_n|) \in (0,\infty)$ if and only if none of the factors a_n is equal to -1 and $\sum_{n=1}^\infty |a_n| < \infty$.

Proof. Assume all a_n's are not equal to -1, and $\sum_n |a_n| < \infty$. In particular all a_n's for large enough n satisfy $|a_n| < 1/2$, so we can assume without loss of generality that this holds for *all* n. We therefore have

$$\prod_{k=1}^n (1+a_k) = \prod_{k=1}^n \exp\left(\mathrm{Log}(1+a_k)\right) = \exp\left(\sum_{k=1}^n \mathrm{Log}(1+a_k)\right),$$

where $\mathrm{Log}(z)$ denotes the principal branch of the logarithm function, which has the Taylor expansion around $z = 1$

$$\mathrm{Log}(z) = \sum_{m=1}^\infty \frac{(-1)^{m-1}}{m}(z-1)^m \qquad (|z| < 1).$$

In particular, for z near 0 we have

$$\text{Log}(1+z) = z + O(z^2),$$

i.e., for example $|z|-10|z|^2 \le |\text{Log}(1+z)| \le |z|+10|z|^2$ in some sufficiently small neighborhood of $z=0$. It follows that if $\sum_n |a_n| < \infty$ then also $\sum_n |\text{Log}(1+a_n)| < \infty$, so the product $\prod_{k=1}^n (1+a_k)$ converges. Conversely, it is easily seen from the same inequality that if $\sum_n |\text{Log}(1+a_n)| < \infty$ *and* $a_n \to 0$ as $n \to \infty$ (which would be a consequence of $\prod_n (1+a_n)$ converging to a number in $(0,\infty)$) then $\sum_n |a_n| < \infty$ $\square$

Lemma. Let $(f_n)_{n=1}^\infty$ be a sequence of functions that are holomorphic and nonzero on some region Ω. Then $\prod_{n=1}^\infty (1+f_n)$ converges absolutely uniformly on compact subsets in Ω to a nonzero holomorphic function if and only if the series $\sum_{n=1}^\infty f_n$ also converges absolutely uniformly on compact subsets in Ω.

Proof. Use the same estimates in the previous proof together with the uniformity of the convergence on compacts to ensure that the inequalities hold uniformly so the limiting function is holomorphic.

129. **Proof that $\prod_{n=1}^\infty \left(1+\frac{z}{n}\right) e^{-z/n}$ is an entire function.**

$$\begin{aligned}\sum_{n=1}^\infty \left| \left(1+\frac{z}{n}\right) e^{-z/n} - 1 \right| &= \sum_{n=1}^\infty \left| \left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}+O\left(\frac{z^2}{n^2}\right)\right) - 1 \right| \\ &= \sum_{n=1}^\infty \left| O\left(\frac{z^2}{n^2}\right) \right| < \infty,\end{aligned}$$

The convergence is uniform on compacts on $\mathbb{C}$, but to apply the previous result (which requires the functions to be nonzero) one needs to be a bit more careful and separate out the zeros: for a fixed disc $D_{N+1/2}(0)$ of radius $N+1/2$ around 0, consider only the product starting at $n=N+1$ — those functions are nonzero in the disc so the previous result applies to give a function that's holomorphic and nonzero in $D_N(0)$. Then separately the factors $(1+z/n)$, $n=1,\dots,N$ contribute simple zeros at $z=-1,\dots,-N$. $\square$

130. **Corollary (the reflection formula).** $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin \pi s}$.

Proof.

$$\begin{aligned}\frac{1}{\Gamma(s)\Gamma(1-s)} &= \Gamma(s)^{-1}(-s)^{-1}\Gamma(-s)^{-1} \\ &= \frac{-1}{s}\cdot se^{\gamma s}\prod_{n=1}^\infty \left(1+\frac{s}{n}\right)e^{-s/n}\cdot(-s)e^{-\gamma s}\prod_{n=1}^\infty\left(1-\frac{s}{n}\right)e^{s/n} \\ &= s\prod_{n=1}^\infty\left(1-\frac{s^2}{n^2}\right) = s\frac{\sin(\pi s)}{\pi s} = \frac{\sin(\pi s)}{\pi},\end{aligned}$$

where we used the product representation $\sin(\pi z) = \pi z \prod_{n=1}^\infty (1-z^2/n^2)$ for the sine function derived in a homework problem.

131. **Alternative derivation of the reflection formula ([Stein-Shakarchi], page 164).** By analytic continuation, it is enough to prove the formula for real s in $(0,1)$. For such s we have

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t} t^{-s}\Gamma(s)\,dt \\
&= \int_0^\infty e^{-t}t^{-s}\left(t\int_0^\infty e^{-vt}(vt)^{s-1}\,dv\right)dt \\
&= \int_0^\infty \int_0^\infty e^{-t(1+v)}v^{s-1}\,dv\,dt = \int_0^\infty \left(\int_0^\infty e^{-t(1+v)}\,dt\right)v^{s-1}\,dv \\
&= \int_0^\infty \frac{v^{s-1}}{1+v}\,dv = \int_{-\infty}^\infty \frac{e^{sx}}{1+e^x}\,dx \quad \text{(by setting } v=e^x\text{)}.
\end{aligned}$$

So it is enough to prove that for $0<s<1$ we have

$$\int_{-\infty}^\infty \frac{e^{sx}}{1+e^x}\,dx = \frac{\pi}{\sin(\pi s)}.$$

This integral can be evaluated using residue calculus; see Example 2 in Section 2.1, Chapter 3, pages 79–81 of [Stein-Shakarchi] for the details. □

132. Note that by combining the alternative derivation of the reflection formula given above with the infinite product representation for the gamma function, we get a new proof of the infinite product representation for $\sin(\pi z)$.

15 The Riemann zeta function

133. The **Riemann zeta function** (often referred to simply as the zeta function when there is no risk of confusion), like the gamma function is considered one of the most important special functions in "higher" mathematics. However, the Riemann zeta function is a lot more mysterious than the gamma function, and remains the subject of many famous open problems, including the most famous of them all: the **Riemann hypothesis**, considered by many (including myself) as the most important open problem in mathematics.

134. The main reason for the zeta function's importance is its connection with prime numbers and other concepts and quantities from number theory. Its study, and in particular the attempts to prove the Riemann hypothesis, have also stimulated an unusually large number of important developments in many areas of mathematics.

135. As with the gamma function, the Riemann zeta function is usually defined on only part of the complex plane and its definition is then extended by analytic continuation. Again, I will formulate this as a theorem asserting the existence of the zeta function and its various properties.

136. **Theorem.** There exists a unique function, denoted $\zeta(s)$, of a complex variable s, having the following properties:

(a) $\zeta(s)$ is a meromorphic function on $\mathbb{C}$.

(b) For $\text{Re}(s) > 1$, $\zeta(s)$ is given by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots.$$

(c) **Euler product formula:** for $\text{Re}(s) > 1$, $\zeta(s)$ also has an infinite product representation

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where the product ranges over the prime numbers $p = 2, 3, 5, 7, 11, \ldots$.

(d) $\zeta(s)$ has no zeros in the region $\text{Re}(s) > 1$.

(e) $\zeta(s)$ has no zeros on the line $\text{Re}(s) = 1$ (this requires a separate proof from the previous claim).

(f) **The "trivial" zeros:** the zeros of $\zeta(s)$ in the region $\text{Re}(s) \leq 0$ are precisely at $s = -2, -4, -6, \ldots$.

(g) $\zeta(s)$ has a unique pole located at $s = 1$. It is a simple pole with residue 1.

(h) **The "Basel problem" and its generalizations:** the values of $\zeta(s)$ at even positive integers are given by Euler's formula

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n} \qquad (n = 1, 2, \ldots),$$

where $(B_m)_{m=0}^{\infty}$ are the Bernoulli numbers, defined as the coefficients in the Taylor expansion

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m.$$

Many of the properties of these amazing numbers were discussed in our homework problem sets.

(i) **Values at negative odd integers:** we have

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

(j) **Functional equation:** the zeta function satisfies

$$\zeta^*(1 - s) = \zeta^*(s),$$

where we denote by $\zeta^*(s)$ the **symmetrized zeta function**

$$\zeta^*(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

(k) **Mellin transform representation:** an expression for $\zeta(s)$ valid for all $s \in \mathbb{C}$ is

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$
$$= -\frac{1}{1-s} - \frac{1}{s} + \frac{1}{2} \int_1^{\infty} \left(t^{-\frac{s+1}{2}} + t^{\frac{s-2}{2}}\right) (\vartheta(t) - 1)\, dt,$$

where the function $\vartheta(t)$ is one of **Jacobi theta series**, defined as

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

(l) **Contour integral representation:** another expression for $\zeta(s)$ valid for all $s \in \mathbb{C}$ is

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x},$$

where C is a keyhole contour coming from $+\infty$ to 0 slightly above the positive x-axis, then circling the origin in a counterclockwise direction around a circle of small radius, then going back to $+\infty$ slightly below the positive x-axis.

(m) **Connection to prime number enumeration — the "explicit formula of number theory":** define Von Mangoldt's weighted prime counting function

$$\psi(x) = \sum_{p^k \le x} \log p,$$

where the sum is over all prime powers less than or equal to x. Then for non-integer $x > 1$,

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi),$$

where the sum ranges over all zeros ρ of the Riemann zeta function. (In most textbooks the sum is separated into two sums, one ranging over the trivial zeros which can be evaluated explicitly, and the other ranging over the much less trivial zeros in the strip $0 < \mathrm{Re}(s) < 1$.)

137. The explicit formula of number theory illustrates that knowing where the zeros of $\zeta(s)$ has important consequences for prime number enumeration. In particular, proving that $\mathrm{Re}(s)$ has no zeros in $\mathrm{Re}(s) \geq 1$ will enable us to prove one of the most famous theorems in mathematics.

The prime number theorem. Let $\pi(x)$ denote the number of prime numbers less than or equal to x. Then we have

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

138. **The Riemann hypothesis.** All the nontrivial zeros of $\zeta(s)$ are on the "critical strip" $\mathrm{Re}(s) = 1/2$.

139. **Proofs.** To begin the proof, again, let's take as the definition of $\zeta(s)$ the standard representation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Since $\sum_n |n^{-s}| = \sum_n n^{-\operatorname{Re}(s)}$, we see that the series converges absolutely precisely when $\operatorname{Re}(s) > 1$, and that the convergence is uniform on any half-plane of the form $\operatorname{Re}(s) > \alpha$ where $\alpha > 1$. In particular, it is uniform on compact subsets, so $\zeta(s)$ is holomorphic in this region.

140. Similarly, the Euler product $Z(s) := \prod_p (1 - p^{-s})^{-1}$ converges absolutely if and only if the series $\sum_p |p^{-s}| = \sum_p p^{-\operatorname{Re}(s)}$ converges, and in particular if $\operatorname{Re}(s) > 1$. It follows that $Z(s)$ is well-defined, holomorphic and nonzero for $\operatorname{Re}(s) > 1$.

141. We now prove that $Z(s) = \zeta(s)$. This can be done by manipulating the partial products associated with the infinite product defining $Z(s)$, as follows:

$$\begin{aligned} \zeta_N(s) &:= \prod_{p \le N} \frac{1}{1-p^{-s}} = \prod_{p\le N} (1 + p^{-s} + p^{-2s} + p^{-3s} + \ldots) \\ &= \sum_{\substack{n = p_1^{j_1} \cdots p_k^{j_k} \\ p_1,\ldots,p_k \text{ primes } \le N}} \frac{1}{n^s}, \end{aligned}$$

where the last equality follows from the fundamental theorem of arithmetic, together with the fact that when multiplying two (or a finite number of) infinitely convergent series, the summands can be rearranged and summed in any order we desire. So, we have represented $\zeta_N(s)$ as a series of a similar form as the series defining $\zeta(s)$, but involving terms of the form n^{-s} only for those positive integers n whose prime factorization contains only primes $\le N$. It follows that

$$|\zeta(s) - \zeta_N(s)| \le \sum_{n>N} \frac{1}{n^s}.$$

Taking the limit as $N \to \infty$ shows that $Z(s) = \lim_{N\to\infty} \zeta_N(s) = \zeta(s)$. This proves the validity of the Euler product formula. $\square$

142. **Corollary:** $\zeta(s)$ has no zeros in the region $\operatorname{Re}(s) > 1$.

Proof. The Euler product formula gives a convergent product for $\zeta(s)$ in this region where each factor $(1 - p^{-s})^{-1}$ has no zeros. $\square$

143. **Theorem (the Poisson summation formula).** For a sufficiently well-behaved function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k),$$

where

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x}\, dx.$$

is the Fourier transform of f.

Proof. Define a function $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + n),$$

the "periodiciziation" of f. Assume that $f(x)$ is sufficiently well-behaved (i.e., decays fast enough as $x \to \pm\infty$ so that $g(x)$ is in turn well-behaved, and has reasonable smoothness properties). In that case, $g(x)$ will have a convergent Fourier expansion of the form

$$g(x) = \sum_{k=-\infty}^{\infty} \hat{g}(k) e^{2\pi i k x},$$

where the Fourier coefficients $\hat{g}(k)$ can be computed as

$$\hat{g}(k) = \int_0^1 g(x) e^{-2\pi i k x}\, dx.$$

In particular, setting $x = 0$ in the formula for $g(x)$ gives the standard result that

$$g(0) = \sum_{k=-\infty}^{\infty} \hat{g}(k).$$

However, note that $g(0) = \sum_{n=-\infty}^{\infty} f(n)$, the quantity on the left-hand side of the Poisson summation formula. On the other hand, the Fourier coefficient $\hat{g}(k)$ can be expressed in terms of the Fourier coefficients of the original function $f(x)$:

$$\begin{aligned}
\hat{g}(k) &= \int_0^1 g(x) e^{-2\pi i k x}\, dx = \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n) e^{-2\pi i k x}\, dx \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i k x}\, dx = \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(u) e^{-2\pi i k u}\, du \\
&= \int_{-\infty}^{\infty} f(u) e^{-2\pi i k u}\, du = \hat{f}(k).
\end{aligned}$$

Combining these observations gives the result, modulo a few details we've glossed over concerning the precise assumptions that need to be made about $f(x)$ (we will only apply the Poisson summation formula for one extremely well-behaved function, so I will not bother discussing those details). $\square$

144. **Theorem.** The Jacobi theta function $\vartheta(t)$ satisfies the functional equation

$$\vartheta(t) = \frac{1}{\sqrt{t}} \vartheta(1/t) \qquad (t > 0).$$

(**Note:** equations of this form are studied in the theory of **modular forms**, an area of mathematics combining number theory, complex analysis and algebra in a very surprising and beautiful way.)

Proof. The idea is to apply the Poisson summation formula to the function

$$f(x) = e^{-\pi t x^2},$$

for which it can be checked that

$$\hat{f}(k) = t^{-1/2} e^{-\pi k^2 / t},$$

using a simple change of variables from the standard integral evaluation

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x u}\, du = e^{-\pi u^2}$$

(that is, the fact that the function $e^{-\pi x^2}$ is its own Fourier transform); this evaluation appears in Example 1, Chapter 2, pages 42–44 in [Stein-Shakarchi]. With the above substitution for $f(x)$ and $\hat{f}(k)$, the Poisson summation formula becomes precisely the functional equation for $\vartheta(t)$. $\square$

145. **Exercise.** (a) Use the residue theorem to evaluate the contour integral

$$\oint_{\gamma_N} \frac{e^{-\pi z^2 t}}{e^{2\pi i z} - 1}\, dz,$$

where γ_N is the rectangle with vertices $\pm(N+1/2)\pm i$ (with N a positive integer), then take the limit as $N \to \infty$ to derive the integral representation

$$\vartheta(t) = \int_{-\infty - i}^{\infty - i} \frac{e^{-\pi z^2 t}}{e^{2\pi i z} - 1}\, dz - \int_{-\infty + i}^{\infty + i} \frac{e^{-\pi z^2 t}}{e^{2\pi i z} - 1}\, dz$$

for the Jacobi theta function.

(b) In this representation, expand the factor $(e^{2\pi i z} - 1)^{-1}$ as a geometric series in $e^{-2\pi i z}$ (for the first integral) and as a geometric series in $e^{2\pi i z}$ (for the second integral). Evaluate the resulting infinite series, rigorously justifying all steps, to obtain an alternative proof of the functional equation for $\vartheta(t)$.

146. **Lemma.** The asymptotic behavior of $\vartheta(t)$ near $t = 0$ and $t = +\infty$ is given by

$$\begin{aligned} \vartheta(t) &= O\left(\frac{1}{\sqrt{t}}\right) && (t \to 0+), \\ \vartheta(t) &= 1 + O(e^{-\pi t}) && (t \to \infty). \end{aligned}$$

Proof. The asymptotics as $t \to \infty$ is immediate from

$$\vartheta(t) - 1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t} \le 2\sum_{n=1}^{\infty} e^{-\pi n t} = \frac{2e^{-\pi t}}{1 - e^{-\pi t}},$$

which is bounded by $Ce^{-\pi t}$ if $t > 10$. Using the functional equation now gives that $\vartheta(t) = t^{-1/2}(1 + O(e^{-\pi/t})) = O(t^{-1/2})$ as $t \to 0+$. $\square$

147. **Proof of the analytic continuation of $\zeta(s)$.** Start with the formula

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-x} x^{s/2-1}\, dx,$$

valid for $\mathrm{Re}(s) > 0$. A linear change of variable $x = \pi n^2 t$ brings this to the form

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{\infty} e^{-\pi n^2 t} t^{s/2-1}\, dt.$$

Summing the left-hand side over $n = 1, 2, \ldots$ gives $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ — the function we denoted $\zeta^*(s)$ — adding the stronger assumption that $\mathrm{Re}(s) > 1$. For the right-hand side we have that

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 t} t^{s/2-1}\,dt. = \int_0^{\infty}\left(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right) t^{s/2-1}\,dt$$
$$= \int_0^{\infty} \frac{\vartheta(t)-1}{2} t^{s/2-1}\,dt,$$

where the estimates in the lemma are needed to justify interchanging the order of the summation and integration, and show that the integral converges for $\mathrm{Re}(s) > 1$. Thus we have obtained the representation

$$\zeta^*(s) = \frac{1}{2}\int_0^{\infty}(\vartheta(t)-1)t^{s/2-1}\,dt = \int_0^{\infty}\varphi(t)t^{s/2-1}\,dt,$$

where we denote $\varphi(t) = \frac{1}{2}(\vartheta(t)-1)$. Next, the idea is to use the functional equation for $\vartheta(t)$ to bring this to a new form that can be seen to be well-defined for all $s \in \mathbb{C}$ except $s = 1$. Specifically, we note that the functional equation for can be expressed in the form

$$\varphi(t) = t^{-1/2}\varphi(1/t) + \tfrac{1}{2}t^{-1/2} - \tfrac{1}{2}.$$

We can therefore write

$$\begin{aligned}
\zeta^*(s) &= \int_0^1 \varphi(t)t^{s/2-1}\,dt + \int_1^{\infty}\varphi(t)t^{s/2-1}\,dt \\
&= \int_0^1 \left(t^{-1/2}\varphi(1/t) + \tfrac{1}{2}t^{-1/2} - \tfrac{1}{2}\right)t^{s/2-1}\,dt + \int_1^{\infty}\varphi(t)t^{s/2-1}\,dt \\
&= -\frac{1}{1-s} - \frac{1}{s} + \int_1^{\infty}\left(t^{-s/2-1/2} + t^{s/2-1}\right)\varphi(t)\,dt.
\end{aligned}$$

We have derived a formula for $\zeta^*(s)$ (one of the formulas claimed in the main theorem above) that is now seen to define a meromorphic function on all of $\mathbb{C}$ — the integrand decays rapidly as $t \to \infty$ so actually defines an entire function, so the only poles are due to the two terms $-1/s$ and $1/(s-1)$. We have therefore proved that $\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$. □

148. **Corollary.** The zeta function satisfies the functional equation

$$\zeta^*(1-s) = \zeta^*(s).$$

Equivalently, because of the reflection formula satisfied by the gamma function, it is easy to check that the functional equation can be rewritten in the form

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Proof. The representation we derived for $\zeta^*(s)$ is manifestly symmetric with respect to replacing each occurrence of s by $1-s$. □

149. **Corollary.** The only pole of $\zeta(s)$ is a simple pole at $s = 1$ with residue 1.

Proof. Our representation for $\zeta^*(s)$ expresses it as a sum of $-\frac{1}{s}$, $\frac{1}{s-1}$, and an entire function. Thus the poles of $\zeta^*(s)$ are simple poles at $s = 0, 1$ with residues -1 and 1, respectively. It follows that

$$\zeta(s) = \pi^{s/2}\Gamma(s/2)^{-1}\zeta^*(s)$$

has a pole at $s = 1$ with residue $\pi^{1/2}\Gamma(1/2)^{-1} = 1$, and a pole (that turns out to be a removable singularity) at $s = 0$ with residue $\pi^0\Gamma(0)^{-1} = 0$. (That is, the pole of $\zeta^*(s)$ at $s = 0$ is cancelled out by the zero of $\Gamma(s/2)$.) $\square$

150. **Corollary.** $\zeta(-n) = -B_{n+1}/(n+1)$ for $n = 1, 2, 3, \ldots$.

Proof. Using the functional equation, we have that

$$\begin{aligned}\zeta(-n) &= 2^{-n}\pi^{-n-1}\sin(-\pi n/2)\Gamma(n+1)\zeta(n+1) \\ &= 2^{-n}\pi^{-n-1}\sin(-\pi n/2)n!\zeta(n+1).\end{aligned}$$

If $n = 2k$ is even, then $\sin(-\pi n/2) = \sin(-\pi k) = 0$, so we get that $\zeta(-2k) = 0$ (that is, $n = 2k$ is one of the so-called "trivial zeros"). We also know that $B_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$, so the formula $\zeta(-n) = B_{n+1}/(n+1)$ is satisfied in this case.

If on the other hand $n = 2k-1$ is odd, then $\sin(-\pi(2k-1)/2) = (-1)^k$, and therefore we get, using the formula expressing $\zeta(2k)$ in terms of the Bernoulli numbers (derived in the homework and in the textbook), that

$$\begin{aligned}\zeta(-n) &= (-1)^k 2^{-2k+1}\pi^{-2k}(2k-1)!\zeta(2k) \\ &= (-1)^k 2^{-2k+1}\pi^{-2k}(2k-1)!\frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!}B_{2k} \\ &= -\frac{B_{2k}}{2k} = -\frac{B_{n+1}}{n+1},\end{aligned}$$

so again the formula is satisfied. $\square$

151. **Corollary.** The zeros of $\zeta(s)$ in the region $\mathrm{Re}(s) < 0$ are precisely the trivial zeros $s = -2, -4, -6, \ldots$.

152. **Proof.** We already established the existence of the trivial zeros. The fact that there are no other zeros also follows easily from the functional equation and is left as an exercise. $\square$

153. **An alternative approach to the analytic continuation of $\zeta(s)$.** There is a more "down-to-earth" approach to the analytic continuation of $\zeta(s)$ based on the standard idea from numerical analysis of approximating an integral by a sum (or in this case going in the other direction, approximating a sum by an integral). The technical name for this procedure, when it is done in a more

systematic way, is **Euler-Maclaurin summation**.

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \left(\int_n^{n+1} \frac{dx}{x^s} + \left(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s} \right) \right) \\ &= \int_1^{\infty} \frac{dx}{x^s} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \\ &= \frac{1}{s-1} - \int_1^{\infty} \left(x^{-s} - \lfloor x \rfloor^{-s} \right) dx.\end{aligned}$$

This representation is certainly valid for $\mathrm{Re}(s) > 1$. However, note that we have the bound

$$\left| x^{-s} - \lfloor x \rfloor^{-s} \right| \le |s| \cdot \lfloor x \rfloor^{-\mathrm{Re}(s)-1} \qquad (x \ge 1)$$

by the mean value theorem. Thus, the integral is actually an absolutely convergent integral in the larger region $\mathrm{Re}(s) > 0$, and the representation we derived gives an analytic continuation of $\zeta(s)$ to a meromorphic function on $\mathrm{Re}(s) > 0$, which has a single pole at $s = 1$ (a simple pole with residue 1) and is holomorphic everywhere else.

154. An elaboration of this idea using what is known as the Euler-Maclaurin summation formula can be used to perform the analytic continuation of $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ by extending it inductively from each region $\mathrm{Re}(s) > -n$ to $\mathrm{Re}(s) > -n-1$, as we saw could be done for the gamma function. Another approach is to use the analytic continuation for $\mathrm{Re}(s) > 0$ shown above, then prove that the functional equation $\zeta^{(}1-s) = \zeta^*(s)$ holds in the region $0 < \mathrm{Re}(s) < 1$, and then use the functional equation to analytically continue $\zeta(s)$ to $\mathrm{Re}(s) \le 0$ (which is the reflection of the region $\mathrm{Re}(s) \ge 1$ under the transformation $s \mapsto 1 - s$).

155. Next, we prove a nontrivial and very important fact about the zeta function that will play a critical role in our proof of the prime number theorem.

Theorem. $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$.

This theorem can also be thought of as a “toy” version of the Riemann hypothesis. If you ever want to try solving this famous open problem, getting a good understanding of its toy version seems like a good idea...

Proof. For this proof, denote $s = \sigma + it$, where we assume $\sigma > 1$ and t is real and nonzero. The proof is based on investigating simultaneously the behavior of $\zeta(\sigma + it)$, $\zeta(\sigma + 2it)$, and $\zeta(\sigma)$, for fixed t as $\sigma \searrow 1$. Consider the following somewhat mysterious quantity

$$X = \log \left| \zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it) \right|.$$

We can evaluate "X" as

$$\begin{aligned}
&\log|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \\
&\quad = 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\
&\quad = 3\log\left(\prod_p |1-p^{-\sigma}|^{-1}\right) + 4\log\left(\prod_p |1-p^{-\sigma-it}|^{-1}\right) \\
&\qquad\quad + \log\left(\prod_p |1-p^{-\sigma-2it}|^{-1}\right) \\
&\quad = \sum_p \Big(-3\log|1-p^{-\sigma}| - 4\log|1-p^{-\sigma-it}| - \log|1-p^{-\sigma-2it}|\Big) \\
&\quad = \sum_p \Big(-3\,\mathrm{Re}\big[\mathrm{Log}(1-p^{-\sigma})\big] - 4\,\mathrm{Re}\big[\mathrm{Log}(1-p^{-\sigma-it})\big] \\
&\qquad\qquad\qquad - \mathrm{Re}\,\mathrm{Log}\big[1-p^{-\sigma-2it}\big]\Big),
\end{aligned}$$

where $\mathrm{Log}(\cdot)$ denotes the principal branch of the logarithm function. Now note that for $z = a + ib$ with $a > 1$ and p prime we have $|p^{-z}| = p^{-a} < 1$, so

$$-\mathrm{Log}(1-p^{-z}) = \sum_{m=1}^{\infty} \frac{p^{-mz}}{m},$$

and

$$\begin{aligned}
-\mathrm{Re}\Big[\mathrm{Log}(1-p^{-z})\Big] &= \sum_{m=1}^{\infty} \frac{p^{-ma}}{m}\,\mathrm{Re}\Big[\cos(mb\log p) + i\sin(mb\log p)\Big] \\
&= \sum_{m=1}^{\infty} \frac{p^{-ma}}{m}\cos(mb\log p).
\end{aligned}$$

So we can rewrite X as

$$X = \sum_{n=1}^{\infty} c_n n^{-\sigma}(3 + 4\cos\theta_n + \cos(2\theta_n))$$

where $\theta_n = t\log n$ and $c_n = 1/m$ if $n = p^m$ for some prime p. We can now use a simple trigonometric identity

$$3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2,$$

to rewrite X yet again as

$$X = 2\sum_{n=1}^{\infty} c_n n^{-\sigma}(1+\cos\theta_n)^2.$$

We have proved a crucial fact, namely that $X \geq 0$, or equivalently that

$$e^X = |\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \geq 1.$$

We now claim that this innocent-looking inequality is incompatible with the existence of a zero of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$. Indeed, assume by contradiction that $\zeta(1+it)=0$ for some real $t\neq 0$. Then the three quantities $\zeta(\sigma)$, $\zeta(\sigma+it)$ and $\zeta(\sigma+2it)$ have the following asymptotic behavior as $\sigma \searrow 1$:

$$\begin{aligned}
|\zeta(\sigma)| &= \frac{1}{\sigma-1}+O(1) && \text{(since } \zeta(s) \text{ has a pole at } s=1\text{)},\\
|\zeta(\sigma+it)| &= O(\sigma-1) && \text{(since } \zeta(s) \text{ has a zero at } s=1+it\text{)},\\
|\zeta(\sigma+2it)| &= O(1) && \text{(since } \zeta(s) \text{ is holomorphic at } s=1+2it\text{)}.
\end{aligned}$$

Combining these results we have that

$$e^X = |\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| = O((\sigma-1)^{-3}(\sigma-1)^4) = O(\sigma-1).$$

In particular, $e^X \to 0$ as $\sigma \searrow 1$, in contradiction to the result we proved above that $e^X \geq 1$. This proves the claim that $\zeta(s)$ cannot have a zero on the line $\mathrm{Re}(s)=1$. □

156. **Exercise.** The above proof that $e^X \geq 1$ (which immediately implied the claim of the theorem) relied on showing that for any prime number p, the corresponding factors in the Euler product formula satisfy the inequality

$$(1-p^{-\sigma})^{-3}|1-p^{-\sigma}p^{-it}|^{-4}|1-p^{-\sigma}p^{-2it}|^{-1} \geq 1,$$

and this was proved by taking the logarithm of the left hand-side, expanding in a power series and using the elementary trigonometric identity $3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2$. However, one can imagine a more direct approach that starts as follows: denote $x=p^{-\sigma}$ and $z=p^{-it}=e^{-it\log p}$. Then the inequality reduces to the claim that

$$(1-x)^3|1-zx|^4|1-z^2x| \leq 1$$

for all $x\in[0,1]$ and z satisfying $|z|=1$. Since this is an elementary inequality, it seems like it ought to have an elementary proof (i.e., a proof that does not involve logarithms and power series expansions). Can you find such a proof?

16 The prime number theorem

157. The prime number theorem was proved in 1896 by Jacques Hadamard and independently by Charles Jean de la Vallée Poussin, using the groundbreaking ideas from Riemann's famous 1859 paper in which he introduced the use of the Riemann zeta function as a tool for counting prime numbers. (This was the only number theory paper Riemann wrote in his career!) The history (including all the technical details) of these developments is described extremely well in the classic textbook *Riemann's Zeta Function* by H. M. Edwards, which I highly recommend.

The original proofs of the prime number theorem were very complicated and relied on the "explicit formula of number theory" (that I mentioned in the previous section) and some of its variants. Throughout the 20th century, mathematicians worked hard to find simpler ways to derive the prime number theorem. This resulted in several important developments (such as the Wiener tauberian theorem and the Hardy-Littlewood tauberian theorem) that advanced not just the state of analytic number theory but also complex analysis, harmonic analysis and functional analysis. Despite all the efforts and the discovery of several paths to a proof that were simpler than the original approach, all proofs remained quite difficult... until 1980, when the mathematician Donald Newman discovered a wonderfully simple way to derive the theorem using a completely elementary use of complex analysis. It is Newman's proof (as presented in the note *Newman's short proof of the prime number theorem* by Don Zagier, *Amer. Math. Monthly* 104 (1997), 705–708) that I present here.

158. Define the weighted prime counting functions

$$\pi(x) = \#\{p \text{ prime} : p \le x\} = \sum_{p\le x} 1,$$
$$\psi(x) = \sum_{p^k \le x} \log p = \sum_{p\le x} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor,$$

with the convention that the symbol p in a summation denotes a prime number, and p^k denotes a prime power, so that summation over $p \le x$ denotes summation over all primes $\le x$, and the summation over p^k denotes summation over all prime powers $\le x$. Another customary way to write the function $\psi(x)$ is as

$$\psi(x) = \sum_{n\le x} \Lambda(n),$$

where the function $\Lambda(n)$, called the von Mangoldt function, is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ p \text{ prime}, \\ 0 & \text{otherwise.} \end{cases}$$

159. **Lemma.** The prime number theorem $\pi(x) \sim \frac{x}{\log x}$ is equivalent to the statement that $\psi(x) \sim x$.

Proof. Note the inequality

$$\psi(x) = \sum_{p\le x} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \le \sum_{p\le x} \log p \frac{\log x}{\log p} = \sum_{p\le x} \log x = \log x \cdot \pi(x).$$

In the opposite direction, we have a similar (but slightly less elegant) inequality, namely that for any $0 < \epsilon < 1$ and $x \ge 2$,

$$\begin{aligned}\psi(x) &\ge \sum_{p\le x} \log p \ge \sum_{x^{1-\epsilon}<p\le x} \log p \ge \sum_{x^{1-\epsilon}<p\le x} \log\left(x^{1-\epsilon}\right) \\ &= (1-\epsilon)\log x \Big(\pi(x) - \pi(x^{1-\epsilon})\Big) \ge (1-\epsilon)\log x\Big(\pi(x) - x^{1-\epsilon}\Big).\end{aligned}$$

Now assume that $\psi(x) \sim x$ as $x \to \infty$. Then the first of the two bounds above implies that

$$\pi(x) \ge \frac{\psi(x)}{\log x},$$

so

$$\liminf_{x\to\infty} \pi(x)/\left(\frac{x}{\log x}\right) \geq 1.$$

On the other hand, the second of the two bounds implies that

$$\pi(x) \leq \frac{1}{1-\epsilon}\cdot\frac{\psi(x)}{\log x} + x^{1-\epsilon},$$

which implies that $\limsup_{x\to\infty} \pi(x)/\left(\frac{x}{\log x}\right) \leq \frac{1}{1-\epsilon} + \limsup_{x\to\infty}\frac{\log x}{x^\epsilon} = \frac{1}{1-\epsilon}$. Since ϵ was an arbitrary number in $(0,1)$, it follows that

$$\limsup_{x\to\infty} \pi(x)/\left(\frac{x}{\log x}\right) \leq 1.$$

Combining the two results about the lim inf and the lim sup gives that $\pi(x) \sim x/\log x$.

Now assume that $\pi(x) \sim \frac{x}{\log x}$, and apply the inequalities we derived above in the opposite direction from before. That is, we have

$$\psi(x) \leq \log x \cdot \pi(x),$$

so

$$\limsup_{x\to\infty} \psi(x)/x \leq 1.$$

On the other hand,

$$\psi(x) \geq (1-\epsilon)\log x(\pi(x) - x^{1-\epsilon})$$

implies that

$$\liminf_{x\to\infty} \psi(x)/x \geq \lim_{x\to\infty}(1-\epsilon)\left(1-\frac{\log x}{x^\epsilon}\right) = 1-\epsilon.$$

Again, since $\epsilon\in(0,1)$ was arbitrary, it follows that $\liminf_{x\to\infty}\frac{\psi(x)}{x} = 1$. Combining the two results about the lim inf and lim sup proves that $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$, as claimed. $\square$

160. **Lemma.** For $\mathrm{Re}(s) > 1$ we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^\infty \Lambda(n) n^{-s}.$$

Proof. Using the Euler product formula and taking the logarithmic derivative (which is an operation that works as it should when applied to infinite products of holomorphic functions that are uniformly convergent on compact subsets), we have

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= \sum_p \frac{\frac{d}{ds}(1-p^{-s})}{1-p^{-s}} = \sum_p \frac{\log p\cdot p^{-s}}{1-p^{-s}} \\
&= \sum_p \log p\,(p^{-s}+p^{-2s}+p^{-3s}+\ldots) = \sum_{p\text{ prime}}\sum_{k=1}^\infty \log p\cdot p^{-ks} \\
&= \sum_{n=1}^\infty \Lambda(n)n^{-s}.
\end{aligned}$$

□

161. **Lemma.** There is a constant $C > 0$ such that $\psi(x) < Cx$ for all $x \geq 1$.

Proof. The idea of the proof is that the binomial coefficient $\binom{2n}{n}$ is not too large on the one hand, but is divisible by many primes (all primes between n and $2n$) on the other hand — hence it follows that there cannot be too many primes, and in particular the weighted prime-counting function $\psi(x)$ can be easily bounded from above using such an argument. Specifically, we have that

$$\begin{aligned} 2^{2n} &= (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} > \binom{2n}{n} \geq \prod_{n<p\leq 2n} p = \exp\left(\sum_{n<p\leq 2n} \log p \right) \\ &= \exp\Big(\psi(2n) - \psi(n) - \sum_{n<p^k\leq 2n,\ k>1} \log p \Big). \\ &\geq \exp\Big(\psi(2n) - \psi(n) - O(\sqrt{n}\log^2 n) \Big). \end{aligned}$$

(The estimate $O(\sqrt{n}\log^2 n)$ for the sum of $\log p$ for prime powers higher than 1 is easy and is left as an exercise.) Taking the logarithm of both sides, this gives the bound

$$\psi(2n) - \psi(n) \leq 2n\log 2 + C_1\sqrt{n}\log n \leq C_2 n,$$

valid for all $n \geq 1$ with some constant $C_2 > 0$. It follows that

$$\begin{aligned} \psi(2^m) &= (\psi(2^m) - \psi(2^{m-1})) \\ &\qquad + (\psi(2^{m-1}) - \psi(2^{m-2})) + \ldots + (\psi(2^1) - \psi(2^0)) \\ &\leq C_2(2^{m-1} + \ldots + 2^0) \leq C_2 2^m, \end{aligned}$$

so the inequality $\psi(x) \leq C_2 x$ is satisfied for $x = 2^m$. It is now easy to see that this implies the result also for general x, since for $x = 2^m + \ell$ with $0 \leq \ell < 2^m$ we have

$$\psi(x) = \psi(2^m + \ell) \leq \psi(2^{m+1}) \leq C_2 2^{m+1} \leq 2C_2 2^m \leq 2C_2 x.$$

□

162. **Newman's tauberian theorem.** Let $f : [0, \infty) \to \mathbb{R}$ be a bounded function that is integrable on compact intervals. Define a function $g(z)$ of a complex variable z by

$$g(z) = \int_0^\infty f(t)e^{-zt}\,dt$$

(g is known as the **Laplace transform** of f). Clearly $g(z)$ is defined and holomorphic in the open half-plane $\mathrm{Re}(z) > 0$. Assume that $g(z)$ has an analytic continuation to an open region Ω containing the *closed* half-plane $\mathrm{Re}(z) \geq 0$. Then $\int_0^\infty f(t)\,dt$ exists and is equal to $g(0)$ (the value at $z = 0$ of the analytic continuation of g).

Proof. Define a truncated version of the integral defining $g(z)$, namely

$$g_T(z) = \int_0^T f(t)e^{-zt}\,dt$$

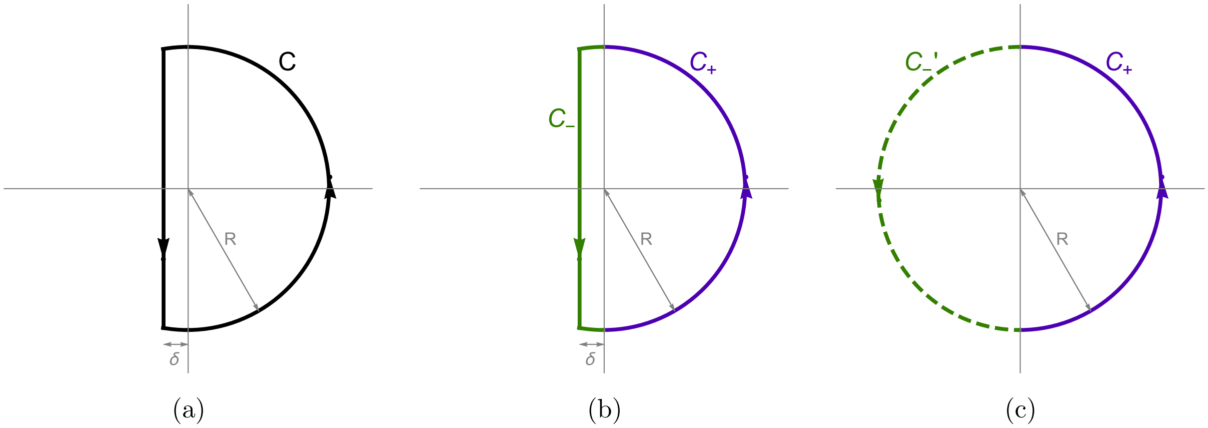


Figure 6: The contours C, C_+, C_- and C'_-.

for $T > 0$, which for any T is an entire function of z. Our goal is to show that $\lim_{T\to\infty} g_T(0) = g(0)$. This can be achieved using a clever application of Cauchy's integral formula. Fix some large $R > 0$ and a small $\delta > 0$ (which depends on R in a way that will be explained shortly), and consider the contour C consisting of the part of the circle $|z| = R$ that lies in the half-plane $\mathrm{Re}(z) \geq -\delta$, together with the straight line segment along the line $\mathrm{Re}(z) = -\delta$ connecting the top and bottom intersection points of this circle with the line (see Fig. 6(a)). Assume that δ is small enough so that $g(z)$ (which extends analytically at least slightly to the right of $\mathrm{Re}(z) = 0$) is holomorphic in an open set containing C and the region enclosed by it. Then by Cauchy's integral formula we have

$$\begin{aligned} g(0) - g_T(0) &= \frac{1}{2\pi i}\int_C (g(z) - g_T(z))e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z} \\ &= \frac{1}{2\pi i}\left(\int_{C_+} + \int_{C_-}\right)(g(z) - g_T(z))e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}, \end{aligned}$$

where we separate the contour into two parts, a semicircular arc C_+ that lies in the half-plane $\mathrm{Re}(z) > 0$, and the remaining part C_- in the half-plane $\mathrm{Re}(z) < 0$ (Fig. 6(b)). We now bound the integral separately on C_+ and on C_-. First, for z lying on C_+ we have

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\,dt\right| \leq B\int_T^\infty |e^{-zt}|\,dt = \frac{Be^{-\mathrm{Re}(z)T}}{\mathrm{Re}(z)},$$

where $B = \sup_{t\geq 0}|f(t)|$, and

$$\left|e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\right| = e^{\mathrm{Re}(z)T}\,\frac{2\,\mathrm{Re}(z)}{R}$$

(by the trivial identity $|1 + e^{it}|^2 = |e^{it}(e^{it} + e^{it})|^2 = 2\cos(t)$, valid for $t \in \mathbb{R}$).

So in combination we have

$$\left| \frac{1}{2\pi i} \int_{C_+} (g(z) - g_T(z)) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right| \le (\pi R) \frac{2B}{2\pi R^2} = \frac{B}{R}.$$

Next, for C_-, we bound the integral by bounding the contributions from $g(z)$ and $g_T(z)$ separately. In the case of $g_T(z)$, the function is entire, so we can deform the contour, replacing it with the semicircular arc $C'_- = \{|z| = R, \mathrm{Re}(z) < 0\}$ (Fig. 6(c)). On this contour we have the estimate

$$|g_T(z)| = \left| \int_0^T f(t) e^{-zt}\, dt \right| \le B \int_{-\infty}^T |e^{-zt}|\, dt = \frac{Be^{-\mathrm{Re}(z)T}}{|\mathrm{Re}(z)|},$$

which leads using a similar calculation as before to the estimate

$$\frac{1}{2\pi i} \int_{C'_-} \left| g_T(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \right| \frac{|dz|}{|z|} \le \frac{B}{R}.$$

The remaining integral

$$\frac{1}{2\pi i} \int_{C_-} \left| g(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \right| \frac{|dz|}{|z|}$$

tends to 0 as $T \to \infty$, since the dependence on T is only through the factor e^{Tz}, which converges to 0 uniformly on compact sets in $\mathrm{Re}(z) < 0$ as $T \to \infty$.

Combining the above estimates, we have shown that

$$\limsup_{T \to \infty} |g(0) - g_T(0)| \le \frac{2B}{R}.$$

Since R was an arbitrary positive number, the lim sup must be 0, and the theorem is proved. □

163. **An application of Newman's theorem.** Take

$$f(t) = \psi(e^t) e^{-t} - 1 \qquad (t \ge 0),$$

which is bounded by the lemma we proved above, as our function $f(t)$. The associated function $g(z)$ is then

$$\begin{aligned} g(z) &= \int_0^\infty (\psi(e^t) e^{-t} - 1) e^{-zt}\, dt = \int_1^\infty \left(\frac{\psi(x)}{x} - 1 \right) x^{-z-1}\, dx \\ &= \int_1^\infty \psi(x) x^{-z-2}\, dx - \frac{1}{z} = \int_1^\infty \left(\sum_{n \le x} \Lambda(n) \right) x^{-z-2}\, dx - \frac{1}{z} \\ &= \sum_{n=1}^\infty \Lambda(n) \left(\int_n^\infty x^{-z-2}\, dx \right) - \frac{1}{z} = \sum_{n=1}^\infty \Lambda(n)\, \frac{x^{-z-1}}{-z-1} \bigg|_n^\infty - \frac{1}{z} \\ &= \frac{1}{z+1} \sum_{n=1}^\infty \Lambda(n) n^{-z-1} - \frac{1}{z} = -\frac{1}{z+1} \cdot \frac{\zeta'(z+1)}{\zeta(z+1)} - \frac{1}{z} \qquad (\mathrm{Re}(z) > 0). \end{aligned}$$

Recall that $-\zeta'(s)/\zeta(s)$ has a simple pole at $s = 1$ with residue 1 (because $\zeta(s)$ has a simple pole at $s = 1$; it is useful to remember the more general fact that if a holomorphic function $h(z)$ has a zero of order k at $z = z_0$ then the logarithmic derivative $h'(z)/h(z)$ has a simple pole at $z = z_0$ with residue k). So $-\frac{1}{z+1} \cdot \frac{\zeta'(z+1)}{\zeta(z+1)}$ has a simple pole with residue 1 at $z = 0$, and therefore $-\frac{1}{z+1} \cdot \frac{\zeta'(z+1)}{\zeta(z+1)} - \frac{1}{z}$ has a *removable* singularity at $z = 0$. Thus, the identity $g(z) = -\frac{1}{z+1} \cdot \frac{\zeta'(z+1)}{\zeta(z+1)} - \frac{1}{z}$ shows that $g(z)$ extends analytically to a holomorphic function in the set

$$\{z \in \mathbb{C} : \zeta(z+1) \neq 0\}.$$

By the "toy Riemann Hypothesis" — the theorem we proved according to which $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$, $g(z)$ in particular extends holomorphically to an open set containing the half-plane $\mathrm{Re}(z) \geq 0$. Thus, $f(t)$ satisfies the assumption of Newman's theorem. We conclude from the theorem that the integral

$$\begin{aligned}\int_0^\infty f(t)\,dt = \int_0^\infty (\psi(e^t)e^{-t} - 1)dt = \int_1^\infty \left(\frac{\psi(x)}{x} - 1\right)\frac{dx}{x} \\ = \int_1^\infty \frac{\psi(x) - x}{x^2}\,dx\end{aligned}$$

converges.

164. **Proof of the prime number theorem.** We will prove that $\psi(x) \sim x$, which we already showed is equivalent to the prime number theorem. Assume by contradiction that $\limsup_{x\to\infty} \frac{\psi(x)}{x} > 1$ or $\liminf_{x\to\infty} \frac{\psi(x)}{x} < 1$. In the first case, that means there exists a number $\lambda > 1$ such that $\psi(x) \geq \lambda x$ for arbitrarily large x. For such values of x it then follows that

$$\int_x^{\lambda x} \frac{\psi(t) - t}{t^2}\,dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt = \int_1^\lambda \frac{\lambda - t}{t^2}\,dt =: A > 0,$$

but this is inconsistent with the fact that the integral $\int_1^\infty (\psi(x) - x)x^{-2}\,dx$ converges.

Similarly, in the event that $\liminf_{x\to\infty} \frac{\psi(x)}{x} < 1$, that means that there exists a $\mu < 1$ such that $\psi(x) \leq \mu x$ for arbitrarily large x, in which case we have that

$$\int_{\lambda x}^x \frac{\psi(t) - t}{t^2}\,dt \leq \int_{\lambda x}^x \frac{\lambda x - t}{t^2}\,dt = \int_\lambda^1 \frac{\lambda - t}{t^2}\,dt =: B < 0,$$

again giving a contradiction to the convergence of the integral. □

17 Introduction to asymptotic analysis

165. In this section we'll learn how to use complex analysis to prove asymptotic formulas such as

$$\begin{aligned} n! &\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n && \text{(Stirling's formula)}, \\ p(n) &\sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}} && \text{(the Hardy-Ramanujan formula)}, \\ \mathrm{Ai}(x) &\sim \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right) && \text{(asymptotics for the Airy function)}, \end{aligned}$$

and more. At the heart of many such results is an important technique known as the **saddle point method**. Some related techniques (that are all minor variations on the same theme) are **Laplace's method**, the **steepest descent method** and the **stationary phase method**.

166. **A toy estimate for $n!$.** For $x > 0$ real, we have

$$\frac{x^n}{n!} \le \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

which gives a lower bound

$$n! \ge e^{-x} x^n.$$

for $n!$. It makes sense to try to get the best lower bound possible by looking for the x where the lower-bounding function is maximal. This happens when

$$0 = \frac{d}{dx}\left(e^{-x}x^n\right) = e^{-x}\left(-x^n + nx^{n-1}\right) = e^{-x}x^{n-1}(-x+n),$$

i.e., when $x = n$. Plugging this value into the inequality gives the bound

$$n! \ge (n/e)^n \qquad (n \ge 1).$$

This is of course a standard and very easy result. The point of this computation is that, as we shall see below, there is something special about the value $x = n$ that resulted from this maximization operation; when interpreted in the context of complex analysis, it corresponds to a so-called "saddle point," since it is a local minimum of e^x/x^n as one moves along the real axis, but it will be a local maximum when one moves in the orthogonal direction parallel to the imaginary axis.

167. **First example: Stirling's formula.** Start with the power series expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

As we know very well from our study of Cauchy's integral formula and the residue theorem, the nth Taylor coefficient can be extracted from the function by contour integration, namely by writing

$$\frac{1}{n!} = \frac{1}{2\pi i}\oint_{|z|=r} \frac{e^z}{z^{n+1}}\,dz,$$

where the radius r of the circle chosen as the contour of integration is an arbitrary positive number. It turns out that some values of r are better than others when one is trying to do asymptotics. We select $r = n$ (I'll explain later where that seemingly inspired choice comes from), to get

$$\begin{aligned}\frac{1}{n!} &= \frac{1}{2\pi i}\oint_{|z|=n} \frac{e^z}{z^{n+1}}dz = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \exp\left(ne^{it}\right) n^{-n}e^{-int} i\,dt \\ &= \frac{1}{2\pi\, n^n}\int_{-\pi}^{\pi} \exp\left(n(e^{it} - it)\right)\, dt \\ &= \frac{e^n}{2\pi\, n^n}\int_{-\pi}^{\pi} \exp\left(n(e^{it} - 1 - it)\right)\, dt,\end{aligned}$$

where we have strategically massaged the integrand (by pulling out the factor e^n) to cancel out a term in the Taylor expansion of e^{it}, in addition to a term that was already canceled out. For convenience, rewrite this as

$$\frac{n^n}{e^n n!} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(n(e^{it} - 1 - it)\right)\, dt.$$

Now noting that

$$n\left(e^{it} - 1 - it\right) = -\frac{nt^2}{2} + O(nt^3) = \frac{(\sqrt{n}t)^2}{2} + O\left(\frac{(\sqrt{n}t)^3}{\sqrt{n}}\right),$$

for $|t|$ small, we see that a change of variable $u = \sqrt{n}t$ in the integral will enable us to rewrite this as

$$n\left(e^{iu/\sqrt{n}} - 1 - \frac{iu}{\sqrt{n}}\right) = -\frac{u^2}{2} + O\left(\frac{u^3}{\sqrt{n}}\right).$$

Performing the change of variable and moving a factor of $\sqrt{n}$ to the left-hand side, the integral then becomes

$$\frac{\sqrt{n}n^n}{e^n n!} = \frac{1}{2\pi}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \exp\left(n\left(e^{iu/\sqrt{n}} - 1 - \frac{iu}{\sqrt{n}}\right)\right) du.$$

The integrand converges pointwise to $e^{-u^2/2}$ (for u fixed and $n \to \infty$), so it's reasonable to guess that the integral should converge to $\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}$, which would lead to the formula

$$\frac{\sqrt{n}n^n}{e^n n!} \approx \frac{1}{\sqrt{2\pi}},$$

or

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

which is precisely Stirling's formula. However, note that the $O(u^3/\sqrt{n})$ estimate holds whenever $t = u/\sqrt{n}$ is in a neighborhood of 0, and since u actually ranges in $[-\pi\sqrt{n}, \pi\sqrt{n}]$, we need to be more careful to get a precise asymptotic result.

To proceed, it makes sense to divide the integral into two parts. Denote $M = n^{1/10}$, and let

$$I = \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \exp\left(n\left(e^{iu/\sqrt{n}} - 1 - \frac{iu}{\sqrt{n}}\right)\right) du = I_1 + I_2,$$
$$I_1 = \int_{-M}^{M} \exp\left(n\left(e^{iu/\sqrt{n}} - 1 - \frac{iu}{\sqrt{n}}\right)\right) du,$$
$$I_2 = \int_{[-\pi\sqrt{n},\pi\sqrt{n}]\setminus[-M,M]} \exp\left(n\left(e^{iu/\sqrt{n}} - 1 - \frac{iu}{\sqrt{n}}\right)\right) du.$$

We now estimate each of I_1 and I_2 separately. For I_1, we have

$$\begin{aligned}
I_1 &= \int_{-M}^{M} \exp\left(-\frac{u^2}{2} + O\left(\frac{u^3}{\sqrt{n}}\right)\right) du \\
&= \int_{-M}^{M} e^{-u^2/2} \exp\left(O\left(\frac{u^3}{\sqrt{n}}\right)\right) du \\
&= \int_{-M}^{M} \left(1 + O\left(\frac{u^3}{\sqrt{n}}\right)\right) e^{-u^2/2} du = \left(1 + O(n^{-1/5})\right) \int_{-M}^{M} e^{-u^2/2} du \\
&= \left(1 + O(n^{-1/5})\right) \left(\int_{-\infty}^{\infty} -2\int_{M}^{\infty}\right) e^{-u^2/2} du \\
&= \left(1 + O(n^{-1/5})\right) \left(\sqrt{2\pi} - O\left(\exp\left(-n^{-1/5}\right)\right)\right) \\
&= \left(1 + O(n^{-1/5})\right) \sqrt{2\pi}.
\end{aligned}$$

For I_2, we have

$$\begin{aligned}
|I_2| &\le 2\int_{M}^{\pi\sqrt{n}} \left|\exp\left(n\left(e^{iu/\sqrt{n}} - 1 - \frac{iu}{\sqrt{n}}\right)\right)\right| du \\
&= 2\int_{M}^{\pi\sqrt{n}} \exp\left(n \operatorname{Re}\left(e^{iu/\sqrt{n}} - 1\right)\right) du \\
&= 2\int_{M}^{\pi\sqrt{n}} \exp\left[n\left(\cos\left(\frac{u}{\sqrt{n}}\right) - 1\right)\right] du
\end{aligned}$$

Now use the elementary fact that $\cos(t) \le 1 - t^2/8$ for $x \in [-\pi, \pi]$ (see Fig. 7) to infer further that

$$|I_2| \le 2\int_{M}^{\pi\sqrt{n}} \exp\left(-u^2/8\right) du \le 2\pi\sqrt{n}\exp\left(-n^{1/5}\right) = O(n^{-1/5}).$$

Combining the above results, we have proved the following version of Stirling's formula with a quantitative (though suboptimal) bound:

Theorem. As $n \to \infty$ we have $n! = \left(1 + O(n^{-1/5})\right)\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

168. **Second example: the central binomial coefficient.** Let $a_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. A standard way to find the asymptotic behavior for a_n as $n \to \infty$ is to use Stirling's formula. This easily gives that

$$\binom{2n}{n} = (1 + o(1))\frac{4^n}{\sqrt{\pi n}}.$$

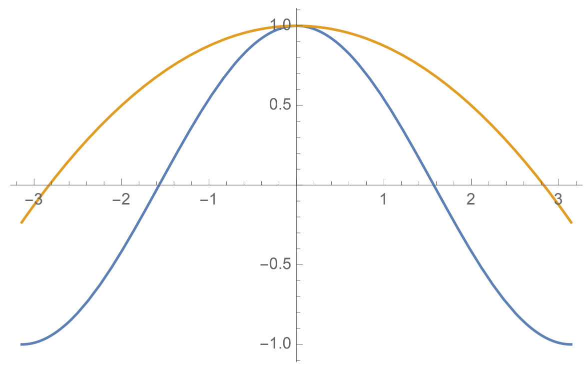


Figure 7: Illustration of the inequality $\cos(t) \leq 1 - t^2/8$.

(Note that this is not too far from the trivial upper bound $\binom{2n}{n} \leq (1+1)^{2n} = 2^{2n}$.) It is instructive to rederive this result using the saddle-point method, starting from the expansion

$$(1+z)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} z^n,$$

which in particular gives the contour integral representation

$$\binom{2n}{n} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{(1+z)^{2n}}{z^{n+1}} dz.$$

By the same trivial method for deriving upper bounds that we used in the case of the Taylor coefficients $1/n!$ of the function e^z, we have that for each $x > 0$,

$$\binom{2n}{n} \leq (1+x)^{2n}/x^n = \exp\left(\log(1+x) - n \log x\right).$$

We optimize over x by differentiating the expression $\log(1+x) - n \log x$ inside the exponent and setting the derivative equal to 0. This gives $x = 1$, the location of the saddle point. For this value of x, we again recover the trivial inequality $\binom{2n}{n} \leq 2^{2n}$.

Next, equipped with the knowledge of the saddle point, we set $r = 1$ in the contour integral formula, to get

$$\begin{aligned}\binom{2n}{n} &= \frac{1}{2\pi i} \oint_{|z|=r} \frac{(1+z)^{2n}}{z^{n+1}} dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+e^{it})^{2n} e^{-int}\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big(n \Big(2\log(1+e^{it}) - t\Big)\Big)\, dt.\end{aligned}$$

Now note that the expression in the exponent has the Taylor expansion

$$n(2\log(1+e^{it}) - t) = 2\log 2 - \frac{1}{4} n t^2 + O(nt^4) \qquad \text{as } t \to 0.$$

Again, we see that a change of variables $u = t/\sqrt{n}$ will bring the integrand to an asymptotically scale-free form. More precisely, we have

$$\binom{2n}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(n\left(2\log 2 - \frac{1}{4}nt^2 + O(nt^4)\right)\right) dt$$
$$= \frac{4^n}{2\pi\sqrt{n}}\int_{-\pi}^{\pi} \exp\left(-\frac{1}{4}u^2 + O\left(\frac{u^4}{n}\right)\right) du.$$

It is now reasonable to guess that in the limit as $n \to \infty$, the pointwise limit of the integrands translates to a limit of the integrals, so that we get the approximation

$$\binom{2n}{n} \approx \frac{4^n}{2\pi\sqrt{n}}\int_{-\infty}^{\infty} e^{-u^2/4}\, du = \frac{4^n}{2\pi\sqrt{n}} 2\sqrt{\pi} = \frac{4^n}{\sqrt{\pi n}},$$

as required. Indeed, this is correct, but it remains to make this argument precise by breaking up the integral into two parts, a "central part" where the $O(u^4/n)$ error term can be shown to be small, and the remaining part that has to be bounded separately.

169. **Exercise.** Complete this analysis to give a rigorous proof using this method of the asymptotic formula $\binom{2n}{n} = (1+o(1))4^n/\sqrt{\pi n}$.

170. **Exercise.** Repeat this analysis for the sequence $(b_n)_{n=1}^{\infty}$ of central *trinomial* coefficients, where b_n is defined as the coefficient of x^n in the expansion of $(1+x+x^2)^n$, a definition that immediately gives rise to the contour integral representation

$$b_n = \frac{1}{2\pi i}\oint_{|z|=r} \frac{(1+z+z^2)^n}{z^{n+1}}\, dz.$$

Like their more famous cousins the central binomial coefficients, these coefficients are important in combinatorics and probability theory. Specifically, a_n and b_n correspond to the numbers of random walks on $\mathbb{Z}$ that start and end at 0 and have n steps, where in the case of the central binomial coefficients the allowed steps of the walk are -1 or $+1$, and in the case of the central trinomial coefficients the allowed steps are -1, 0 or 1; see Fig. 8.

Using a saddle point analysis, show that the asymptotic behavior of b_n as $n \to \infty$ is given by

$$b_n \sim \frac{\sqrt{3}\cdot 3^n}{\sqrt{\pi n}}.$$

171. **A conceptual explanation.** In both the examples of Stirling's formula and the central binomial coefficient we analyzed above, the quantities we were trying to estimate took a particular form, where for some function $g(z)$ we had

$$a(n) = \frac{1}{2\pi i}\oint_{|z|=r} \frac{e^{-ng(z)}}{z^n}\frac{dz}{z} = \frac{1}{2\pi i}\oint_{|z|=r} \exp\Big(-n(g(z)+\log z)\Big)\frac{dz}{z}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ng(re^{it})} r^{-n} e^{-int}\, dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\Big(-n(g(re^{it}) + it - \log r)\Big) dt.$$

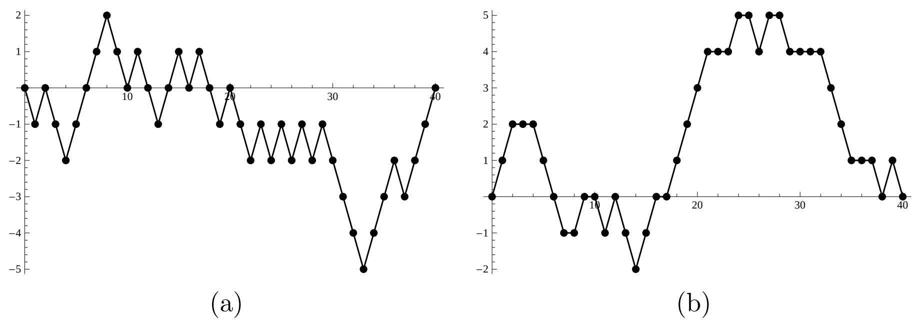


Figure 8: An illustration (with $n = 40$) of the random walks enumerated by (a) the central binomial coefficients and (b) the central trinomial coefficients.

(Sometimes $g(z)$ would actually be $g_n(z)$, a sequence of functions that depends on n.) The idea is to choose the contour radius r as the solution to the equation

$$\frac{d}{dz}\left(g(z) + \log z\right) = g'(z) + \frac{1}{z} = 0.$$

This causes the first-order term in the Taylor expansion of $g(z)+\log z$ around $z = r$ to disappear. One is then left with a constant term, that can be pulled outside of the integral; a second order term, which (in favorable circumstances where this technique actually works) causes the integrand to be well-approximated by a Gaussian density function $e^{-u^2/2}$ near $z = r$; and lower-order terms which can be shown to be asymptotically negligible.

Geometrically, if one plots the graph of $|g(z)+1/z|$ then one finds the emergence of a saddle point at $z = r$, and this is the origin of the term "saddle point method." This phenomenon is illustrated with many beautiful examples and graphical figures in the lecture slides prepared by Sedgewick and Flajolet as an online resource to accompany their excellent textbook Analytic Combinatorics. The lecture slides can be accessed at
`http://ac.cs.princeton.edu/lectures/lectures13/AC08-Saddle.pdf`.

172. **Exercise.** It is instructive to see an example where the saddle point analysis *fails* if applied mindlessly without checking that the part of the integral that is usually assumed to make a negligible contribution actually behaves that way. A simple example illustrating what can go wrong is the function

$$f(z) = e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} = \sum_{n=0}^{\infty} b_n z^n,$$

where the Taylor coefficients are

$$b_n = \begin{cases} \frac{1}{(n/2)!} & n \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly any analysis, asymptotic or not, needs to address and take into account the fact that b_n behaves differently according to whether n is even or odd. Try to apply the method we developed to derive an asymptotic formula for b_n. The method fails, but the failure can easily be turned into a success by noting that there are actually *two* saddle points, each of which makes a contribution to the integral, in such a way that for odd n the contributions cancel and for even n they reinforce each other. This shows that *periodicities* are one common pitfall to look out for when doing asymptotic analysis.

173. **Exercise.** As another amusing example, apply the saddle point method to the function $f(z) = 1/(1-z) = \sum_{n=0}^\infty d_n z^n$, for which the Taylor coefficients $d_n = 1$ are all equal to 1. Can you succeed in deriving an asymptotic formula for the constant function 1?

174. **Third example: Stirling's formula for the gamma function.** Our next goal is to prove a stronger version of Stirling's formula that gives an asymptotic formula for $\Gamma(t)$, the extension of the factorial function to non-integer arguments. Specifically, we will prove.

Theorem. For a real-valued argument t, the gamma function satisfies the asymptotic formula

$$\Gamma(t) = \left(1 + O(t^{-1/5})\right)\sqrt{\frac{2\pi}{t}}\left(\frac{t}{e}\right)^t \qquad (t \to \infty).$$

Proof. We use a method called **Laplace's method**, which is a variant of the saddle-point method adapted to estimating real integrals instead of contour integrals around a circle. Start with the integral formula

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\,dx$$

Performing the change of variables $x = tu$ in the integral gives that

$$\begin{aligned}\Gamma(t) &= t^t \int_0^\infty e^{-tu} u^{t-1}\,du = t^t e^{-t} \int_0^\infty e^{-tu+t} u^{t-1}\,du \\ &= t^t e^{-t} \int_0^\infty e^{-tu+t} u^{t-1}\,du = t^t e^{-t} \int_0^\infty e^{-t\Phi(u)}\frac{du}{u} = \left(\frac{t}{e}\right)^t I(t),\end{aligned}$$

where we define

$$\begin{aligned}\Phi(u) &= u - 1 - \log u, \\ I(t) &= \int_0^\infty e^{-t\Phi(u)}\frac{du}{u}.\end{aligned}$$

(Again, note that we massaged the integrand to cancel the Taylor expansion of $-\log u$ around $u = 1$ up to the first order.) Our goal is to prove that

$$I(t) = \sqrt{\frac{2\pi}{t}} + O(t^{-7/10}) \qquad \text{as } t \to \infty.$$

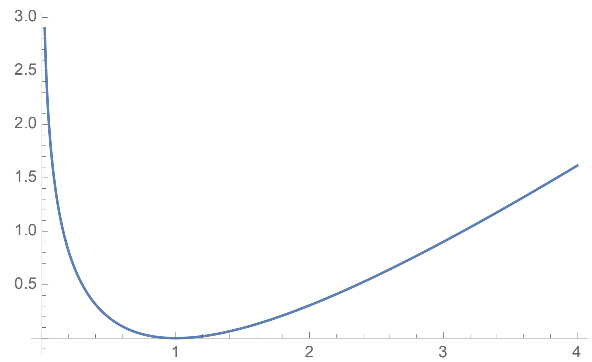


Figure 9: The function $\Phi(u) = u - 1 - \log u$.

As before, this will be done by splitting the integral into a main term and error terms. The idea is that for large t, the bulk of the contribution to the integral comes from a region very near the point where $\Phi(u)$ takes its minimum. It is easy to check by differentiation that this minimum is obtained at $u = 1$, and that we have

$$\Phi(1) = 0, \quad \Phi'(1) = 0, \quad \Phi''(1) = 1,$$

and $\Phi(u) \geq 0$ for all $u \geq 0$. See Fig. 9. Denote

$$\begin{aligned} I_1 &= \int_0^{1/2} e^{-t\Phi(u)} \frac{du}{u}, \\ I_2 &= \int_{1/2}^{2} e^{-t\Phi(u)} \frac{du}{u}, \\ I_3 &= \int_2^{\infty} e^{-t\Phi(u)} \frac{du}{u}, \end{aligned}$$

so that $I(t) = I_1 + I_2 + I_3$. The main contribution will come from I_2, the part of the integral that contains the critical point $u = 1$, so let us examine that term first. Expanding $\Phi(u)$ in a Taylor series around $u = 1$, we have

$$\Phi(u) = \frac{(u-1)^2}{2} + O((u-1)^3)$$

for $u \in [1/2, 2]$ (in fact the explicit bound $\left|\Phi(u) - \frac{(u-1)^2}{2}\right| \leq (u-1)^3$ on this interval can be easily checked). As before, noting that

$$t\left[\frac{(u-1)^2}{2} + O((u-1)^3)\right] = \frac{1}{2}(\sqrt{t}(u-1))^2 + O\left(\frac{(\sqrt{t}(u-1))^3}{\sqrt{t}}\right),$$

we see that it is natural to apply a linear change of variables $v = \sqrt{t}(u-1)$ to bring the integrand to a scale-free, centered form. This results in

$$\begin{aligned} I_2 &= \frac{1}{\sqrt{t}} \int_{-\frac{1}{2}\sqrt{t}}^{\sqrt{t}} \exp\left(-t\Phi\left(1+\frac{v}{\sqrt{t}}\right)\right) \frac{1}{1+v/\sqrt{t}}\, dv \\ &= \frac{1}{\sqrt{t}} \int_{-\frac{1}{2}\sqrt{t}}^{\sqrt{t}} \exp\left(-\frac{v^2}{2}+O\left(\frac{v^3}{\sqrt{t}}\right)\right)\left(1+O\left(\frac{t}{\sqrt{t}}\right)\right) dv. \end{aligned}$$

As before, we actually need to split up this integral into two parts to take into account the fact that the $O(v^3/\sqrt{t})$ term can blow up when v is large enough. Let $M = t^{1/10}$, and denote

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{t}} \int_{-M}^{M} \exp\left(-t\Phi\left(1+\frac{v}{\sqrt{t}}\right)\right) \frac{1}{1+v/\sqrt{t}}\, dv, \\ J_2 &= \frac{1}{\sqrt{t}} \int_{[-\frac{1}{2}\sqrt{t},\sqrt{t}]\setminus[-M,M]} \exp\left(-t\Phi\left(1+\frac{v}{\sqrt{t}}\right)\right) \frac{1}{1+v/\sqrt{t}}\, dv, \end{aligned}$$

so that $I_2 = J_1 + J_2$. For J_1 we have

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{t}} \int_{-M}^{M} \exp\left(-t\Phi\left(1+\frac{v}{\sqrt{t}}\right)\right) \frac{1}{1+v/\sqrt{t}}\, dv \\ &= \frac{1}{\sqrt{t}} \int_{-M}^{M} e^{-v^2/2}\left(1+O\left(\frac{v^3}{\sqrt{t}}\right)\right)\left(1+O\left(\frac{v}{\sqrt{t}}\right)\right) dv \\ &= \frac{1}{\sqrt{t}}\left(1+O(t^{-1/5})\right)\int_{-M}^{M} e^{-v^2/2} dv = \sqrt{\frac{2\pi}{t}}\left(1+O(t^{-1/5})\right), \end{aligned}$$

in the last step using a similar estimate as the one we used in our proof of Stirling's approximation for $n!$. Next, for J_2 we use the elementary inequality (prove it as an exercise)

$$\Phi(u) \geq \frac{(u-1)^2}{2} \qquad (0 \leq u \leq 1),$$

and the more obvious fact that $1/(1+v/\sqrt{t}) \leq 2$ for $v \in [-\frac{1}{2}\sqrt{t}, \sqrt{t}]$ to get that

$$\begin{aligned} J_2 &\leq \frac{2}{\sqrt{t}} \int_{[-\frac{1}{2}\sqrt{t},\sqrt{t}]\setminus[-M,M]} e^{-v^2/2}\, dv \leq \frac{4}{\sqrt{t}}\int_M^\infty e^{-v^2/2}\, dv \\ &= O(e^{-M}) = \frac{1}{\sqrt{t}} O(t^{-1/5}). \end{aligned}$$

as in our earlier proof. Combining the above results, we have shown that

$$I_2 = \left(1+O(t^{-1/5})\right)\sqrt{\frac{2\pi}{t}}.$$

Next, we bound I_1. Here we use a different method since there is a different source of potential trouble near the left end $u = 0$ of the integration interval. Considering first a truncated integral over $[\varepsilon, 1/2]$ and performing an integration by parts, we have

$$\int_\varepsilon^{1/2} e^{-t\Phi(u)}\frac{du}{u} = -\frac{1}{t}\int_\varepsilon^{1/2} \frac{d}{du}\left(e^{-t\Phi(u)}\right)\frac{1}{\Phi'(u)u}\,du$$
$$= -\frac{1}{t}\left[\frac{e^{-t\Phi(u)}}{u-1}\right]_{u=\varepsilon}^{u=1/2} - \frac{1}{t}\int_\varepsilon^{1/2} e^{-t\Phi(u)}\frac{du}{(u-1)^2}.$$

Taking the limit as $\varepsilon \to 0$ (and noting that $\Phi(\varepsilon) \to +\infty$ in this limit) yields the formula

$$I_1 = \frac{2}{t}e^{-t\Phi(1/2)} - \frac{1}{t}\int_0^{1/2} e^{-t\Phi(u)}\frac{du}{(u-1)^2} = O\left(\frac{1}{t}\right) \qquad \text{as } t \to \infty.$$

Finally, I leave it as an exercise to obtain a similar estimate $I_3 = O(1/t)$ for the remaining integral on $[2, \infty)$. Combining the various estimates yields the claimed result that

$$I(t) = I_1 + I_2 + I_3 = \left(1 + O(t^{-1/5})\right)\sqrt{\frac{2\pi}{t}}.$$

□

175. The proof above is a simplified version of the analysis in Appendix A of [Stein-Shakarchi]. The more detailed analysis there shows that the asymptotic formula we proved for $\Gamma(t)$ remains valid for complex t. Specifically, they prove that for complex s in the "Pac-Man shaped" region

$$S_\delta = \{z \in \mathbb{C} \,:\, |\arg z| \geq \pi - \delta\}$$

(for each fixed $0 < \delta < \pi$) the gamma function satisfies

$$\Gamma(s) = \left(1 + O(|s|^{-1/2})\right)\sqrt{2\pi}s^{s-1/2}e^{-s} \qquad \text{as } |s| \to \infty,\ s \in S_\delta.$$

Here, $s^{s-1/2}$ is defined as $\exp((s - 1/2)\operatorname{Log} s)$, where Log denotes as usual the principal branch of the logarithm function.

Additional reading

While preparing these notes I consulted the following sources, which contain a large amount of additional interesting material related to the topics we covered.

[1] H. M. Edwards. Riemann's Zeta Function. Dover Publications, 2001.

[2] P. Flajolet, R. Sedgewick. Analytic Combinatorics. Cambridge University Press, 2009.

[3] P. Flajolet, R. Sedgewick. Analytic Combinatorics Lecture Slides. Online resource: `http://ac.cs.princeton.edu/lectures/`. Accessed March 6, 2016.

[4] B. de Smit, H. W. Lenstra Jr. Artful mathematics: the heritage of M.C. Escher. *Notices Amer. Math. Soc.* **50** (2003), 446–457.

[5] E. M. Stein, R. Shakarchi, Complex Analysis. Princeton University Press, 2003. **104** (1997), 705–708.

[6] A. Ivić. The Riemann Zeta-Function: Theory and Applications. Dover Publications, 2003.

[7] D. Zagier. Newman's short proof of the prime number theorem. *Amer. Math. Monthly* 104 (1997), 705–708.