Lecture 9: Basics of PDEs III

Note Title

☆ The Potential Eqn.

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^d \text{ with some B.C.} \\ \rightarrow \text{Laplace's eqn.} \\ \Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d \text{ with some B.C.} \\ \rightarrow \text{Poisson's eqn.} \end{cases}$$

They show up as the stationary distribution (i.e., $t \to \infty$) of the corresponding heat eqn. But $\exists$ other numerous physical phenomena described by these eqn's.

$\Rightarrow$ See, e.g., R. P. Feynman: Lectures on Physics, vol. II, chap. 12.

beautiful book!

Examples include: electrostatics; heat flow; stretched membrane; diffusion of neutrons; irrotational fluid flow; illumination of a plane by uniform light; Brownian motion (Kakutani; Hersh & Griego); ...

Feyman said: this happens because "it's the underlying unity of nature."

In particular, if "things" are reasonably smooth in space, distributed homogeneously and isotropically, then $\Delta u = 0$ or $\Delta u = f$ show up often!

These eqn's also played significant role in my research on image compression and led to my patents in US & Japan! See, e.g., my papers whose title contain 'polyharmonic', 'PHLST', or 'PHLCT'.

☆ B.C.s & I.C.s for the Heat & Potential Eqn's.

Consider $u_t = \frac{k}{\rho\sigma}\Delta u + \frac{1}{\rho\sigma} f(\mathbb{r}, t)$ in $\Omega \subset \mathbb{R}^d$

* **B.C.:** $\alpha u + \beta \frac{\partial u}{\partial \nu} = \phi(\mathbb{r}, t)$ on $\partial\Omega \subset \mathbb{R}^{d-1}$.

ν : outward unit normal vector on $\partial\Omega$, $d = 2, 3$.

α, β : some const's.

This B.C. is called the Robin bdy cond.

or the impedance " " .

Note that this includes

Dirichelet ($\beta = 0$) & Neumann ($\alpha = 0$)

If $\phi(\mathbb{r}, t) \equiv 0$, i.e., the homogeneous Robin B.C.

then the physical situation is, since $\frac{\partial u}{\partial \nu} = -\frac{\alpha}{\beta} u$ on $\partial\Omega$,

$$\alpha/\beta \begin{cases} > 0 \Rightarrow \text{radiation of heat through } \partial\Omega. \\ < 0 \Rightarrow \text{absorption " " "} \\ = 0 \Rightarrow \text{insulation " at "} \end{cases}$$

the Neumann case

see Strauss's book!

* **I.C.:** $u(\mathbb{r}, t_0) = \psi(\mathbb{r})$ in Ω

In the case of the potential eqn.,

only B.C. is considered, not I.C.

☆ Uniqueness of the Solution of the Heat Eqn.

Since the argument is similar, we only deal with the heat eqn. You can do this for the potential eqn. yourself!

Thm (Uniqueness)

Consider the heat eqn. with B.C. & I.C.

$$(*)\ \begin{cases} u_t = \dfrac{k}{\rho\sigma}\Delta u + \dfrac{1}{\rho\sigma} f(\mathbf{r}, t) & \text{in } \Omega \\ \alpha u + \beta u_\nu = \phi(\mathbf{r}, t) & \text{on } \partial\Omega \\ u(\mathbf{r}, t_0) = \psi(\mathbf{r}) & \text{in } \Omega \end{cases}$$

where $f \in C(\Omega \times [t_0, \infty))$, $\phi \in C(\partial\Omega \times [t_0, \infty))$, $\psi \in C(\Omega)$.

If $u(\mathbf{r}, t) \in C^2(\bar{\Omega})$ in $\mathbf{r}$ and $\in C^1[t_0, \infty)$ in t, is a sol. of $(*)$, then it's unique.

(Proof) Consider another possible sol. of $(*)$, say $v(\mathbf{r}, t)$, and the difference

$$\zeta(\mathbf{r}, t) := u(\mathbf{r}, t) - v(\mathbf{r}, t).$$

Then, we have

$$(**)\ \begin{cases} \zeta_t = \dfrac{k}{\rho\sigma}\Delta\zeta & \text{in } \Omega \\ \alpha\zeta + \beta\zeta_\nu = 0 & \text{on } \partial\Omega \\ \zeta(\mathbf{r}, t_0) = 0 & \text{in } \Omega \end{cases}$$

Since $\zeta \in C^1[t_0, \infty)$ in t variable,

$$Z(t) := \iiint_\Omega \zeta^2 \, dV \geq 0 \text{ is also in } C^1[t_0, \infty).$$

Note $Z(t_0) = 0$ since $\zeta(\mathbf{r}, t_0) = 0$.

Now, $Z'(t) = 2\iiint_\Omega \zeta\zeta_t \, dV$

$$Z''(t) = 2\iiint_\Omega (\zeta_t^2 + \zeta \underbrace{\zeta_{tt}}_{= \frac{k}{\rho\sigma}\Delta\zeta_t \text{ via } (**)}) \, dV$$

So, $\iiint_\Omega \zeta\,\zeta_{tt}\,dV = \frac{k}{\rho\sigma}\iiint_\Omega \zeta\,\Delta\zeta_t\,dV$

Green's 2nd Id.

$$\overset{\text{Green's 2nd Id.}}{=} \frac{k}{\rho\sigma}\Bigg[\iiint_\Omega \zeta_t \underbrace{\Delta\zeta}_{=\frac{\rho\sigma}{k}\zeta_t}\,dV + \underbrace{\iint_{\partial\Omega}\Big(\zeta\underbrace{\frac{\partial\zeta_t}{\partial\nu}}_{=-\frac{\alpha}{\beta}\zeta_t} - \zeta_t\underbrace{\frac{\partial\zeta}{\partial\nu}}_{=-\frac{\alpha}{\beta}\zeta}\Big)\,dS}_{=0}\Bigg]$$

via (**)

$$= \iiint_\Omega \zeta_t^2\,dV$$

Hence, $Z''(t) = 4\iiint_\Omega \zeta_t^2\,dV$

Now, consider

$$Z\cdot Z'' - (Z')^2 = 4\iiint_\Omega \zeta^2\,dV\cdot\iiint_\Omega \zeta_t^2\,dV - 4\left(\iiint_\Omega \zeta\zeta_t\,dV\right)^2$$

$$= 4\left\{\|\zeta\|^2\cdot\|\zeta_t\|^2 - |\langle\zeta,\zeta_t\rangle|^2\right\}$$

≥ 0 by Cauchy-Schwarz!

So, $\begin{cases} Z\cdot Z'' - (Z')^2 \geq 0, & \forall t \geq t_0 \text{ and} \\ Z(t_0) = 0, \quad Z(t)\geq 0, & \forall t\geq t_0. \end{cases}$

Now, we can see that $Z(t)$ is logarithmically convex, i.e., $(\log Z)'' = \frac{Z\cdot Z'' - (Z')^2}{Z^2} \geq 0$.

So, $Z(\theta t_0 + (1-\theta)t_1) \leq \underbrace{Z(t_0)}_{=0}{}^{\theta}\, Z(t_1)^{1-\theta}$, $\forall \theta\in[0,1]$, $\forall t_1 \geq t_0$

But $Z(t)\geq 0$. Hence we must have $Z(t)\equiv 0$ $\forall t\geq t_0$, i.e., $\zeta(\mathbf{r},t)\equiv 0$ in Ω & $\forall t\geq t_0$

$\Rightarrow u(\mathbf{r},t)$ is a unique sol. ///

☆ Well-Posed Problems (J. Hadamard, 1902)

— consist of a PDE in a domain together with a set of I.C. and/or B.C. or some other auxiliary cond's that enjoy the following fundamental properties:

(i) Existence : $\exists$ at least one sol. satisfying all these cond's.

(ii) Uniqueness : $\exists$ at most one sol.

(iii) Stability : The unique sol. depends in a stable manner on the data (= I.C./B.C.)
$\Longleftrightarrow$ If data are changed a little, the corresponding sol. changes only a little.

Notice that if you impose too many auxiliary cond's, then there may not be any sol. (non-existence) while if there are too few aux. cond's, then there may be many sol's (non-uniqueness).

String, heat, potential eqn's with those I.C. & B.C. we considered are all well-posed.

Then what are examples of ill-posed problems?

Ex. 1 Consider the following potential eqn.:

$$(*)\begin{cases} \Delta u = 0 & \text{in } \Omega = \mathbb{H} = \text{the upper half-plane} \subset \mathbb{R}^2, \\ u|_{\partial\Omega} = 0 & \partial\Omega = \{y=0\} \end{cases}$$

Specifying $\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$ (i.e., overspecification) leads to ill-posedness!!

Why??

$\Rightarrow$ Consider $u_n(x,y) := \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny$, $n \in \mathbb{N}$.
This fcn satisfies (*), but

$$\frac{\partial u_n}{\partial \nu}(x,0) = -\frac{\partial u_n}{\partial y}(x,0) = -e^{-\sqrt{n}} \sin nx \xrightarrow{n\to\infty} 0$$

But, $u_n(x,y) = \frac{1}{2}\sin nx \left(\frac{e^{ny-\sqrt{n}}}{n} - \frac{e^{-ny-\sqrt{n}}}{n} \right) \overset{n\to\infty}{\not\to} 0 \quad \forall y \gneq 0$

Similarly, $\frac{\partial u_n}{\partial \nu} = -e^{-\sqrt{n}} \sin nx \cosh ny \overset{n\to\infty}{\not\to} 0 \quad \forall y \gneq 0$

$\Rightarrow$ Inconsistent with B.C. $\Rightarrow$ ill-posed!

Ex. 2 Backward Heat Eqn.

$$\begin{cases} u_t = u_{xx} & \text{in } \Omega = (0, \pi) \\ \text{I.C.}: u(x,0) = f(x) & \text{in } \Omega \\ \text{B.C.}: u(0,t) = u(\pi,t) = 0 \end{cases}$$

$\Rightarrow$ of course, it's well-posed for $t \geq 0$.

But consider $u(x,t)$, $t < 0$, i.e., suppose we want to know the past temp. that could have led up to the concentration $f(x)$ at $t = 0$, which is antidiffusion.

$$\begin{cases} \text{Diffusion} \Rightarrow \text{smoothing} \\ \text{Antidiffusion} \Rightarrow \text{sharpening} \end{cases}$$, clearly unstable!

Yet, people have been developing algorithms (and tricks) to stabilize antidiffusion problems.

$\Rightarrow$ image enhancement, sharpening, deblurring!

See the reference page for more.