MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lectures 12+13: Laplacian Eigenvalue Problems for General Domains: IV. Asymptotics of the Eigenvalues

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The basic references for this lecture are [1, Sec. 11.6], [2, Sec VI.2] and [3, Sec. 11.2].

1 Asymptotics of the Eigenvalues

Our main purpose here is to show $\lambda_n \uparrow \infty$ and $\nu_n \uparrow \infty$ as $n \to \infty$ and also to show how fast the eigenvalues go to infinity.

Theorem 1.1 (Weyl)**.** *Consider the following Dirichlet-Laplacian (D-L) problem.*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$, Ω is open, and $|\Omega| < \infty$. Then,

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|}. \tag{1}$$

For general domains in higher dimensions, the following are also true:

- *For* $\Omega \subset \mathbb{R}^3$, $\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{|\Omega|}$.
- *For* $\Omega \subset \mathbb{R}^d$, $\lim_{n\to\infty} \frac{\lambda_n^{d/2}}{n} = \frac{\widetilde{C}_d}{|\Omega|}$, *where* $\widetilde{C}_d = (4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)$.

Before discussing a rough proof, let us list a couple of examples.

Example 1.2 (1D interval)**.**
Given $\Omega = (0, \ell)$ for the D-L problem, then

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$$

and

$$\lim_{n\to\infty} \frac{\lambda_n^{1/2}}{n} = \frac{\pi}{\ell} = \frac{\pi}{|\Omega|}.$$

Notice that for Neumann-Laplacian (N-L) and Robin-Laplacian (R-L) problems, we have the same results.

Example 1.3 (2D rectangle of sides a and b)**.**
Let $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < a, 0 < y < b\}$ for the D-L problem, then

$$\lambda_{\boldsymbol{n}} = \lambda_{\ell,m} = \left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2,\ \ell, m = 1, 2, \dots \tag{2}$$

Because these are naturally indexed by (ℓ, m), it is difficult to see the relationship between (1) and (2). So we'll introduce the so-called enumeration function:

$$N(\lambda) \triangleq \#\{n \in \mathbb{N} \,|\, \lambda_n \leq \lambda\},$$

e.g., if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \dots$ then $N(\lambda_n) = n$.

In this particular case, $N(\lambda)$ is the number of $(\ell, m) \in \mathbb{N}^2$ satisfying

$$\frac{\ell^2}{a^2} + \frac{m^2}{b^2} \leq \frac{\lambda}{\pi^2},$$

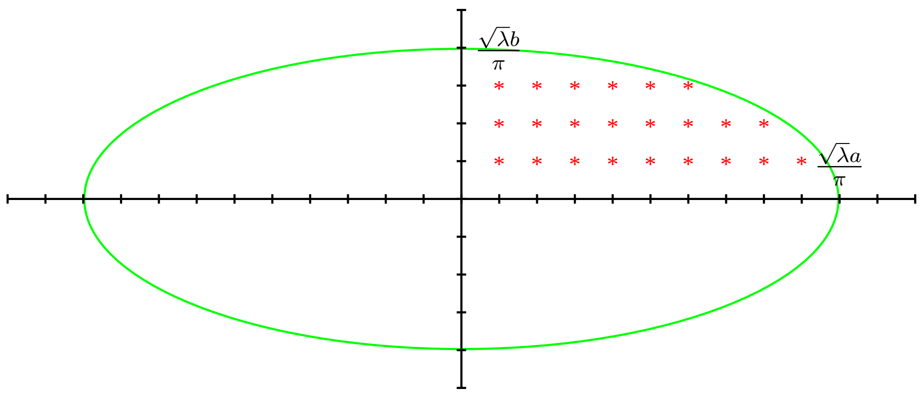

Figure 1: Lattice points contained inside the ellipse in the first quadrant. The ellipse equation: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{\lambda}{\pi^2}$.

see Figure 1.

For each (ℓ, m) we can associate a square of area 1 which implies

$$N(\lambda) \leq \frac{\text{area of the ellipse}}{4} = \pi \frac{\sqrt{\lambda}a}{\pi} \frac{\sqrt{\lambda}b}{\pi} \frac{1}{4} = \frac{\lambda ab}{4\pi}.$$

Now for large λ, the difference between $N(\lambda)$ and $\frac{\lambda ab}{4\pi}$ is proportional to the length of the perimeter. So

$$\frac{\lambda ab}{4\pi} - c\sqrt{\lambda} \leq N(\lambda) \leq \frac{\lambda ab}{4\pi}$$

for some $c > 0$. By setting $\lambda = \lambda_n$, we get

$$\frac{\lambda_n ab}{4\pi} - c\sqrt{\lambda_n} \leq n \leq \frac{\lambda_n ab}{4\pi},$$

where c is independent of n. Now we have

$$\frac{4\pi}{ab} \leq \frac{\lambda_n}{n} \leq \frac{4\pi}{ab} + \frac{4\pi}{ab} c \frac{\sqrt{\lambda_n}}{n}.$$

Then

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{ab} = \frac{4\pi}{|\Omega|}.$$

One can get the same thing for the N-L case. To proceed further, we still need several other theorems.

Theorem 1.4 (Maximin Principle)**.**
Fix $n \in \mathbb{N}$ with $n \geq 2$. Fix $n-1$ arbitrary trial functions,

$$\begin{aligned} &y_1, \ldots, y_{n-1} \in C_0^2(\Omega)^1 \quad \textit{for the D-L problem,} \\ &y_1, \ldots, y_{n-1} \in C^2(\Omega) \quad \textit{for the N-L problem.} \end{aligned}$$

Define

$$\lambda_{n*} \triangleq \min_{\substack{w \in C_0^2(\Omega) \\ \langle w, y_j \rangle = 0 \\ j=1,\ldots,n-1 \\ w \neq 0}} \frac{\|\nabla w\|^2}{\|w\|^2},$$

$$\nu_{n*} \triangleq \min_{\substack{w \in C^2(\Omega) \\ \langle w, y_j \rangle = 0 \\ j=1,\ldots,n-1 \\ w \neq 0}} \frac{\|\nabla w\|^2}{\|w\|^2}.$$

Then,

$$\lambda_n = \max_{\substack{y_j \in C_0^2(\Omega) \\ j=1,\ldots,n-1}} \lambda_{n*},$$

$$\nu_n = \max_{\substack{y_j \in C^2(\Omega) \\ j=1,\ldots,n-1}} \nu_{n*}.$$

Proof. We will prove this theorem for D-L case first. The N-L case can be proved similarly. Note that $y_1, \ldots, y_{n-1}$ are fixed at the moment. Given the eigenfunctions $\{\varphi_j\}_{j=1}^n$ of D-L problem, let

$$w = \sum_{j=1}^{n} c_j \varphi_j(x).$$

[1][2] uses $f \in C_0(\Omega)$ and $\dfrac{\partial f}{\partial x_j} \in \mathrm{PC}(\Omega)$.

From the assumption, we have $\langle w, y_k \rangle = 0$ for $k = 1, \ldots, n-1$. Assume also $\|\varphi_j\| = 1$ for $j = 1, \ldots, n$. Hence, c_j must satisfy

$$0 = \left\langle \sum_{j=1}^{n} c_j \varphi_j, y_k \right\rangle = \sum_{j=1}^{n} c_j \left\langle \varphi_j, y_k \right\rangle, \ k = 1, \ldots, n-1.$$

Since there are $n-1$ equations and n unknowns, we can choose $c_1, \ldots, c_n$ so that not all of them are zeroes. In particular, $\sum_{j=1}^{n} c_j^2 \neq 0$.

Then by the definition of λ_{n*}, we have

$$\lambda_{n*} \leq \frac{\|\nabla w\|^2}{\|w\|^2} \overset{(a)}{=} \frac{\sum_j \sum_k c_j c_k \left\langle -\Delta \varphi_j, \varphi_k \right\rangle}{\sum_j \sum_k c_j c_k \left\langle \varphi_j, \varphi_k \right\rangle} = \frac{\sum_j \sum_k c_j c_k \lambda_j \delta_{j,k}}{\sum_j \sum_k c_j c_k \delta_{j,k}}$$

$$= \frac{\sum_{j=1}^{n} \lambda_j c_j^2}{\sum_{j=1}^{n} c_j^2} \leq \lambda_n.$$

where (a) is derived by Green's first identity. This inequality holds for each choice of $\{y_1, \ldots, y_{n-1}\}$. Hence we have

$$\max_{\{y_1, \ldots, y_{n-1}\} \subset C_0^2(\Omega)} \lambda_{n*} \leq \lambda_n. \tag{3}$$

To show (3) is in fact equal, we only need to find a special choice of $\{y_1, \ldots, y_{n-1}\}$ that attains equality in (3). So let $y_j = \varphi_j, \ j = 1, \ldots, n-1$. By the minimum principle MP_n, and the definition of λ_{n*}, we know that for this choice of $y_j = \varphi_j$ we have $\max \lambda_{n*} = \lambda_n$. □

Theorem 1.5.

$$\nu_j \leq \lambda_j, \ j = 1, 2, \ldots$$

Note that this is different from the Friedlander Theorem that claims $\nu_{j+1} \leq \lambda_j$ for $j = 1, 2, \ldots$, whose proof is much more difficult.

Proof. By the minimum principle, we have

$$\lambda_1 = \min_{\substack{w \in C_0^2(\Omega) \\ w \not\equiv 0}} \frac{\|\nabla w\|^2}{\|w\|^2}, \qquad \nu_1 = \min_{\substack{w \in C^2(\Omega) \\ w \not\equiv 0}} \frac{\|\nabla w\|^2}{\|w\|^2}.$$

Now $C_0^2(\Omega) \subset C^2(\Omega)$, so the search space for the Neumann case is larger. Therefore, $\lambda_1 \geq \nu_1$. Now let $n \geq 2$. For the same reason, we have

$$\lambda_{n*} \geq \nu_{n*}.$$

This holds for each set of $n-1$ trial functions. So by the *Maximin principle*,

$$\lambda_n = \max_{\substack{y_j \in C_0^2(\Omega) \\ j=1,\ldots,n-1}} \lambda_{n*} \geq \max_{\substack{y_j \in C^2(\Omega) \\ j=1,\ldots,n-1}} \nu_{n*} = \nu_n.$$

$\square$

Remark 1.6. Any additional constraint will increase the value of the maximin.

Example 1.7 (1D String)**.**
Let $\Omega = (0, \ell)$. We have

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \geq \nu_n = \left(\frac{(n-1)\pi}{\ell}\right)^2, \qquad n = 1, 2, \ldots.$$

Theorem 1.8. *If $\Omega \subset \Omega'$, then $\lambda_n(\Omega) \geq \lambda_n(\Omega')$.*

Proof. For simplicity, let's write $\lambda_n = \lambda_n(\Omega),\ \lambda_n' = \lambda_n(\Omega')$.

Let $w \in C_0^2(\Omega)$ be an arbitrary trial function in Ω. Define $w' \in C_0^2(\Omega')$ such that

$$w'(\boldsymbol{x}) \triangleq \begin{cases} w(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega, \\ 0 & \text{if } \boldsymbol{x} \in \Omega' \setminus \Omega. \end{cases}$$

So every trial function in Ω corresponds to a trial function in Ω', but not conversely (i.e., $\exists$ trial functions for Ω' that do not satisfy the Dirichlet boundary condition for Ω). So compared to the trial function for Ω', the trial function for Ω have the extra constraint of vanishing on $\partial\Omega$. So by Remark 1.6, we get

$$\lambda_n \geq \lambda_n'.$$

Here, we avoided to show $w' \in C_0^2(\Omega)$, but for the details see [2, Sec. VI.1]. $\square$

For the Neumann case there exists a counterexample (see [4, Sec. 1.3.2]) as follows.

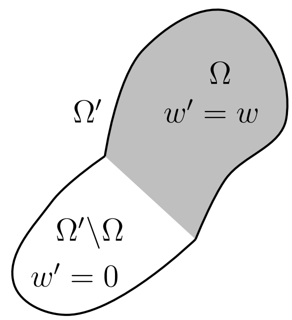

Figure 2: $w'(\boldsymbol{x})$

Example 1.9.
Consider a $2D$ rectangle of sides a and b with $a > b$. See Figure 3.

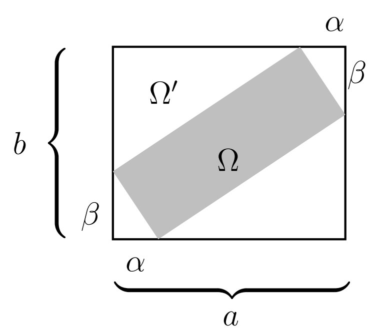

Figure 3: Neumann Case Counter Example

Let $\Omega' = \{(x,y)\,|\,0 < x < a,\ 0 < y < b\}$ and Ω be the inscribed thin rectangle as shown in Figure 3. Clearly $\Omega \subset \Omega'$. We already know the Neumann eigenvalues and eigenfunctions for a rectangle:

$$\nu_n = \nu_{\ell,m} = \pi^2 \left[\left(\frac{\ell}{a}\right)^2 + \left(\frac{m}{b}\right)^2 \right],$$
$$\psi_n(x,y) = \psi_{\ell,m}(x,y) = \text{const} \cdot \cos\left(\frac{\pi \ell x}{a}\right) \cos\left(\frac{m\pi y}{b}\right),$$

where $\ell, m = 0, 1, 2, \ldots$

Clearly, $\nu_1 = \nu_{0,0} = 0, \quad \psi_1 \equiv c = \frac{2}{\sqrt{ab}}$. Since $a > b$, the second smallest eigenvalue and its corresponding eigenfunction are

$$\nu_2 = \nu_{1,0} = \left(\frac{\pi}{a}\right)^2, \quad \psi_2 = \psi_{1,0} = c \cdot \cos\left(\frac{\pi}{a}x\right).$$

For ν_3, we have several possibilities, depending on the relationship between a and b. Here are just two examples:

(i) If $\frac{2}{a} > \frac{1}{b}$, i.e., $b < a < 2b$, we have

$$\nu_3 = \nu_{0,1} = \left(\frac{\pi}{b}\right)^2, \quad \psi_3 = \psi_{0,1} = c \cdot \cos\left(\frac{\pi}{b}y\right),$$

(ii) If $\frac{2}{a} < \frac{1}{b}$, i.e., $a > 2b$, we have

$$\nu_3 = \nu_{2,0} = \left(\frac{2\pi}{a}\right)^2, \quad \psi_3 = \psi_{2,0} = c \cdot \cos\left(\frac{2\pi}{a}x\right).$$

The point is that the second smallest eigenvalue ν_2 of a $2D$ rectangle only depends on the longer side of the rectangle, in this case a.

Now the longer side of Ω is equal to $\sqrt{(a-\alpha)^2 + (b-\beta)^2}$. By choosing appropriate $\alpha > 0,\ \beta > 0$ we can have $\sqrt{(a-\alpha)^2 + (b-\beta)^2} > a$. In other words, we can have $\nu_2 < \nu_2'$, even if $\Omega < \Omega'$.

2 Subdomains

The next step toward the proof of $\lambda_n \to \infty,\ \frac{\lambda_n}{n} \to \frac{4\pi}{|\Omega|}$ as $n \to \infty$ is to divide Ω into a finite number of subdomains $\Omega_1, \ldots, \Omega_m$ by introducing smooth boundary surfaces (partitions) $\Gamma_1, \Gamma_2, \ldots$. See Figure 4.

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues for Ω. Let $0 < \widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \ldots$ be the collection of all the eigenvalues $\{\lambda_j(\Omega_k)\}_{1\leq k\leq m,\ j\in\mathbb{N}}$ in the ascending order.

By the Maximin principle, each $\widetilde{\lambda}_n$ can be obtained as

$$\widetilde{\lambda}_n = \max_{\{y_1,\cdots,y_{n-1}\}} \min_{\substack{w\in C_0^2(\Omega) \\ w\perp\{y_1,\cdots,y_{n-1}\}}} \frac{\|\nabla w\|^2}{\|w\|^2}.$$

But each $y_j,\ j = 1, \ldots, n-1$ are supported on only one of the subdomains $\Omega_1, \ldots, \Omega_m$. So $\lambda_n \leq \widetilde{\lambda}_n$ by Remark 1.6.

As for the Neumann case, again list all the eigenvalues of the subdomains as $0 = \widetilde{\nu}_1 = \widetilde{\nu}_2 = \cdots = \widetilde{\nu}_m < \widetilde{\nu}_{m+1} \leq \ldots$. Now in the *Maximin principle* the trial functions $\{y_1, \ldots, y_{n-1}\}$ for $\widetilde{\nu}_n$ do not have to vanish at $\partial\Omega$ and $\Gamma_j,\ j = 1, 2, \ldots$ So, there exist less constraints than in the Dirichlet case for Ω, hence we have $\widetilde{\nu}_n \leq \lambda_n$. Summarizing all the results so far, we have:

Theorem 2.1.

$$\nu_n \leq \lambda_n \leq \widetilde{\lambda}_n,$$
$$\widetilde{\nu}_n \leq \lambda_n \leq \widetilde{\lambda}_n.$$

Now let $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \cdots \cup \overline{\Omega_m}$, where Ω_j are all all rectangles, see Figure 5.

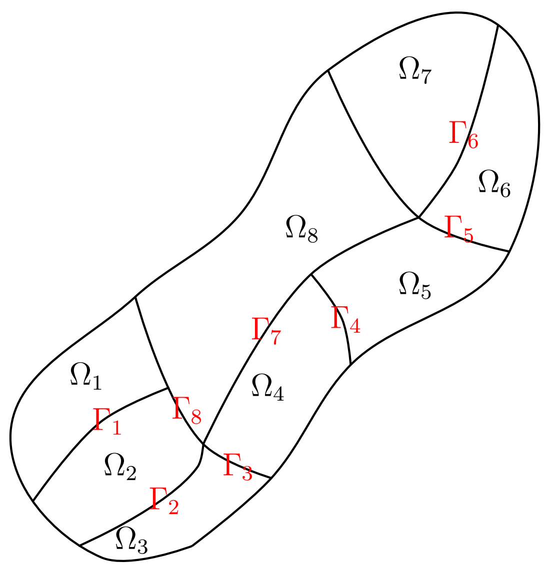

Figure 4: Division of Ω into a finite number of subdomains, $\Omega_1, \Omega_2, \ldots, \Omega_m$ with smooth boundary surfaces, $\Gamma_1, \Gamma_2, \ldots$.

Let

$$M(\widetilde{\lambda}) \triangleq \# \left\{ n \in \mathbb{N} \,|\, \widetilde{\lambda}_n \leq \widetilde{\lambda} \right\}.$$

Then by counting integer lattice points in each rectangle Ω_j and Ω, we have

$$\lim_{\widetilde{\lambda}\to\infty} \frac{M(\widetilde{\lambda})}{\widetilde{\lambda}} = \sum_j \frac{|\Omega_j|}{4\pi} = \frac{|\Omega|}{4\pi}.$$

Since $M(\widetilde{\lambda}_n) = n$, we get

$$\lim_{n\to\infty} \frac{\widetilde{\lambda}_n}{n} = \frac{4\pi}{|\Omega|}.$$

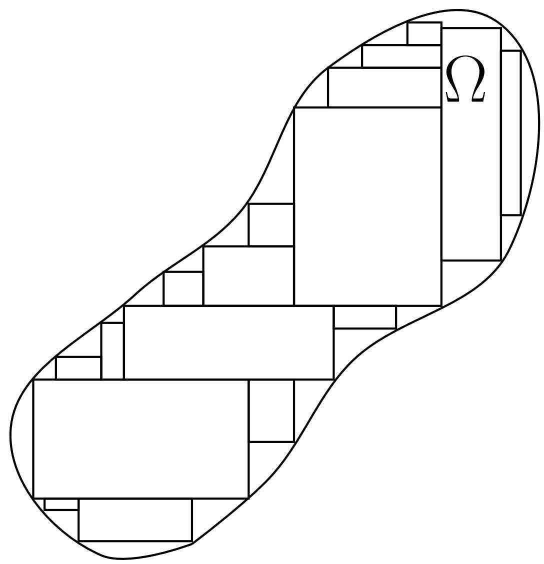

Figure 5: Ω represented as a collection of rectangles.

Similarily, we can get

$$\lim_{n\to\infty} \frac{\widetilde{\nu}_n}{n} = \frac{4\pi}{|\Omega|}.$$

By the sandwich theorem, we have

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|}.$$

Remark 2.2. For a more general domain, it can be approximated by unions of rectangles. Using the similar arguments as before it is possible to prove

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|} \text{ (2D version of Weyl's asymptotic formula).}$$

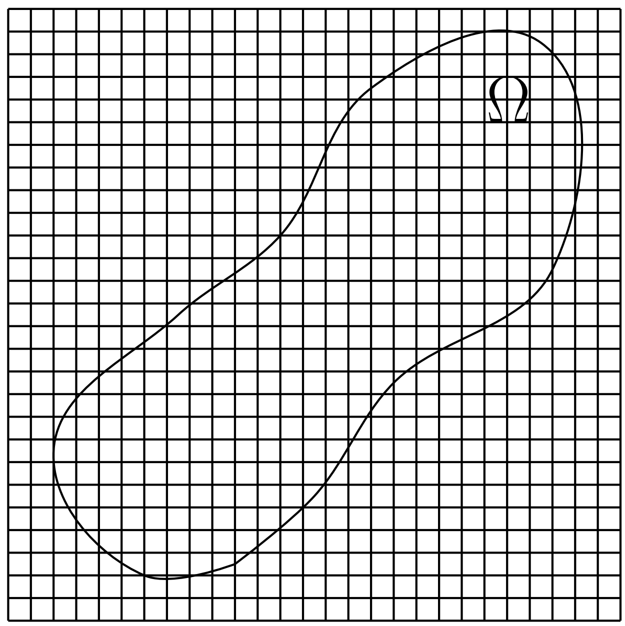

Figure 6: Example of an approximation of Ω by the union of a uniform squares.

For the details, see [2, Sec. VI. 2] and [3, Sec. 11.2]

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