Lecture 1: Variational Problems I

Note Title

☆ Newton's Equations of Motion

We'll use the following notation:

$\mathbb{r} = \mathbb{r}(t) = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$
$= (x_1(t), \cdots, x_n(t))^T$

a point (or particle) in $\mathbb{R}^n$
it's a position vector.

$\mathbb{e}_i = (0, 0, \cdots, 0, \underset{\uparrow i}{1}, 0, \cdots, 0)^T \in \mathbb{R}^n$

The ith standard (or canonical) basis vector of $\mathbb{R}^n$

$$\dot{\mathbb{r}} = \frac{d\mathbb{r}}{dt} = (\dot{x}_1, \cdots, \dot{x}_n)^T$$

the time derivative, i.e., the velocity vector of the particle.

For $n = 3$, we often use $(x, y, z)^T$ instead of $(x_1, x_2, x_3)^T$, and $\mathbb{i}, \mathbb{j}, \mathbb{k}$ instead of $\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3$.

Consider a particle of mass m with its position $\mathbb{r}(t) \in \mathbb{R}^3$ moving under an external force $\mathbb{F} \in \mathbb{R}^3$.

Then, it satisfies the following famous Newton's Equation of Motion:

$$\underbrace{m\underbrace{\ddot{\mathbb{r}}}_{\text{acceleration}}}_{\text{inertial force}} = \mathbb{F}$$

$\Rightarrow$ Can solve for $\mathbb{r}(t)$ given $\mathbb{r}(0)$ & $\dot{\mathbb{r}}(0)$.

Now, let's consider a special force

$$\mathbb{F} = -\nabla V, \qquad V = V(x, y, z).$$

$$= \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right)^T$$

a potential function (or potential energy)

Now suppose a particle moves under such a potential field with no other external forces.

Suppose we do not know Newton's eqn. What can we say?

Leibniz's viewpoint!

<u>Def.</u> The kinetic energy of a particle is

$$T := \frac{1}{2} m |\dot{\mathbb{r}}|^2, \quad |\cdot| \text{ is the Euclidean norm.}$$

The total energy of the particle is

$$E := T + V$$

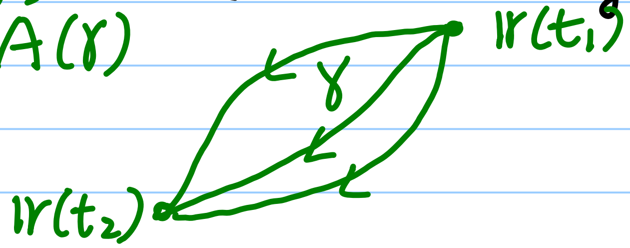
If $E \equiv$ const. indep. of time t, then its field is called conservative.
(i.e., Conservation of energy)

<u>Q</u>: Show that if the particle satisfies Newton's eqn of Motion, $m\ddot{\mathbb{r}} = -\nabla V$, then it's a conservative field.

<u>Def.</u> The action of the particle along a continuous path γ starting at $\mathbb{r}(t_1)$ and arriving at $\mathbb{r}(t_2)$, $t_1 \le t_2$ is defined as

$$\underbrace{A}_{\substack{\uparrow \\ \text{depends on} \\ \text{a path } \gamma, \\ \text{so } A = A(\gamma)}} := \int_{t_1}^{t_2} (\underbrace{T - V}_{=: L}) \, dt = \int_{t_1}^{t_2} L \, dt$$

L: the Lagrangian (or the Lagrange fcn)



$\Rightarrow$ Euler & Lagrange thought

an actual path $= \arg\min_{\gamma \in \Gamma} A(\gamma)$

Γ: a set of admissible curves

or more precisely,

$$\Gamma = \{ \gamma(t) = (x(t), y(t), z(t))^T, \ t_1 \le t \le t_2 \mid \underbrace{\gamma(t_i) = \mathbb{r}(t_i)}, \ i = 1, 2, \ x, y, z \in C^1[t_1, t_2] \}.$$

Remark: Under the conservative field,

$E = T + V = \text{const.}$, say C.

So, $L = T - V = 2T - C$

$\Rightarrow \underset{\gamma \in \Gamma}{\arg\min}\, A(\gamma)$ amounts to

$$\underset{\gamma \in \Gamma}{\arg\min} \int_{t_1}^{t_2} T\, dt$$

minimization of the time integral of the kinetic energy

$\Rightarrow$ "Nature always minimizes action."

"God made the world in the most economical way."

See: Hildebrandt & Tromba: The Parsimonious Universe, Springer, 1996.

- Principle of Least Action

(Leibniz 1696; Maupertuis 1746; Euler 1744; Lagrange 1760; Hamilton 1834; ...)

A particle with mass m and potential energy $V(\mathbf{r})$ takes, in the time interval $[t_1, t_2]$, the path γ^* s.t.

$$\gamma^* = \underset{\gamma \in \Gamma}{\arg\min}\, A(\gamma) = \underset{\gamma \in \Gamma}{\arg\min} \int_{t_1}^{t_2} (T - V)\, dt$$

Let's check this!

$\mathbb{r}(t) = (x(t), y(t), z(t))^T$

(*) $A = \int_{t_1}^{t_2} \left\{ \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z) \right\} dt$

Let $\mathbb{r}(t_i) = P_i = (x_i, y_i, z_i)^T$, $i=1,2$.

As Euler & Lagrange did, let's derive the necessary cond. for a path γ^* to be a minimizer of (*).

Let $\gamma^* = \gamma^*(t) = (x^*(t), y^*(t), z^*(t))^T$

$t \in [t_1, t_2]$, to be such a minimizer and assume $\gamma^* \in \Gamma$.

γ^* is said to be an extremal.

Now consider a path slightly deviating from γ^*, i.e., $\gamma^*_\varepsilon = (x^*(t) + \varepsilon\xi(t), y^*(t), z^*(t))^T \in \Gamma$

So, $\xi(t_i) = 0$, $i=1,2$.

γ^*_ε is called a variation of γ^*.

$\Rightarrow$ Hence the name Calculus of variations.

$$A(\gamma^*_\varepsilon) = A(\varepsilon) = \int_{t_1}^{t_2} \left\{ \frac{m}{2}\left((\dot{x}^* + \varepsilon\dot{\xi})^2 + \dot{y}^{*2} + \dot{z}^{*2}\right) - V(x^* + \varepsilon\xi, y^*, z^*) \right\} dt$$

Since $\gamma^* = \gamma^*_0$ is the extremal, minimizer, we must have $\left.\dfrac{dA}{d\varepsilon}\right|_{\varepsilon=0} = 0$

$\delta A := \left.\dfrac{dA}{d\varepsilon}\right|_{\varepsilon=0}$ is called the first variation of A.

$$\delta A = \frac{d}{d\varepsilon}\int_{t_1}^{t_2}\left\{\frac{m}{2}\left(\dot{x}^{*2} + 2\varepsilon\dot{x}^*\dot{\xi} + \varepsilon^2\dot{\xi}^2 + \dot{y}^{*2} + \dot{z}^{*2}\right) - V(x^* + \varepsilon\xi, y^*, z^*)\right\} dt \,\Big|_{\varepsilon=0}$$

$$= \int_{t_1}^{t_2}\left(m\dot{x}^*\dot{\xi} - \xi\frac{\partial V}{\partial x}(x^*, y^*, z^*)\right) dt$$

Int. by Parts

$$\overset{\downarrow}{=} \underbrace{m\dot{x}^*\xi\Big|_{t_1}^{t_2}}_{=0 \text{ since } \xi(t_i)=0} - \int_{t_1}^{t_2}\left(m\ddot{x}^*\xi + \xi\frac{\partial V}{\partial x}\right) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\xi\left(m\ddot{x}^* + \frac{\partial V}{\partial x}\right) dt = 0$$

This must hold for all admissible $\xi(t)$, i.e., $\xi \in \Gamma_0 := \Gamma$ with $P_1 = P_2 = 0$

By the Fundamental Lemma of the Calculus of Variations (to be discussed later), we must have

$$m\ddot{x}^* + \frac{\partial V}{\partial x}(x^*, y^*, z^*) = 0$$

$$\text{i.e., } m\ddot{x}^* = -\frac{\partial V}{\partial x}(x^*, y^*, z^*)$$

Similarly, by the variation w.r.t. y^* & z^*, we recover Newton's eqn. of Motion:

$$m\ddot{\mathbf{r}}(t) = -\nabla V(\mathbf{r}) \;!!$$

Note that $\delta A = 0$ is just a necessary cond., just like the usual calculus. The sufficient cond. for the minimizer involves the second variation, and we won't discuss in this course.
See, e.g., Gelfand & Fomin or Kot if you want to know the detail.

It's interesting to note that the only necessity, i.e., $\delta A = \left.\frac{dA}{d\varepsilon}\right|_{\varepsilon=0} = 0$,

was sufficient to derive Newton's eqn. of Motion! We really didn't have to find an absolute minimizer. Just finding a stationary path was sufficient for N. eqn.

Def. If a path $\gamma \in \Gamma$ satisfies $\delta A = 0$, then such γ is called a stationary path, and we say the action takes on a stationary value.

Hamilton generalized such Principle of Least Action for the mechanics/dynamics of continua/body and reached:

Hamilton's Principle of Stationary Action (1834–35):

A mechanical system with the kinetic energy T & the potential energy V behaves within time $t \in [t_1, t_2]$ for a given initial & end position s.t.
$$A = \int_{t_1}^{t_2} \underbrace{L}_{= T - V} \, dt$$
assumes a stationary value.