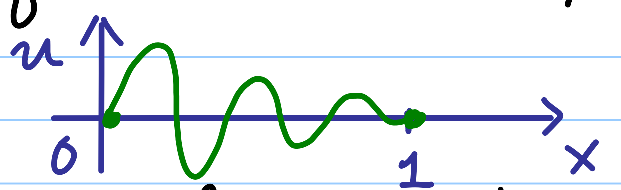
Lecture 3: Variational Problems III

Note Title

☆ Vibration of a Stretched String

Now we deal with variational problems involving more than one indep. variables, e.g., both x and t.

Consider a vibration of a string made of material with uniform density ρ, fixed at two pts $(0,0)$ & $(1,0)$

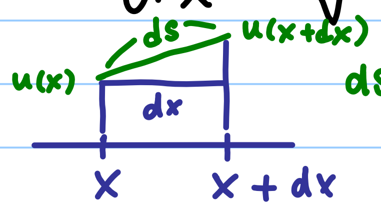


Let $u(x,t)$ be the displacement of this string at position x, time t.

We only consider deformations with small $\frac{\partial u}{\partial x}$. Consider an element of length ds of this string. The corresponding kinetic energy dT of ds is: $dT = \frac{1}{2} dm\, v^2 = \frac{\rho\, ds}{2}\left(\frac{\partial u}{\partial t}\right)^2$

On the other hand,

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} = 1 + \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + O\left(\left(\frac{\partial u}{\partial x}\right)^4\right)}_{\text{negligible!}}$$



$$ds = \sqrt{(dx)^2 + (du)^2}$$

$$\Longrightarrow ds \approx dx.$$

So, $dT \approx \frac{\rho\, dx}{2}\left(\frac{\partial u}{\partial t}\right)^2$. For the whole string,

$$T = \int_0^1 \frac{\rho\, dx}{2}\left(\frac{\partial u}{\partial t}\right)^2 = \frac{\rho}{2}\int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dx$$

ρ is a const. here!

How about the potential energy V?

$V =$ (the total external force) $\times$ (the increase in length)

If we only consider the const. tension τ as the ext. force, then

$$V = \tau\left(\int_0^1 \sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\,dx - \underset{\sim}{1}\right)$$

original length of the string

$$= \tau \int_0^1 \left(\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} - 1\right) dx$$

$$\approx \frac{\tau}{2}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx$$

So, $$I = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (T - V)\,dt$$

$$= \frac{1}{2}\int_{t_1}^{t_2}\int_0^1 \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}\right)^2\right] dx\,dt$$

Hamilton's Principle leads us to find the stationary value of I with

$$\begin{cases} \text{B.C.} & u(0,t) = u(1,t) = 0 \\ \text{I.C.} & u(x,t_1) = u_1(x),\ u(x,t_2) = u_2(x) \end{cases}$$

$\hookrightarrow$ rather than I.C., these are B.C. in the time variable!

In general, we have the following variational problem:

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} f\left(x, t, u(x,t), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) dx\,dt \to \min!$$

with certain B.C. & I.C. for u.

Let's derive the E-L eqn. for this general 2D problem first. Then, deal with the string.

☆ The E-L Eqn. for 2D Problem

Instead of $u(x,t)$, let's use $z(x,y)$, and consider the following problem.

Find $z = z(x,y)$ s.t. $z(x,y) = z_0(x,y)$ on C ← given data!

(C is a closed curve in the x-y plane) and

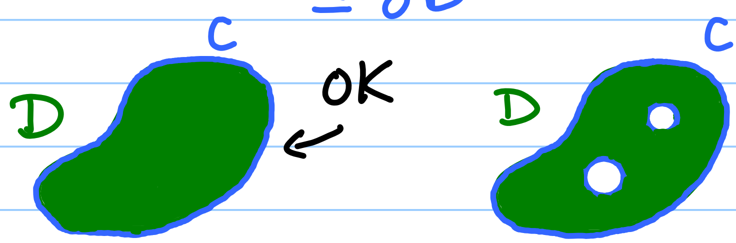
$$I = \iint_D f\left(x, y, z(x,y), \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\, dy \to \min!$$

D is the region (or domain) bdd. by C.

Assume D : simply connected

C : rectifiable $\overset{\text{def}}{\Longleftrightarrow}$ arclength$(C) < \infty$

$C = \partial D$



not simply conn. (it's called multiply connected)

Let $E : z = z^*(x,y)$, where z, $z_x := \frac{\partial z}{\partial x}$, $z_y := \frac{\partial z}{\partial y}$ are all in $C(D)$, ↳ continuous fcns in D

be the minimizer of I satisfying

B.C. $z^*(x,y) = z_0(x,y)$ on C.

Now consider a one-param. variation of the form: $E_\varepsilon : z = z(x,y,\varepsilon)$ with $E_0 = E$

i.e., $z(x,y,0) = z^*(x,y)$ in D

$z(x,y,\varepsilon) = z_0(x,y)$ on C.

For convenience, let $\frac{\partial z}{\partial \varepsilon} =: \zeta$

Then, $\zeta(x,y,\varepsilon) = 0$ on C because $z_0(x,y)$ doesn't depend on ε.

As before, we assume $\zeta, \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y} \in C(D)$.

$$I(\varepsilon) = \iint_D f(x, y, z(x,y,\varepsilon), z_x(x,y,\varepsilon), z_y(x,y,\varepsilon))\,dx\,dy$$

$$\delta I = \frac{dI}{d\varepsilon}(0) = \iint_D \left(\frac{\partial f}{\partial z}\frac{\partial z}{\partial \varepsilon} + \frac{\partial f}{\partial z_x}\frac{\partial z_x}{\partial \varepsilon} + \frac{\partial f}{\partial z_y}\frac{\partial z_y}{\partial \varepsilon} \right) dx\,dy \Big|_{\varepsilon=0}$$

$$= \iint_D \left(\frac{\partial f}{\partial z}\zeta + \frac{\partial f}{\partial z_x}\zeta_x + \frac{\partial f}{\partial z_y}\zeta_y \right) dx\,dy \Big|_{\varepsilon=0} = 0$$

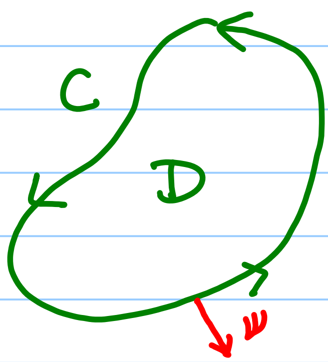
Again, we want to do Integration by Parts here, but now we are in 2D!

$\Rightarrow$ Need Vector Calculus (MAT 21D).

- Green's Thm (2D version of the Divergence Thm in 3D)

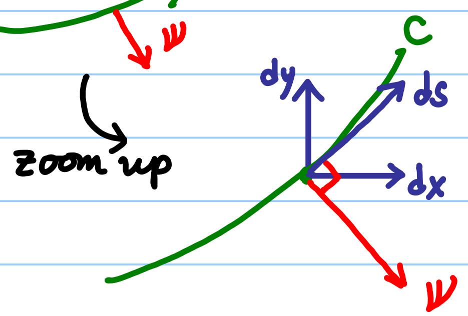
$\mathbb{F}$: a vector-valued fcn in D

ν : the outward unit normal vector on C



$$\iint_D \nabla \cdot \mathbb{F}\,dx\,dy = \oint_C \mathbb{F} \cdot \nu\,ds$$

Flux going out from D or flux across C



$\Rightarrow \nu\,ds = (dy, -dx)^T$

So, for $\mathbb{F} = (f_1, f_2)^T$, we have

$$\iint_D \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \right) dx\,dy = \oint_C (f_1\,dy - f_2\,dx)$$

Now, let $\mathbb{F} = \left(\zeta \frac{\partial f}{\partial z_x}, \zeta \frac{\partial f}{\partial z_y} \right)^T$

Using Green's Thm, we have

$$\iint_D \left(\underbrace{\frac{\partial}{\partial x}\left(\zeta \frac{\partial f}{\partial z_x} \right) + \frac{\partial}{\partial y}\left(\zeta \frac{\partial f}{\partial z_y} \right)}_{= \zeta \left(\frac{\partial^2 f}{\partial z_x \partial x} + \frac{\partial^2 f}{\partial z_y \partial y} \right) + \zeta_x \frac{\partial f}{\partial z_x} + \zeta_y \frac{\partial f}{\partial z_y}} \right) dx\,dy = \oint_C \left(\zeta \frac{\partial f}{\partial z_x} dy - \zeta \frac{\partial f}{\partial z_y} dx \right)$$

$$\Rightarrow \iint_D \left(\zeta_x \frac{\partial f}{\partial z_x} + \zeta_y \frac{\partial f}{\partial z_y} \right) dx\,dy = -\iint_D \zeta \left(\frac{\partial^2 f}{\partial z_x \partial x} + \frac{\partial^2 f}{\partial z_y \partial y} \right) dx\,dy$$

So, adding $\iint_D \zeta \frac{\partial f}{\partial z} dx\,dy \quad + \oint_C \left(\zeta \frac{\partial f}{\partial z_x} dy - \zeta \frac{\partial f}{\partial z_y} dx \right)$

to both sides, δI is simplified as

$$\delta I = \iint_D \zeta \left(\frac{\partial f}{\partial z} - \frac{\partial^2 f}{\partial z_x \partial x} - \frac{\partial^2 f}{\partial z_y \partial y} \right) dx\,dy + \underbrace{\oint_C \zeta \left(\frac{\partial f}{\partial z_x} dy - \frac{\partial f}{\partial z_y} dx \right)}_{=0 \text{ because } \zeta = 0 \text{ on } C.}$$

$$= 0$$

Invoking the 2D version of F.L.C.V.,
we have

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} = 0$$

→ The E-L eqn. for $n=2$.

You can now guess what the E-L eqn. for general n.

$z = z(x_1, \cdots, x_n)$ over a domain $D \subset \mathbb{R}^n$

$$I = \int \cdots \int_D f(x_1, \cdots, x_n, z, z_{x_1}, \cdots, z_{x_n})\, dx_1 \cdots dx_n$$

→ min! subj. to the B.C. $z = z_0$ on ∂D.

$\Rightarrow$ The E-L eqn: $\frac{\partial f}{\partial z} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial z_{x_k}} = 0.$

Now, let's go back to the vibrating string problem.

$$I = \frac{1}{2}\int_{t_1}^{t_2}\int_0^1 \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}\right)^2\right] dx\,dt$$

$$(x, y, z(x,y)) \longleftrightarrow (x, t, u(x,t))$$

So, the E-L eqn. is

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial t}\frac{\partial f}{\partial u_t} = 0$$

$$f = \frac{1}{2}(\rho u_t^2 - \tau u_x^2)$$

So, $\frac{\partial f}{\partial u} = 0$, $\frac{\partial f}{\partial u_x} = -\tau u_x$, $\frac{\partial f}{\partial u_t} = \rho u_t$

Since τ, ρ : const's., we have

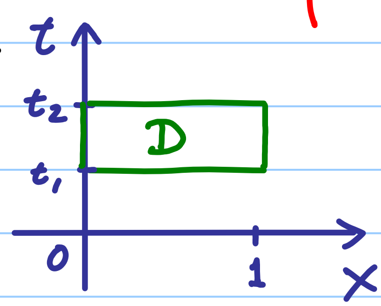
$$\tau u_{xx} - \rho u_{tt} = 0$$

So, the E-L eqn. leads to

$$\rho u_{tt} = \tau u_{xx},$$

or $u_{tt} = \frac{\tau}{\rho} u_{xx}$ i.e., the wave eqn!!

<u>Note</u>



B.C. $u(0,t) = u(1,t) = 0$.

I.C. $u(x,t_1) = u_1(x)$

$u(x,t_2) = u_2(x)$

Often, a true I.C. is imposed only at $t = t_1$, e.g.,

$$\begin{cases} u(x,t_1) = u_1(x) \\ u_t(x,t_1) = v_1(x) \end{cases}$$