Lecture 17: Basics of L^2 Theory I

Note Title

In order to study the Sturm-Liouville Theory, Green's fcns, & integral eqn's, we need to know the very basics of the L^2 theory!

☆ Convergence & Completeness

<u>Def.</u> Let f_n be a sequence in $PC[a,b]$.
f_n converges to f in norm $\overset{def}{\Longleftrightarrow}$ $\|f_n - f\| \to 0$ as $n \to \infty$

Of course, the norm here is $\|f\| = \|f\|_2 := \sqrt{\int_a^b |f(x)|^2 dx}$.

<u>Axioms of norm</u>: Let $\mathfrak{X}$ be a vector space over $\mathbb{C}$. For $\forall f, g \in \mathfrak{X}$, $\forall \alpha \in \mathbb{C}$,
i) $\|f\| \geq 0$ with $\|0\| = 0$; ii) $\|\alpha f\| = |\alpha| \|f\|$; iii) $\|f+g\| \leq \|f\| + \|g\|$.

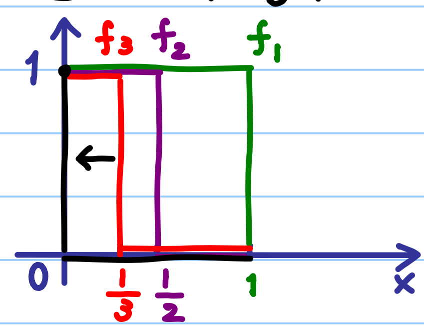
So, the norm conv. is equiv. to $\int_a^b |f_n(x) - f(x)|^2 dx \underset{n\to\infty}{\to} 0$.

Relationship among various convergences:

- L²-Norm conv. $\overset{①}{\not\Rightarrow}$ Pointwise conv.
- L²-Norm conv. $\overset{②}{\not\Leftarrow}$ Pointwise conv.
- L²-Norm conv. $\overset{③}{\not\Rightarrow}$ Uniform conv.
- Uniform conv. $\overset{④}{\Rightarrow}$ L²-Norm conv.
- Uniform conv. $\Rightarrow$ Pointwise conv.; Pointwise conv. $\not\Rightarrow$ Uniform conv. } → These are obvious.

<u>①: Example</u> WLOG, let $[a,b] = [0,1]$.

$$f_n = \begin{cases} 1 & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise}. \end{cases}$$



$$\|f_n\|^2 = \int_0^{1/n} 1^2 dx = \frac{1}{n} \underset{n\to\infty}{\to} 0$$

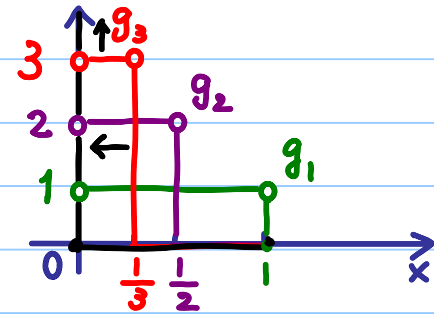
So, $f_n \to 0$ in norm.
But, $f_n(0) \equiv 1$, $\forall n \in \mathbb{N}$.
So, f_n cannot conv. to 0 pointwise.
$f_n \to$ the unit impulse at $x = 0$. #

$$L^2\text{-Norm conv.} \quad \overset{①}{\not\Rightarrow} \quad \underset{②}{\not\Leftarrow} \quad \text{Pointwise conv.}$$

$$L^2\text{-Norm conv.} \ \underset{④}{\Leftarrow}\ \overset{③}{\not\Rightarrow}\ \text{Uniform conv.} \ \Rightarrow\ \not\Leftarrow \ \text{Pointwise conv.}$$

②: <u>Example</u> WLOG, let $[a,b]=[0,1]$.

$$g_n = \begin{cases} n & 0<x<\frac{1}{n} \\ 0 & \text{otherwise}. \end{cases}$$



$$\begin{cases} g_n(0) \equiv 0 & \forall n \in \mathbb{N}. \\ g_n(x) = 0 & \forall x>0 \text{ with } n \geq \frac{1}{x} \end{cases}$$

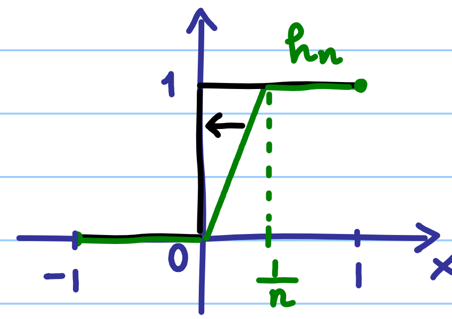
So, $g_n \to 0$ pointwise as $n\to\infty$

But, $\|g_n\|^2 = \int_0^{1/n} n^2\,dx = n \to \infty$.

So it doesn't conv. to 0 in norm. #

③: <u>Example</u> WLOG, let $[a,b]=[-1,1]$.

$$h_n(x) = \begin{cases} 0 & -1\leq x\leq 0 \\ nx & 0\leq x\leq \frac{1}{n} \\ 1 & \frac{1}{n}\leq x\leq 1. \end{cases}$$



$$\|h_n - \chi_{[0,1]}\|^2 = \int_0^{\frac{1}{n}} |nx-1|^2\,dx$$

the characteristic fcn of $[0,1]$.

$$= \int_0^{\frac{1}{n}} (n^2x^2 - 2nx + 1)\,dx$$

$$= \left[\frac{n^2}{3}x^3 - nx^2 + x\right]_0^{\frac{1}{n}} = \frac{1}{3n} \to 0 \text{ as } n\to\infty$$

So, $h_n \to \chi_{[0,1]}$ in norm.

But $\chi_{[0,1]}$ is a discontinuous fcn while each h_n is continuous. Hence, h_n cannot conv. uniformly to $\chi_{[0,1]}$. #

④: <u>Thm</u> $f_n \to f$ uniformly on $[a,b] \Rightarrow f_n \to f$ in norm.

(Proof) $f_n\to f$ unif. $\iff \exists M_n>0$, s.t. $|f_n(x)-f(x)|<M_n$, $\forall x\in[a,b]$
$M_n \to 0$ as $n\to\infty$

But, $\|f_n - f\|^2 = \int_a^b |f_n(x)-f(x)|^2\,dx \leq M_n^2(b-a) \xrightarrow[n\to\infty]{} 0$. ///

Remark: If $f_n \to f$ in norm, then
$\|f_n\| \to \|f\|$, $\langle f_n, g\rangle \to \langle f, g\rangle$, $\forall g \in \mathcal{X}$

(Proof) $\big|\, \|f_n\| - \|f\| \,\big| \leq \|f_n - f\| \to 0$.
why this inequality holds?
Let $f_n - f =: g_n$. Then $f_n = g_n + f$
$\Rightarrow \|f_n\| = \|g_n + f\| \leq \|g_n\| + \|f\|$.
So, $\|f_n\| - \|f\| \leq \|g_n\| = \|f_n - f\|$.
Similarly, $\|f\| - \|f_n\| \leq \|f - f_n\|$.
Hence $\big|\, \|f_n\| - \|f\| \,\big| \leq \|f_n - f\| \to 0$. ///
$\langle f_n, g\rangle \to \langle f, g\rangle$, $\forall g \in \mathcal{X}$ can be shown similarly.

Now, let's discuss the completeness of a fcn space.
Def. $\mathcal{X}$ is complete w.r.t. $\|\cdot\|$
$\underset{\text{narrow def.}}{\Longleftrightarrow}$ $\forall \{f_n\} \subset \mathcal{X}$ with $\|f_n - f\| \to 0 \Rightarrow f \in \mathcal{X}$
So, $C[a,b]$ is not complete with $\|\cdot\|_2 = \sqrt{\int_a^b |\cdot|^2 dx}$
See Ex. 3! $PC[a,b]$ is not complete w.r.t. this norm either (will show this later).
A better def. of the completeness is the following:
any Cauchy seq. $\{f_n\} \subset \mathcal{X}$ conv. to a limit $f \in \mathcal{X}$.
$\{f_n\} \subset \mathcal{X}$ is a Cauchy seq. $\overset{\text{def}}{\Longleftrightarrow}$ $\|f_m - f_n\| \xrightarrow[m,n\to\infty]{} 0$.

For a general vector space $\mathcal{X}$,
$\{f_n\}$: a convergent seq. in $\mathcal{X}$ $\overset{⑤}{\underset{⑥}{\substack{\Longrightarrow \\ \not\Longleftarrow}}}$ $\{f_n\}$: Cauchy in $\mathcal{X}$

$\because$ Ex 3 implies ⑥ for $\mathcal{X} = C[-1,1]$.
For ⑤, Let $f_n \to f \in \mathcal{X}$ in norm. Then,
$\|f_m - f_n\| = \|f_m - f + f - f_n\| \leq \|f_m - f\| + \|f - f_n\| \xrightarrow[n\to\infty]{} 0$. ///

Hence, if every Cauchy seq. in $\mathcal{X}$ is a convergent seq. in $\mathcal{X}$, then such $\mathcal{X}$ is special, and is called complete.

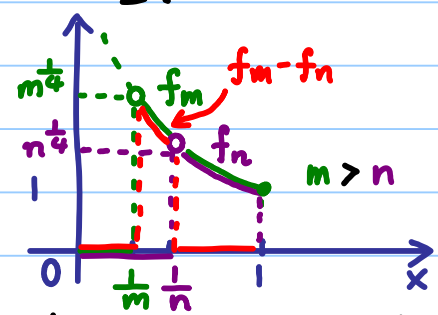
* already showed : $C[a,b]$: not complete w.r.t. $\|\cdot\|_2$
yet 〃 : complete w.r.t. $\|\cdot\|_\infty$

$\|f\|_\infty := \sup_{a\le x\le b} |f(x)|$

* $PC[a,b]$ is not complete w.r.t. $\|\cdot\|_2$.

<ins>why?</ins> Let $[a,b]=[0,1]$.

Define $f_n(x) = \begin{cases} x^{-1/4} & \frac{1}{n} < x \le 1 \\ 0 & 0 \le x \le \frac{1}{n} \end{cases}$



So, $f_m(x) - f_n(x) = x^{-\frac14}\chi_{(\frac1m,\frac1n)}(x)$

$$\Rightarrow \|f_m - f_n\|_2^2 = \int_0^1 x^{-\frac12}\chi_{(\frac1m,\frac1n)}(x)\,dx = \int_{1/m}^{1/n} x^{-\frac12}\,dx = 2\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right) \xrightarrow[m,n\to\infty]{} 0.$$

But, $f_n(x) \to f(x) = \begin{cases} x^{-\frac14} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$

So, $f(0+) = +\infty \Rightarrow f \notin PC[0,1]$, i.e., PC is not complete w.r.t. $\|\cdot\|_2$! ///

In order to have a fcn space $\mathcal{X}$ complete w.r.t. $\|\cdot\|_2$, we need Lebesgue integral instead of Riemann integral. However, we won't discuss Lebesgue integrals in detail in this course.

Denote $L^2[a,b] := \left\{ f : \int_a^b |f(x)|^2 dx < \infty \right\}$.

Lebesgue

This is a space of square-integrable (L^2) fcns.

In $L^2[a,b]$, the following inner product is well defined: $\langle f, g\rangle := \int_a^b f(x)\overline{g(x)}\,dx$

Why? Thanks to the Cauchy-Schwarz ineq., we have $|\langle f, g\rangle| \leq \|f\|_2 \|g\|_2 < \infty$ for $\forall f, g \in L^2[a,b]$. //

Remark: $\|f\| = 0$ in L^2-norm does not mean $f(x) \equiv 0$ everywhere in $[a,b]$ but means $f(x) = 0$ almost everywhere (a.e.) in $[a,b]$.

An extreme case:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ (rational numbers)} \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow \|f\| = 0$!! in terms of the Lebesgue integral. Riemann integral cannot be applied to such a fcn.

Thm (a) $L^2[a,b]$ is complete w.r.t. $\|\cdot\|_2$.
(b) $\forall f \in L^2[a,b]$, $\exists \{f_n\} \subset C[a,b]$ s.t. $\|f_n - f\| \to 0$.
In fact, one can also take $\{f_n\} \subset C^\infty[a,b]$, $\|f_n - f\| \to 0$.

Bessel's Ineq.: If $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2[a,b]$, then $\sum_1^\infty |\langle f, \phi_n\rangle|^2 \leq \|f\|_2^2$, $\forall f \in L^2[a,b]$.

(Proof) Essentially the same as before. The only difference is $\phi_n(x)$ instead of $\frac{1}{\sqrt{2\pi}} e^{in\theta}$. //

We know that
- $f \in PS(\mathbb{R})$, 2π-periodic $\Rightarrow S_N[f]$ conv. pointwise.
- $f \in PS(\mathbb{R}) \cap C(\mathbb{R})$, 2π-per $\Rightarrow S_N[f]$ conv. abs/unif.

Question: $f \in L^2[a,b] \Rightarrow \sum_1^\infty \langle f, \phi_n\rangle \phi_n \to f$ in norm?