MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 19: Introduction to Spectral Graph Theory–II. Graph Laplacians and Eigenvalues of Adjacency Matrices and Laplacians

Lecturer: Naoki Saito
Scribe: Ashley Evans/Allen Xue

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The previous lecture introduced basic ideas of graph theory and defined the adjacency and transition matrices of a graph. In this lecture, we will further introduce the Laplacian of a graph, and the eigenvalues associated with these fundamental matrices of graphs. This lecture is based on [2] and the texts by Van Lint and Wilson [1, Chap. 31] and Chung [3, Sec. 1.1-1.3].

1 Graph Laplacians and Derivatives

There are two versions of the Laplacian of a graph associated with the adjacency and transition matrices which will be characterized. Also the derivative of a graph function and the resulting discrete version of Green's identity will be introduced.

1.1 The Laplacian of a Graph

Recall that given a set of nodes (or vertices) V and edges E, a graph G is defined by $G = (V, E)$. The degree of a node is denoted by $m(u)$.

Definition 1.1. Given a graph G with adjacency matrix A and transition matrix P as defined in the previous lecture, define the **degree matrix** of G as

$$D \triangleq \operatorname{diag}(m(u)) \in \mathbb{R}^{N \times N}.$$

The **adjacency Laplacian** of G is defined as

$$\varDelta_A \triangleq D - A. \tag{1}$$

The **transition Laplacian** of G (or **normalized Laplacian** of G) is defined as

$$\varDelta_P \triangleq I - P, \tag{2}$$

where I is the identity matrix. Note the following relationship:

$$\varDelta_A = D^{1/2} \varDelta_P D^{1/2},$$

where

$$D^{1/2} = \operatorname{diag}(\sqrt{m(u)}).$$

Notice that this definition in the discrete case corresponds to $-\varDelta$ in the continuum.

Figure 1: 1-D lattice graph.

Example 1.2. Consider the simple 1D lattice graph in Figure 1 with four nodes. The adjacency and transition matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The degree matrix is $\mathrm{diag}(1,2,2,1)$. Thus,

$$\varDelta_A = D - A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

$$\varDelta_P = I - P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

It is interesting to note that the eigenvectors of the 1D lattice adjacency matrix are the basis vectors for the Discrete Cosine Transform Type II (DCT-II). The basis vectors for the DCT-I, III, ... can be generated with subtle changes in the discrete boundary conditions. For more information about the relationships between descrete boundary value problems and the DCT and Discrete Sine Transform (DST), see [4].

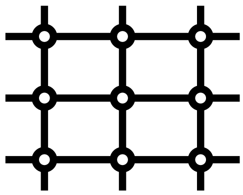


Figure 2: 2-D lattice graph.

Now, consider a function f on V ($f : V \to \mathbb{R}$). Recall from the previous lecture that

$$C(V) \triangleq \{\text{all functions defined on } V\text{'s}\},$$

and

$$L^2(V) \triangleq \left\{ f \in C(V) \middle| \|f\| = \sqrt{\langle f, f \rangle} < \infty \right\},$$

where

$$\langle f, g \rangle \triangleq \sum_{u \in V} m(u) f(u) g(u).$$

If $f \in L^2(V)$, then the corresponding Laplacians are applied to f as follows:

$$\Delta_A f(u) = m(u)f(u) - \sum_{v\sim u} f(v), \tag{3}$$

$$\Delta_P f(u) = f(u) - \frac{1}{m(u)} \sum_{v\sim u} f(v), \tag{4}$$

where $\sum_{v\sim u}$ is the sum over all v's that are adjacent to u. Looking at this last transition Laplacian, we can see that it is essentially the function value minus the average of the function values at the nodes connected to u. This can also be re-written as follows:

$$\Delta_P f(u) = \frac{1}{m(u)} \sum_{v\sim u} \left(f(u) - f(v)\right).$$

Definition 1.3. A function $f \in C(V)$ is called **harmonic** if

$$\Delta_A f = 0 \quad \text{or} \quad \Delta_P f = 0.$$

$f \in C(V)$ is called **superharmonic** at $x \in V$ if

$$\Delta_A f(x) \geq 0 \quad \text{or} \quad \Delta_P f(x) \geq 0.$$

Also,

$$f(x) \text{ superharmonic at } x \Leftrightarrow \frac{1}{m(x)} \sum_{y\sim x} f(y) \leq f(x).$$

The last equivalence essentially means that when $f(x)$ is superharmonic, it is larger than or equal to the surrounding function averages. Note that these definitions of Δ_A and Δ_P correspond to $-\Delta$ in $\mathbb{R}^d$.

1.2 Derivatives and Green's Identity

Many times in the continuum the results have greatly benefited from Green's identity, and we will see that there is a useful discrete version. Recall that the set of directed edges is denoted by $\mathbb{E}$ and that for an edge e with direction, $\bar{e}$ denotes a reversal of the direction.

Definition 1.4. Let

$$C(\mathbb{E}) \triangleq \{\varphi \text{ defined on } \mathbb{E} \,|\, \varphi(\bar{e}) = -\varphi(e), e \in \mathbb{E}\} .$$

For $f \in C(V)$, define the **derivative** of f as follows:

$$\mathrm{d}f(e) \triangleq \mathrm{d}f([x, y]) = f(y) - f(x), \quad \mathrm{d}f \in C(\mathbb{E}). \tag{5}$$

where $e = [x, y]$ is the edge connecting nodes x and y.

Recall from the previous lecture that

$$C_0(V) \triangleq \{f \in C(V) \,|\, \operatorname{supp} f \text{ is a finite subset of } V\} ,$$

which indicates that f has a compact support. In the interest of time the proof of the following theorem will be left out, but for more information see [6].

Theorem 1.5 (The discrete version of Green's first identity)**.** *For all $f_1, f_2 \in C_0(V)$, we have*

$$\langle\, \mathrm{d}f_1,\, \mathrm{d}f_2\rangle = \langle \Delta_P f_1, f_2\rangle = \sum_{u\in V} (\Delta_A f_1)(u) f_2(u). \tag{6}$$

Corollary 1.6. *Note that Δ_A and Δ_P are both positive operators.*
$\forall f \in C_0(V)$, we have

$$\langle \Delta_P f, f\rangle = \sum_{u\in V} (\Delta_A f)(u) f(u) = \langle\, \mathrm{d}f,\, \mathrm{d}f\rangle \geq 0. \tag{7}$$

Theorem 1.7 (The discrete version of the minimum principle)**.** *Let $f \in C(V)$ be superharmonic at $x \in V$. If*

$$f(x) \leq \min_{y\sim x} f(y) \qquad (\star)$$

Then any $z \sim x$, we have $f(z) = f(x)$.

Proof. We have

$$\frac{1}{m(x)} \sum_{y\sim x} f(y) \leq f(x) \leq \min_{y\sim x} f(y) \leq \frac{1}{m(x)} \sum_{y\sim x} f(y).$$

So $\frac{1}{m(x)} \sum_{y\sim x} f(y) = f(x)$, but because of $(\star)$, this can only happen when $f(y) = f(x), \forall y \sim x$. $\square$

2 Eigenvalue Problems of Finite Graphs

The properties of eigenvalues of finite graphs can now be defined. In the interest of time, infinite graphs will not be considered.

2.1 Eigenvalues and the Chromatic Number

The following theorem of Perron-Frobenius for the adjacency matrix of a *connected* graph (finite) is well known, see [5, Chap. 8].

Theorem 2.1 (Perron-Frobenius)**.** *Let A be an adjacency matrix of a connected finite graph. The following hold.*

1. *For any $\lambda(A)$, an eigenvalue of A,*

$$|\lambda(A)| \leq \lambda_{\max}(A). \tag{8}$$

   Note that λ_{max} is always positive, but other eigenvalues could be negative. Yet the above still holds.

2. $\lambda_{\max}(A)$ *has multiplicity one.*

3. *There exists an eigenvector corresponding to $\lambda_{\max}(A)$ whose entries are all positive.*

Example 2.2. Figure 3 contains examples of graphs and their eigenvalues that illustrate Theorem 2.1.

Theorem 2.3. *Let $p = \#(V)$, $q = \#(E)$ for $G = (V, E)$.*
Then:

1. $\lambda_{\max}(A) \leq m_\infty(G) =$ *the largest degree of G.*

2. $\frac{2q}{p} \leq \lambda_{\max}(A) \leq \sqrt{\frac{2q(p-1)}{p}}$*, with the first equality holds if and only if G is regular and the second equality holds if and only if $G = K_p$.*

3. *If $G \neq K_p$, $p \geq 3$, then*

$$-\sqrt{2} \leq \lambda_{\min}(A) \leq \lambda_{\max}(A) \leq \sqrt{2}.$$

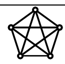
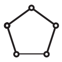
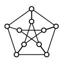
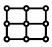
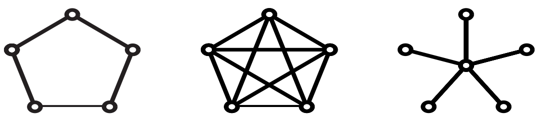
Graph G	$\lambda(A)$
 K_5	$4, -1, -1, -1, -1$
 $K_{3,3}$	$3, 0, 0, 0, 0, -3$
 3-D cube	$3, 1, 1, 1, -1, -1, -1, 3$
 P_5	$2, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$
 Peterson Graph	$3, 1, 1, 1, 1, 1, -2, -2, -2, -2$
 $L_2(3)$ (lattice graph)	$4, 1, 1, 1, 1, -2, -2, -2, -2$

Figure 3: Examples of graphs and their eigenvalues.

This theorem gives some properties of eigenvalues, which reflect information about the structure of a graph.

Definition 2.4. The **chromatic number** $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Some examples of graphs and their corresponding chromatic numbers are given in Figure 4.



(a) $\chi(P_5) = 3$ (b) $\chi(K_5) = 5$ (c) $\chi(S_6) = 2$

Figure 4: Examples of graphs and their chromatic numbers.

Now let's order the eigenvalues of Δ_A, Δ_P as follows:

$$0 = \lambda_1^{(A)} \leq \lambda_2^{(A)} \leq \cdots \leq \lambda_p^{(A)} \tag{9}$$

$$0 = \lambda_1^{(P)} \leq \lambda_2^{(P)} \leq \cdots \leq \lambda_p^{(P)} \tag{10}$$

Lemma 2.5. *The smallest eigenvalues for Δ_A, Δ_P are both zero, and the corresponding eigenvectors are of the form $\alpha(1, 1, 1, \cdots 1)^T$, α is some constant.*

Note the similarity with the Neumann Laplacian problem in the continuum.

The following theorem relates the chromatic number to the eigenvalues.

Theorem 2.6.

1. *Wilf (1967) [7],*
$$\chi(G) \leq 1 + \lambda_{\max}(A),$$
with equality if and only if $G = K_p$ for any $p \in \mathbb{N}$ or $G = P_p$ for p is odd. Here P_p is a circuit with p vertices.

2. *Tan (2000) [8],*
$$\chi(G) \leq \lambda_p^{(A)}$$
with equality if and only if $G = K_p$. If $m_\infty(G) \geq 3$ and $G \neq K_p$, then
$$\chi(G) \leq \frac{p-1}{p}\lambda_p^{(A)}.$$

2.2 Some other characterizations of $\lambda_2^{(P)}$ and $\lambda_j^{(P)}$, $j \geq 2$

Theorem 2.7 (See [3] for more information)**.** *Let $p = |V|$.*

1. $$\sum_{j=1}^{p} \lambda_j^{(P)} \leq p,$$
with equality if and only if G has no isolated vertices.

2. *For $p \geq 2$,*
$$\lambda_2^{(P)} \leq \frac{p}{p-1},$$
with equality if and only if $G = K_p$. Also, for a graph without isolated vertices, we have
$$\lambda_p^{(P)} \geq \frac{p}{p-1}.$$

3. *For $G \neq K_p$, $\lambda_2^{(P)} \leq 1$.*

4. *If G is connected, then $\lambda_2^{(P)} > 0$. If $\lambda_j^{(P)} = 0$ and $\lambda_{j+1}^{(P)} \neq 0$, then G has exactly j connected components.*

5. *$\lambda_j^{(P)} \leq 2$, $j = 1, 2, \cdots, p$. Also, $\lambda_p^{(P)} = 2$ if and only if a connected component of G is bipartite and nontrivial.*

6. *The spectrum of a graph is the union of the spectra of its connected components (the same situation as the continuum we discussed before).*

7. $$\lambda_2^{(P)} \geq \frac{1}{diam(G) \cdot vol(G)},$$
   where
   $$vol(G) \triangleq \sum_{u \in V} m(u).$$

8. *Let $\varphi_2^{(P)}$ be an eigenfunction corresponding to $\lambda_2^{(P)}$. Then for any $x \in V$ we have*
   $$\lambda_2^{(P)} \varphi_2^{(P)}(x) = \frac{1}{m(x)} \sum_{y \sim x} \left(\varphi_2^{(P)}(x) - \varphi_2^{(P)}(y) \right).$$

3 Conclusion

From these characteristics of graphs, we have found that essentially the same ideas of the continuum case carry over to discrete graphs. Suppose you have a very large graph. These results can be used to automate the analysis of that graph through the eigenvalues and eigenfunctions of the Laplacian. Next time we will discuss isospectral graphs sharing the same eigenvalues. It can be seen that the Laplacian eigenvalues of a graph contain certain important information about that graph, but the eigenvalues alone cannot recover or uniquely determine the graph.

References

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