# Lecture 15: Fourier Series IV

Note Title

## ✱ Fourier Series on a general interval

Often, we need to consider the F.S. of a fcn $f(x)$ on a more general interval, say $[\alpha, \beta]$ instead of $[-\pi, \pi]$.

Consider then a linear map from $[-\pi, \pi]$ to $[\alpha, \beta]$ via $x = p\theta + q$. Then find $p, q$ s.t.
$$\theta = -\pi \leftrightarrow x = \alpha$$
$$\theta = +\pi \leftrightarrow x = \beta.$$

$$\Rightarrow \quad p = \frac{\beta - \alpha}{2\pi}, \quad q = \frac{\alpha + \beta}{2}$$

So, $\theta = \dfrac{2\pi}{\beta - \alpha}\left(x - \dfrac{\alpha+\beta}{2}\right)$

Hence,
$$\hat{g}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, e^{-in\theta}\, d\theta$$
$$= \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} f(x)\, e^{-2\pi i n\left(\frac{x - (\alpha+\beta)/2}{\beta - \alpha}\right)}\, dx$$

where $f(x) = g\left(\frac{2\pi}{\beta-\alpha}\left(x - \frac{\alpha+\beta}{2}\right)\right)$

i.e., $f\left(\frac{\beta - \alpha}{2\pi}\theta + \frac{\alpha+\beta}{2}\right) = g(\theta)$

Let us call this $\hat{g}_n$ as the $n$th Fourier coef. of $f$ defined on $[\alpha, \beta]$ with period $\beta - \alpha$, and denote by $\hat{f}_n$ (sorry for notational abuse!)

$$f(x) \sim \sum_{-\infty}^{\infty} \hat{f}_n\, e^{2\pi i n\left(\frac{x - (\alpha+\beta)/2}{\beta - \alpha}\right)}$$

$$\hat{f}_n = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} f(x)\, e^{-2\pi i n\left(\frac{x - (\alpha+\beta)/2}{\beta - \alpha}\right)}\, dx$$

<u>Remark</u>: $[\alpha, \beta] = [0, 1], [-\frac{1}{2}, \frac{1}{2}], [-1, 1]$ are common.

## ☆ Functions of Bounded Variations

Why are we interested in fcns of BVs ?

- Often chosen as a model for piecewise smooth signals & images
- Useful in data compression & statistical estimation
- Provide sharp info on the decay rate of the Fourier coeff's.

Let $g(x)$ be a fcn on a closed interval $I = [a, b]$. ( $I$ could be $\mathbb{R}$ ).
Let $D :=$ a subdivision of $I$, i.e.,

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Now, let's form the sum :

$$T_D(g) := \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})|$$

<u>Def.</u> <u>If</u> $T_D(g) < \infty$ for all possible subdivision $D$, <u>then</u> $g$ is said to be of bdd. var. in $I$, and the total variation of $g$ in $I$ is defined as

$$V_I[g] = V_a^b[g] := \sup_D T_D(g).$$

$BV(I) :=$ a set of all fcns of bdd. var. in $I$.

<u>Fact</u> : • $|g(b) - g(a)| \le V_a^b[g] < \infty$. ← Take $x_0 = a$ $x_1 = b$.

• <u>If</u> $I \subset J$, <u>then</u> $V_I[g] \le V_J[g]$

**Thm 1.** $g \in BV(I) \Rightarrow g$ is bdd. in $I$.

(Pf) $g(x) = g(a) + g(x) - g(a)$

$$\Rightarrow |g(x)| \le |g(a)| + |g(x) - g(a)|$$
$$\le |g(a)| + V_a^x[g]$$
$$\le |g(a)| + V_a^b[g] < \infty. \quad ///$$

One can also show that $BV(I)$ is a Banach space.

**Thm 2.** $g, h \in BV(I) \Rightarrow gh \in BV(I)$.

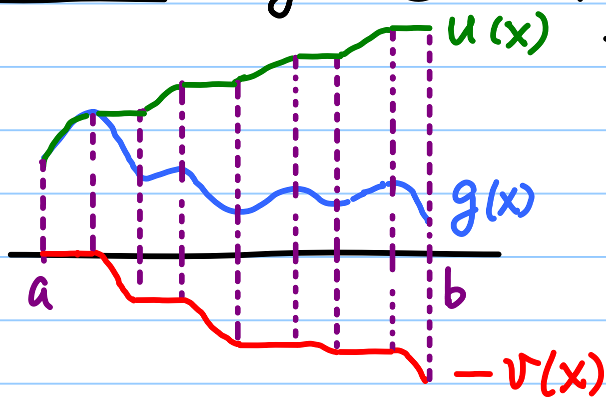
$g, h \in BV(I)$, $|h(x)| \ge \exists\, m > 0$
$\Rightarrow g/h \in BV(I)$.

**Thm 3.** $\forall c \in (a,b)$, $g \in BV[a,b]$
$\Leftrightarrow g \in BV[a,c]$ & $g \in BV[c,b]$.

Moreover, $V_a^b[g] = V_a^c[g] + V_c^b[g]$.

**Remark:** This can be generalized to
$a < c_1 < c_2 < \cdots < c_n < b$.

**Thm. 4** $g \in BV(I) \Leftrightarrow g$ can be written as the difference of two nondecreasing fcns.




say, $u(x) - v(x)$

Previously, we proved the thm:
$f$: $2\pi$-periodic & $\in PS(\mathbb{R})$
$\Rightarrow S_N[f](\theta) \longrightarrow \frac{1}{2}[f(\theta-)+f(\theta+)]$.
Hence if $f$ is also in $C(\mathbb{R})$, then
$S_N[f](\theta) \longrightarrow f(\theta)$, $\forall \theta \in \mathbb{R}$.

This thm is usually attributed to Dirichlet (1829). But it was made sharper thanks to the notion of BV by Jordan (1881):

Thm (Dirichlet–Jordan)
Suppose $f \in BV[\alpha,\beta]$. Then the following hold:
(i) $S_N[f](x) \longrightarrow \frac{1}{2}[f(x+)+f(x-)]$.
(ii) Furthermore, if $f \in C(\alpha,\beta)$, then $S_N[f](x)$ converges to $f(x)$ uniformly $\forall x \in (\alpha+\gamma, \beta-\gamma)$, $\forall \gamma>0$.

In fact, thanks to the Riemann localization principle (1892), the above becomes even sharper as follows:

Thm (Dirichlet–Jordan–Riemann)
(i) If $f|_{N(x)} \in BV(N(x))$ where $N(x)=[x-\delta, x+\delta] \subset [\alpha,\beta]$ $\exists \delta>0$, then $S_N[f](y) \to \frac{1}{2}[f(y+)+f(y-)]$, $\forall y \in N(x)$.
(ii) If $f \in C(\alpha,\beta) \cap BV(\alpha,\beta)$, then $S_N[f](x) \longrightarrow f(x)$ uniformly $\forall x \in (\alpha+\gamma, \beta-\gamma)$, $\forall \gamma>0$

Note: In the above two thm's, of course $\forall \gamma>0$ s.t., $(\alpha+\gamma, \beta-\gamma)$ makes sense, i.e., in fact, $\gamma \in (0, (\beta-\alpha)/2)$.

M. Taibleson's Thm (1967) 1 page paper!

If $f \in BV[0,1]$, $f(x) \sim \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x}$,

then $\alpha_k = O(1/k)$ as $k \to \infty$.

(Pf) Use the fact: $k \neq 0$

$$\int_{j/|k|}^{\frac{j+1}{|k|}} e^{-2\pi i k x}\, dx = 0, \quad j = 0, 1, \cdots, |k|,$$

$$\because \quad = \frac{e^{-2\pi i (j+1)\frac{k}{|k|}} - e^{-2\pi i j \frac{k}{|k|}}}{-2\pi i k}$$

$$= \frac{1}{-2\pi i k}\left(e^{\mp 2\pi i (j+1)} - e^{\mp 2\pi i j}\right) = 0 \ //$$

Now, fix $k$, and let $a_j := \frac{j}{|k|}$, $j = 0, 1, \cdots, |k|$.

Then define

$$g(x) := \sum_{j=0}^{|k|-1} f(a_j)\chi_{[a_j, a_{j+1}]}(x)$$ a step fcn approx. of $f$.

Then, $\alpha_k[g] = \int_0^1 g(x)\, e^{-2\pi i k x}\, dx$

The $k$ th Fourier coeff. of $g$.

$$= \int_0^1 \sum_{j=0}^{|k|-1} f(a_j)\chi_{[a_j, a_{j+1}]}(x)\, e^{-2\pi i k x}\, dx$$

$$= \sum_{j=0}^{|k|-1} f(a_j) \underbrace{\int_{a_j}^{a_{j+1}} e^{-2\pi i k x}\, dx}_{=0} = 0.$$

$$\alpha_k[f] = \int_0^1 f(x)\, e^{-2\pi i k x}\, dx$$

$$|\alpha_k[f]| = |\alpha_k[f] - \alpha_k[g]| = |\alpha_k[f-g]|$$

$$= \left| \int_0^1 (f(x) - g(x))\, e^{-2\pi i k x}\, dx \right|$$

$$\leq \sum_{j=0}^{|k|-1} \int_{a_j}^{a_{j+1}} |f(x) - f(a_j)| \, dx$$

$$\leq \sum_{j=0}^{|k|-1} V_{a_j}^{a_{j+1}}[f] \underbrace{(a_{j+1} - a_j)}_{= 1/|k|}$$

$$\overset{\text{Thm 3}}{=} \frac{1}{|k|} \underbrace{V_0^1 [f]}_{\text{" const.}} = O(1/|k|).$$

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<u>Thm (NS - J.F. Remy, 2006)</u>

Let $f$ be a 1-periodic fcn and $f \in C^m(\mathbb{R})$. Furthermore, let us assume that $f^{(m+1)}$ exists and in $BV[0,1]$. Then its Fourier coeff. $\alpha_k[f] = \hat{f}(k)$ decays as $O(|k|^{-m-2})$, where $m = 0, 1, \cdots$.

(Pf) Use $\begin{cases} \text{the periodicity, i.e., } f^{(\ell)}(0) = f^{(\ell)}(1),\ \ell = 0, \cdots, m. \\ \text{integration by parts!} \end{cases}$

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} dx$$

$$= \cancel{\frac{e^{-2\pi i k x}}{-2\pi i k} f(x) \Big|_0^1} + \frac{1}{2\pi i k} \int_0^1 f'(x) e^{-2\pi i k x} dx$$

$$= \cancel{\frac{e^{-2\pi i k x}}{-(2\pi i k)^2} f'(x) \Big|_0^1} + \frac{1}{(2\pi i k)^2} \int_0^1 f''(x) e^{-2\pi i k x} dx$$

$$= \cdots = \cancel{\frac{e^{-2\pi i k x}}{-(2\pi i k)^{m+1}} f^{(m)}(x) \Big|_0^1} + \underbrace{\frac{1}{(2\pi i k)^{m+1}} \int_0^1 f^{(m+1)}(x) e^{-2\pi i k x} dx}$$

By assumption, $f^{(m+1)} \in BV[0,1]$. So, can use the Taibleson Thm to get:

$$|\hat{f}(k)| \leq V_0^1 [f^{(m+1)}] (2\pi)^{-m-1} \cdot |k|^{-m-2}.$$

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