MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 1: Overture: Motivations, scope and structure of the course

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0 Scope and Structure of This Course

0.1 Scope

- Why Laplacian eigenvalues?
  - Reflect geometric information of a domain Ω
  - Can be used for shape analysis, graph theory, etc.
- Why Laplacian eigenfunctions?
  - Provide an orthonormal basis for $L^2(\Omega)$
  - Can be used for 'spectral analysis' of data defined on $\Omega \Longrightarrow$ generalization of Fourier analysis
  - Can be applied to data approximation, pattern analysis, and image analysis

0.2 Structure of this course

- Laplacian eigenvalue problems in the continuum, i.e., $\Omega \subset \mathbb{R}^d$ with $|\Omega| < \infty$

- 1D vibrations of a string
- 2D/3D vibrations of a membrane of various shape, e.g., rectangle, disk, sphere, etc.
- Necessary functional analysis basics
- Vibrations of a membrane and heat conduction in $\Omega \subset \mathbb{R}^d$
- Application to data analysis, spectral geometry, and shape recognition

- Laplacian eigenvalue problems in the discrete setting
  - Basics of graph theory
  - Laplacian of a graph
  - Laplacian eigenvalues of a graph
  - Random walks and heat propagation on a graph
  - Diffusion maps
  - Application to clustering, image segmentation, statistical learning theory, etc.
- Fast algorithms to compute Laplacian eigenvalues and eigenfunctions.

1 Introduction

Consider a domain $\Omega \subset \mathbb{R}^d$ of finite volume. A **Laplacian Eigenvalue Problem** on Ω is an equation of the form:

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} = \nabla \cdot \nabla$, with one of the following boundary conditions (BCs):

$$\begin{cases} u = 0, & \text{on } \partial\Omega \quad \text{Dirichlet BC} \\ \frac{\partial u}{\partial \nu} = \nu \cdot \nabla u = 0, & \text{on } \partial\Omega \quad \text{Neumann BC} \\ \frac{\partial u}{\partial \nu} + au = 0, a \in \mathbb{R} & \text{on } \partial\Omega \quad \text{Robin (Mixed) BC} \end{cases} \tag{2}$$

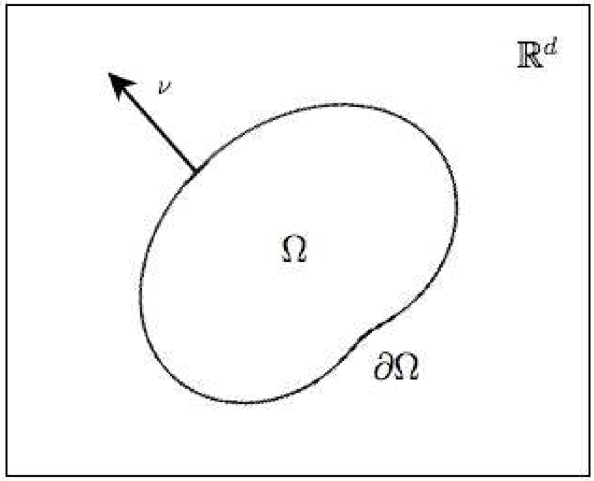

Figure 1: $\Omega \subset \mathbb{R}^d$ with ν being a normal vector.

If $u \neq 0$ in Ω satisfies the Laplacian eigenvalue problem with either of the above BCs, then we say u is an **eigenfunction** and the corresponding λ is called the **eigenvalue**.
Laplacian eigenvalues and eigenfunctions allow us to perform numerous analysis with a given domain Ω. We will see that the eigenvalues of (1) reflect geometric information about Ω. Also, the eigenfunctions can be used for spectral analysis of data defined (or living) on Ω. Furthermore, Laplacian eigenfunctions allow us to generalize Fourier analysis.

2 History of and Introduction to Laplacian Eigenvalues/Eigenfunctions

2.1 Vibrations of a one dimensional string

Around the mid 18th century, d'Alembert, Euler and Daniel Bernoulli examined and created the theory behind vibrations of a one-dimensional string. Consider a perfectly elastic and flexible string resting over the interval $[0, \ell]$ of the x-axis. Let $\rho(x)$ be a mass density for $x \in [0, \ell]$. Assume that the magnitude of the tension $\mathcal{T}$ of the string is constant. If $u(x,t)$ is the vertical displacement of the string at location $x \in [0, \ell]$ and time $t \geq 0$, then the string vibrates according to the one-dimensional wave equation:

$$\rho(x)u_{tt} = \mathcal{T}u_{xx} \quad \text{or} \quad u_{tt} = c^2 u_{xx}, \text{with } c^2 = \frac{\mathcal{T}}{\rho}. \tag{3}$$

See [1, Appendix 1] for more about the derivation of the above wave equation. From now on, for simplicity, we assume the uniform string, i.e., $\rho(x) = \rho =$ constant.

We know (3) has infinitely many solutions. We would need to specify a boundary condition and an initial condition on (3) to obtain the desired solution. For instance, say both ends of the string are held fixed for all time t. This corresponds to the Dirichlet boundary condition $u(0,t) = u(\ell,t) = 0$. If $f(x)$ is the initial vertical displacement at x and $g(x)$ is its initial velocity, then we would have the initial condition (IC) $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ for all $x \in [0,\ell]$. What we have then is:

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } x \in (0,\ell) \text{ and } t > 0; \\ u(0,t) = u(\ell,t) = 0 & \text{for } t \geq 0; \\ u(x,0) = f(x), \;\; u_t(x,0) = g(x) & \text{for } x \in [0,\ell]. \end{cases} \tag{4}$$

2.2 The behavior of the string $u(x,t)$

First we use the method of separation of variables to seek a nontrivial solution of the form $u(x,t) = X(x)T(t)$. Plugging $X(x)T(t)$ into the differential equation (4), we get:

$$XT'' = c^2 X''T \Rightarrow \frac{X''}{X} = \frac{T}{c^2 T} = k$$

where k must be constant. This relation yields the pair of ordinary differential equations:

$$X'' - kX = 0 \quad \text{with } X(0) = X(\ell) = 0, \tag{5}$$

$$T'' - c^2 kT = 0 \tag{6}$$

Case I: $k > 0$

By solving the characteristic equation $r^2 - k = 0$, we get $r = \pm\sqrt{k}$ and

$$X(x) = A\mathrm{e}^{\sqrt{k}x} + B\mathrm{e}^{-\sqrt{k}x} \;\text{ or }\; A\cosh(\sqrt{k}x) + B\sinh(\sqrt{k}x).$$

Applying the BC $X(0) = X(\ell) = 0$ yields $A = B = 0$, thus the case of $k > 0$ is not feasible.

Case II: $k = 0$

$$X'' = 0 \Longrightarrow X(x) = Ax + B,$$

which again leads to $X(x) \equiv 0$.

Case III: $k < 0$
Set $k = -\nu^2$ and $\nu > 0$. We then have the characteristic equation $r^2 + \nu^2 = 0$, i.e., $r = \pm i\nu$. Therefore we get

$$X(x) = A\cos(\nu x) + B\sin(\nu x)$$

By the BC $X(0) = X(\ell) = 0$, we get:

$$\begin{cases} X(0) = 0 & \Longrightarrow \quad A = 0 \\ X(\ell) = B\sin(\nu\ell) = 0 & \Longrightarrow \quad \nu = \frac{n\pi}{\ell}, \ \ \forall n \in \mathbb{N} \end{cases}$$

(Note $n = 0$ leads to $X(x) \equiv 0$ in this case, so it should not be included.) Therefore we have $X(x) = B\sin(\frac{n\pi}{\ell}x)$ and for convenience, we let $B = \sqrt{2/\ell}$ and define:

$$X_n(x) = \phi_n(x) \triangleq \sqrt{\frac{2}{\ell}}\sin\left(\frac{n\pi}{\ell}x\right),$$

so that $\|\phi_n\|_{L^2[0,\ell]} = 1$. Note that $\{\phi_n\}_{n\in\mathbb{N}}$ form an orthonormal basis for $L^2[0,\ell]$. Similarly, by $T'' = -\nu^2 c^2 T$ we obtain the family of solutions

$$T_n(t) = a_n\cos\left(\frac{n\pi c}{\ell}t\right) + b_n\sin\left(\frac{n\pi c}{\ell}t\right).$$

Now, for each $n \in \mathbb{N}$, the function

$$u_n(x,t) = \left\{a_n\cos\left(\frac{n\pi c}{\ell}t\right) + b_n\sin\left(\frac{n\pi c}{\ell}t\right)\right\}\sqrt{\frac{2}{\ell}}\sin\left(\frac{n\pi}{\ell}x\right)$$

satisfies the equation (4), and by the Superposition Principle,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi c}{\ell}t\right) + b_n\sin\left(\frac{n\pi c}{\ell}t\right)\right\}\phi_n(x) \qquad (7)$$

is a general solution with yet undetermined coefficients a_n and b_n.

Next, we specify the coefficients a_n and b_n by matching (7) with the ICs in (4). Thus we get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi}{\ell}x\right) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Then

$$a_n = \langle f, \phi_n \rangle = \sqrt{\frac{2}{\ell}} \int_0^{\ell} f(x) \sin\left(\frac{n\pi}{\ell}x\right) \,\mathrm{d}x,$$

which is a Fourier sine series expansion of f.
Similarly,

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} b_n \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi}{\ell}x\right).$$

We obtain

$$\frac{n\pi c}{\ell} b_n = \langle g, \phi_n \rangle \;\;\Rightarrow\;\; b_n = \frac{\ell}{n\pi c} \langle g, \phi_n \rangle .$$

By construction, the particular solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \langle f, \phi_n \rangle \cos\left(\frac{n\pi c}{\ell}t\right) + \frac{\ell}{n\pi c} \langle g, \phi_n \rangle \sin\left(\frac{n\pi c}{\ell}t\right) \right\} \phi_n(x)$$

We need to check if our solution makes sense physically. Notice that

$$c^2 = \frac{\mathcal{T}}{\rho} \Rightarrow \text{ the sound frequency = } \frac{n\pi}{\ell}\sqrt{\frac{\mathcal{T}}{\rho}}.$$

For instance, if ℓ is short, $\mathcal{T}$ is high, and ρ is small (thin), then such a string generates a high frequency tone. On the other hand, if ℓ is long, $\mathcal{T}$ is low, and ρ is large (thick), then it generates a low frequency tone.

NOTE: The Neumann boundary condition imposes

$$u_x(0,t) = u_x(\ell,t) = 0 \quad \forall t > 0.$$

This leads to the Fourier *cosine* series expansions of f and g. Note that the Neumann problem allows the solution $u(x,t) = a_0 = \text{const}$ and $\nu_0 = 0$.

3 Remarks

Through the process of finding a solution to the boundary value problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } 0 < x < \ell \text{ and } t > 0 \\ u(0,t) = u(\ell,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = f(x) & u_t(x,0) = g(x), \end{cases}$$

we arrive at the system

$$-X'' = \nu^2 X \quad \text{with } X(0) = X(\ell) = 0. \tag{8}$$

Notice that (8) is a one-dimensional version of the Dirichlet-Laplacian eigenvalue problem with $\Omega = (0, \ell)$. More importantly, we were able to obtain two objects, namely:

- Eigenvalues: $\lambda_n = \nu_n^2 = \left(\frac{n\pi}{\ell}\right)^2$
- Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{\ell}} \sin\left(\sqrt{\lambda_n} x\right)$

We see that $\{\lambda_n\}_{n=1}^{\infty}$ contains geometric information of $\Omega = (0, \ell)$. For instance, the size of the first (or the smallest) eigenvalue tells us the volume of Ω (i.e., the length of Ω in this case). If the first eigenvalue is small (under our assumption of constant tension and constant density), then length of the interval is long. Furthermore, the set $\{\phi_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $L^2(\Omega)$, so the eigenfunctions allows us to analyze functions living on Ω.

References

[1] G. B. FOLLAND, *Fourier Analysis and Its Applications*, Brooks/Cole Publishing Company, 1992.