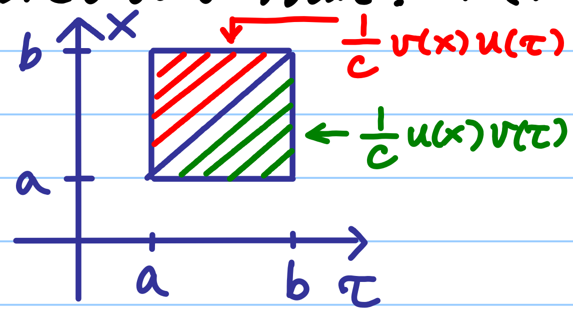
Lecture 23: Green's Functions II

Note Title

☆ Basics of Operator Theory

Recall the form of the Green's fcn in the previous thm: $k(x,\tau) = \begin{cases} \frac{1}{c} v(x)\, u(\tau) & a \le \tau \le x \le b \\ \frac{1}{c} u(x)\, v(\tau) & a \le x \le \tau \le b. \end{cases}$



Continuous on each triangle, and they match along the diagonal $x = \tau$.

$\Rightarrow k(\cdot,\cdot) \in C([a,b]\times[a,b])$.

So, k is bdd., say, $|k(x,\tau)| \le \exists M$, $a \le x, \tau \le b$.

also, $\int_a^b \int_a^b |k(x,\tau)|^2 \, dx\, d\tau \le M^2 (b-a)^2 < \infty$.

This means that $\mathcal{K}$ (the integral op.) is a Hilbert-Schmidt operator!

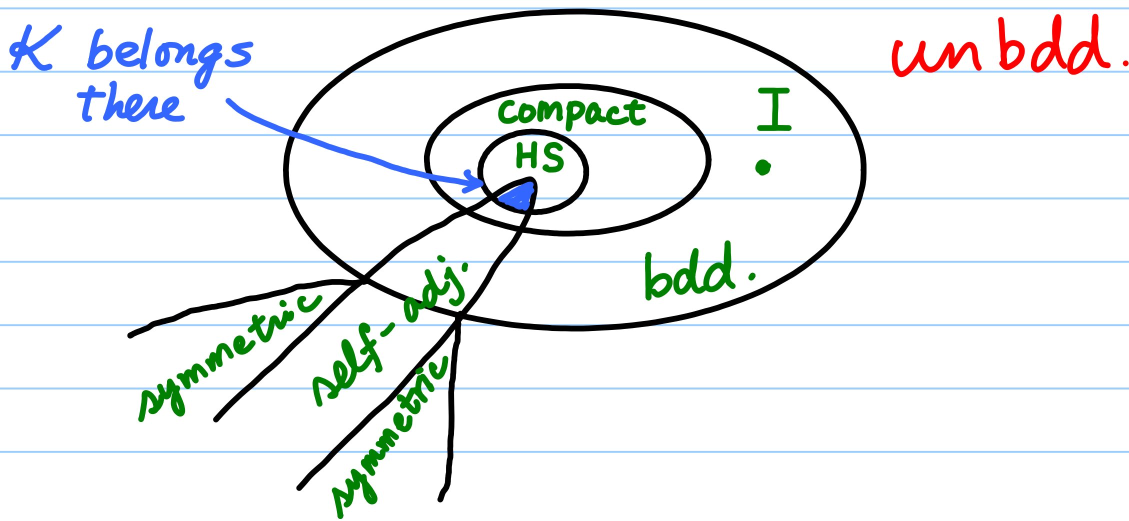
Moreover, in this case, $\mathcal{K}^* = \mathcal{K}$, i.e., $\mathcal{K}$ is self-adjoint!

$\hookrightarrow$ $\mathcal{K}^*$ has a kernel $k^*(x,\tau) = \overline{k(\tau, x)}$, which is $k(x,\tau)$ in this case!

Since {H.S. op's} ⊂ {Compact op's}, $\mathcal{K}$ is a compact self-adj. op.!!

That is, close to a hermitian matrix multiplication, and we can use the Spectral Thm! (∃ eigval's/eigfcns)

World of Linear Operators on a Hilbert space:



Def. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. A bdd. op. $T: \mathcal{H}_1 \to \mathcal{H}_2$ is said to be Hilbert-Schmidt if $\exists \{e_n\}_{n\in\mathbb{N}}$ an ONB in $\mathcal{H}_1$ s.t. $\sum_1^\infty \|Te_n\|_{\mathcal{H}_2}^2 < \infty$.

Thm T: Hilbert-Schmidt $\Rightarrow$ T: compact.

Thm Let $k: (c,d)\times(a,b) \to \mathbb{C}$ be a Lebesgue measurable fcn s.t. $\int_c^d \int_a^b |k(x,y)|^2 dx\, dy < \infty$.
Then, $K: L^2(a,b) \to L^2(c,d)$ defined by
$Kf(x) := \int_a^b k(x,y) f(y) dy$ is Hilbert-Schmidt!

We'll prove these thm's later. See also:
N. Young: An Introduction to Hilbert space, Cambridge Univ. Press, 1988.

☆ Why compactness is important?

In finite dim's, consider a pos. def. hermitian matrix $A \in M_n(\mathbb{C})$. Then the eigval's $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_s$ $(1 \le {}^\exists s \le n)$ are characterized (or computed) via

$$\lambda_{\min} = \lambda_1 = \min_{\|\mathbf{x}\|=1, \mathbf{x}\in\mathbb{C}^n} \langle A\mathbf{x}, \mathbf{x}\rangle$$

$$\lambda_{\max} = \lambda_s = \max_{\|\mathbf{x}\|=1, \mathbf{x}\in\mathbb{C}^n} \langle A\mathbf{x}, \mathbf{x}\rangle$$

Let $S_{\mathbb{C}}^n := \{(z_1, \cdots, z_n) \in \mathbb{C}^n \mid \sum_1^n |z_j|^2 = 1\} \sim S_{\mathbb{R}}^{2n}$ the unit sphere in $\mathbb{C}^n$. This is a compact set!
So, the eigenvalues exist since a continuous fcn defined on a compact set has min/max values!

On the other hand, in a Hilbert space of infinite dim's, say $\mathcal{H}$, consider the unit sphere:

$$S_{\mathcal{H}} := \{ x \in \mathcal{H} \mid \|x\|_{\mathcal{H}} = 1 \}.$$

$\Rightarrow$ $S_{\mathcal{H}}$ is not compact!

(Proof) Let $\{e_n\}_{n \in \mathbb{N}}$ be an ONB of $\mathcal{H}$.
$e_n \in S_{\mathcal{H}}$, $\forall n \in \mathbb{N}$. So $\|e_m - e_n\|_{\mathcal{H}} = \sqrt{2} \neq 0$ for $\forall m \neq n$. $\because$ $\|e_m - e_n\|^2_{\mathcal{H}} = \langle e_m - e_n, e_m - e_n \rangle$

$$= \underbrace{\|e_m\|^2}_{=1} - \underbrace{\cancel{\langle e_m, e_n\rangle}}_{=0} - \underbrace{\cancel{\langle e_n, e_m\rangle}}_{=0} + \underbrace{\|e_n\|^2}_{=1}$$

So, $\{e_n\}_{n \in \mathbb{N}}$ cannot contain a convergent subsequence. $\Longrightarrow$ not compact. ///

What should we do then?
Let $T: \mathcal{H} \to \mathcal{H}$ a bdd. linear op.
(C) Suppose $\{x_n\}_{n \in \mathbb{N}} \subset S_{\mathcal{H}}$ be any seq. on $S_{\mathcal{H}}$.
$\{Tx_n\}_{n \in \mathbb{N}}$ contains a convergent subseq.
i.e., $\exists \{x_{n_j}\}_{j \in \mathbb{N}} \subset \{x_n\}_{n \in \mathbb{N}}$ s.t. $Tx_{n_j} \xrightarrow[j \to \infty]{} {}^{\exists} z_0 \in \mathcal{H}$.

If T satisfies (C), then it's called compact (or completely continuous).
Now, one can show:

Thm If $T: \mathcal{H} \to \mathcal{H}$ is compact & self-adj., then T has at least one eigenvalue. Furthermore,

(i) If $\|T\| = \sup_{x \in S_{\mathcal{H}}} \langle Tx, x \rangle$, then $\|T\|$ is an eigval of T.
(ii) If $-\|T\| = \inf_{x \in S_{\mathcal{H}}} \langle Tx, x \rangle$, then $-\|T\|$ is an eigval of T.

(Proof) See the standard textbook on functional analysis.

Remark: The identity op. I is not a compact op.
$\because$ $Ie_n = e_n$. //

Thm Hilbert-Schmidt op's are compact!

(Proof) Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be an HS op.
Let $\{e_n\}_{n \in \mathbb{N}}$ be an ONB in $\mathcal{H}_1$.
We'll show T is compact by expressing it as a norm limit of finite rank op's.

i.e. $\dim(\text{range space}) < \infty$.

Define $T_k : \mathcal{H}_1 \to \mathcal{H}_2$, $k \in \mathbb{N}$ via

$$T_k x := \sum_{n=1}^{k} x_n T e_n \quad \text{where } \forall x = \sum_{1}^{\infty} x_n e_n \in \mathcal{H}_1, \quad x_n = \langle x, e_n \rangle$$

T_k agrees with T in $\operatorname{span}\{e_1, \cdots, e_k\}$
" becomes 0 in $\operatorname{span}\{e_{k+1}, e_{k+2}, \cdots\}$
So, the rank of $T_k \leq k$, i.e., T_k : compact.
Now, $(T - T_k)x = \sum_{k+1}^{\infty} x_n T e_n$

$$\Rightarrow \quad \|(T - T_k)x\| \leq \sum_{k+1}^{\infty} |x_n| \, \|T e_n\|$$

Cauchy-Schwarz

$$\leq \left\{ \sum_{k+1}^{\infty} |x_n|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k+1}^{\infty} \|T e_n\|^2 \right\}^{\frac{1}{2}}$$

$$\leq \|x\| \cdot \left\{ \sum_{k+1}^{\infty} \|T e_n\|^2 \right\}^{\frac{1}{2}}$$

$$\Rightarrow \frac{\|(T - T_k)x\|}{\|x\|} \leq \left\{ \sum_{k+1}^{\infty} \|T e_n\|^2 \right\}^{\frac{1}{2}}$$

So, by the def. of the operator norm (i.e., taking $\sup_{x \in \mathcal{H}_1}$ above),

$$\|T - T_k\| \leq \left\{ \sum_{k+1}^{\infty} \|T e_n\|^2 \right\}^{\frac{1}{2}}$$

Now, T is HS, so $\sum_{1}^{\infty} \|T e_n\|^2 < \infty$.

$\Rightarrow \sum_{k+1}^{\infty} \|T e_n\|^2 \to 0$ as $k \to \infty$ (because it's a tail of a conv. series)

$\Rightarrow T_k \xrightarrow[k \to \infty]{} T$ in operator norm, so T is compact because a set of cpt op's is closed in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ w.r.t. operator norm. ///

a set of bdd. linear op's from $\mathcal{H}_1$ to $\mathcal{H}_2$

Thm Let $k \in L^2([c,d]\times[a,b])$, i.e.,

$$\int_c^d \int_a^b |k(x,y)|^2 \, dy\, dx < \infty$$

Then the integral op. $K : L^2(a,b) \to L^2(c,d)$ with k as its kernel is an HS op, hence compact!

(Note: (a,b), (c,d) could be $\mathbb{R}$.)

(Proof) Pick any ONB $\{e_n\}_{n\in\mathbb{N}}$ in $L^2(a,b)$.

For $x \in [c,d]$,

$$K e_n(x) = \int_a^b k(x,y)\, e_n(y)\, dy = \langle k(x,\cdot), \overline{e_n} \rangle$$

$$\Rightarrow \|K e_n\|^2 = \int_c^d |K e_n(x)|^2 dx = \int_c^d |\langle k(x,\cdot), \overline{e_n}\rangle|^2 dx$$

Since $\{e_n\}_{n\in\mathbb{N}}$ is an ONB of $L^2(a,b)$, we have

$$\sum_1^\infty \|K e_n\|^2 = \sum_1^\infty \int_c^d |\langle k(x,\cdot), \overline{e_n}\rangle|^2 dx$$

$$\overset{\text{Fubini-Tonelli}}{=} \int_c^d \sum_1^\infty |\langle k(x,\cdot), \overline{e_n}\rangle|^2 dx$$

$$\overset{\text{Parseval}}{=} \int_c^d \|k(x,\cdot)\|^2_{L^2(a,b)} dx$$

$$= \int_c^d \int_a^b |k(x,y)|^2 dy\, dx < \infty \Rightarrow K \text{ is HS.} ///$$

Remark: $\{\text{HS op's}\} \subset \{\text{Compact op's}\}$

$\Rightarrow \exists$ a compact op that is not HS!

Ex. $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ having a matrix rep. $\operatorname{diag}(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots)$ w.r.t. $\exists \{e_n\}_{n\in\mathbb{N}}$ of $\ell^2(\mathbb{N})$.

This T is compact, but $\sum_1^\infty \|T e_n\|^2 = \sum_1^\infty \|\frac{1}{\sqrt{n}} e_n\|^2 = \sum_1^\infty \frac{1}{n} = \infty$. #