Lecture 2: Variational Problems II

Note Title

☆ The Euler-Lagrange Eqn.

Consider a more general problem than the particle system in Lecture 1.

Let $\Gamma := \{ y \in C^1[x_1, x_2] \mid y(x_i) = y_i,\ i = 1, 2 \}$.

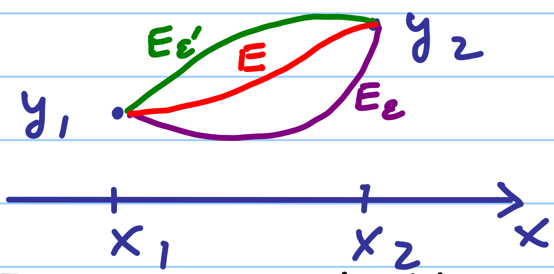
Find $y \in \Gamma$ s.t.

$$(*) \quad I = \int_{x_1}^{x_2} f(x, y, y')\, dx \to \min!$$

where $f \in C^2$ in each variable.

As before, assume $E : y = y^*(x)$ is the extremal (minimizer) and consider a one-parameter variation of E, i.e., $E_\varepsilon : y = y^*(x, \varepsilon) = y^*_\varepsilon(x)$ with $\frac{\partial y^*}{\partial x}, \frac{\partial y^*}{\partial \varepsilon}, \frac{\partial^2 y}{\partial x \partial \varepsilon}$ are all cont. fcns in x, and $y^*(x, 0) = y^*(x)$, $y^*(x_i, \varepsilon) = y_i$, $i = 1, 2$.

← given const's.



For the notational convenience, let

$\frac{\partial y^*}{\partial \varepsilon}(x, \varepsilon) =: \eta(x, \varepsilon)$. Then clearly

$\eta(x_i, \varepsilon) = 0$, $i = 1, 2$.

Putting these in $(*)$, we get

$$I(\varepsilon) := \int_{x_1}^{x_2} f(x, y^*(x, \varepsilon), y^{*\prime}(x, \varepsilon))\, dx$$

For (*) to be min., $\delta I = \frac{dI}{d\varepsilon}(0) = 0$ is a must.

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \cdot \frac{\partial y_\varepsilon^*}{\partial \varepsilon} + \frac{\partial f}{\partial y'}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \cdot \frac{\partial y_\varepsilon^{*\prime}}{\partial \varepsilon} \right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \cdot \eta + \frac{\partial f}{\partial y'}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \cdot \eta' \right) dx$$

Int. by Parts

$$\overset{\downarrow}{=} \int_{x_1}^{x_2} \frac{\partial f}{\partial y}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \eta \, dx + \underbrace{\eta \frac{\partial f}{\partial y'}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \Big|_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2} \eta \frac{d}{dx} \frac{\partial f}{\partial y'}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \, dx$$

$$= \int_{x_1}^{x_2} \eta \left\{ \frac{\partial f}{\partial y}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) - \frac{d}{dx} \frac{\partial f}{\partial y'}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \right\} dx$$

$$\xrightarrow[\varepsilon = 0]{\text{set}} \int_{x_1}^{x_2} \eta \left\{ \frac{\partial f}{\partial y}(x, y^*, y^{*\prime}) - \frac{d}{dx} \frac{\partial f}{\partial y'}(x, y^*, y^{*\prime}) \right\} dx = 0$$

Since η is arbitrary as long as $\eta \in \Gamma_0$, by the F.L.C.V., we must have

admissible variation Γ & $\eta(x_i) = 0$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

The Euler-Lagrange Eqn.

Note that we assumed $\eta' \in C[x_1, x_2]$ for Int. by Parts & exchange of $\frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial x}$ to be valid. Also, $\frac{d}{dx} \frac{\partial f}{\partial y'}$ must be in $C[x_1, x_2]$.

$$\frac{d}{dx} \frac{\partial f}{\partial y'} = \frac{\partial^2 f}{\partial y' \partial x} + \frac{\partial^2 f}{\partial y' \partial y} y' + \frac{\partial^2 f}{\partial y'^2} y''$$

$$\Rightarrow y'' \in C[x_1, x_2], \text{ i.e., } y \in C^2[x_1, x_2].$$

Also, we will assume $\frac{\partial^2 f}{\partial y'^2} \not\equiv 0$ on $[x_1, x_2]$.

If $\frac{\partial^2 f}{\partial y'^2} \equiv 0$ on $[x_1, x_2]$, Then it's called a singular variational problem.

a regular variational problem

In this case, the derived differential eqn. will be of the first order, not the second.

So far, in deriving the E-L eqn. the important things were:

$\begin{cases} ① \text{ Integration by Parts + Bdry. cond.} \\ ② \text{ F.L.C.V.} \end{cases}$

* Fundamental Lemma of the Calc. of Var.

If $M \in C[x_1, x_2]$, $\eta \in C'[x_1, x_2]$ with $\eta(x_i) = 0$, $i = 1, 2$, and $\int_{x_1}^{x_2} \eta(x) M(x)\, dx = 0$ — (*) for all such η, then $M(x) \equiv 0$ on $[x_1, x_2]$.

again, this is a necessary cond.

(Proof) assume $M(x) \not\equiv 0$ on $x \in [x_1, x_2]$.

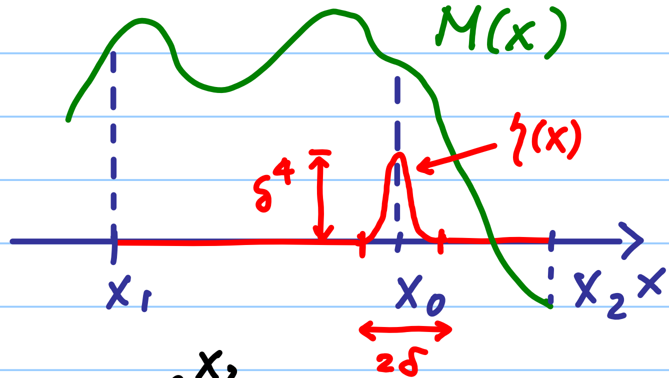
Then, $\exists\, x_0 \in [x_1, x_2]$ s.t. $M(x_0) \neq 0$.

WLOG, suppose $M(x_0) > 0$.

$M \in C[x_1, x_2] \Rightarrow \exists\, \delta > 0$ s.t. $M(x) > 0$ $\forall x$ in $|x - x_0| < \delta$.

Now consider the following fcn η:

$$\eta(x) := \begin{cases} 0 & \text{if } |x - x_0| > \delta \\ (x - x_0 + \delta)^2 (x - x_0 - \delta)^2 & \text{if } |x - x_0| \leq \delta. \end{cases}$$



This η satisfies the cond., i.e., $\eta \in C^1[x_1, x_2]$ with $\eta(x_i) = 0$, $i = 1, 2$

But $\eta \notin C^2[x_1, x_2]$.

also $\eta(x) > 0$ for $|x - x_0| < \delta$.

Now, $\displaystyle\int_{x_1}^{x_2} \eta(x) M(x)\, dx = \int_{x_0-\delta}^{x_0+\delta} (x - x_0 + \delta)^2 (x - x_0 - \delta)^2 M(x)\, dx$

> 0

This contradicts the premise (*). #

* <u>A generalization to n unknown fcns</u>

Suppose $y_k = y_k(x)$, $k = 1, \cdots, n$

$y_k \in C^1[x_1, x_2]$, $y_k(x_i) = y_k^{(i)}$, $k = 1:n$, $i = 1, 2$.

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx$$

where f is in C^2 in each variable.

Then using the same argument, we can derive n Euler-Lagrange eqn's.

$$\frac{\partial f}{\partial y_k} - \frac{d}{dx}\frac{\partial f}{\partial y_k'} = 0, \quad k = 1, \cdots, n.$$

<u>Ex.</u> Newton's eqn's of Motion in $\mathbb{R}^3$, i.e., $n = 3$.

$$f = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

Here, $(x, y, z, t) \longleftrightarrow (y_1, y_2, y_3, x)$

$\cdot \longleftrightarrow \dfrac{d}{dx}$

So, $\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0 \iff -\frac{\partial V}{\partial x} - m\ddot{x} = 0$

$\iff m\ddot{x} = -\frac{\partial V}{\partial x}$

Similarly for y & z,
we get $\quad m\,\ddot{\mathbb{r}}(t) = -\nabla V(\mathbb{r}(t))$

Now, how about the same system in $\mathbb{R}^3$, but in the polar coordinates (r, θ, φ) ? $\implies$ HW problem!

$$\begin{cases} \frac{\partial f}{\partial r} - \frac{d}{dt}\frac{\partial f}{\partial \dot{r}} = 0 \\ \frac{\partial f}{\partial \theta} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\theta}} = 0 \\ \frac{\partial f}{\partial \varphi} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\varphi}} = 0 \end{cases} \implies \quad ??$$