MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 2: Sturm-Liouville Theorem, (Pre)History of the Laplacian Eigenvalue Problem in $\mathbb{R}^d$

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0 Review: Vibrations of A One Dimensional String

In Lecture 1, for the problem of vibration of 1D string, depending on the type of boundary condition (BC), we consider the following:

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } x \in (0,\ell) \text{ and } t > 0; \\ \left.\begin{array}{ll} u(0,t) = u(\ell,t) = 0 & \text{Dirichlet BC} \\ \text{or } \ u_x(0,t) = u_x(\ell,t) = 0 & \text{Neumann BC} \end{array}\right\} & \text{for } t \geq 0; \\ u(x,0) = f(x), \ \ u_t(x,0) = g(x) & \text{for } x \in [0,\ell] \end{cases} \tag{1}$$

To solve for $u(x,t)$, we assume that the solution is independent in time and space. That is, we can write our solution as $u(x,t) = X(x)T(t)$, where $X(x)$ and $T(t)$ do not depend on each other. After separating T part, we had:

$$\begin{cases} -X'' = \lambda X & \text{for } x \in (0,\ell) \text{ and } \lambda \geq 0; \\ X(0) = X(\ell) = 0 & \text{Dirichlet BC} \\ \text{or } \ X'(0) = X'(\ell) = 0 & \text{Neumann BC} \end{cases} \tag{2}$$

This is a 1-D version of the **Laplacian eigenvalue problem**: given a general shape $\Omega \subset \mathbb{R}^d$

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \subset \mathbb{R}^d \\ u = f & \text{on } \partial\Omega, \quad \text{Dirichlet BC} \\ \text{or } \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \quad \text{Neumann BC} \end{cases} \tag{3}$$

REMARK: In 1D problem (2), we get

$$\begin{cases} \lambda_n = \nu_n^2 = \left(\dfrac{n\pi}{\ell}\right)^2 & \\ n = 1, 2, \ldots & \text{Dirichlet BC} \\ n = 0, 1, 2, \ldots & \text{Neumann BC} \end{cases}$$

Notice that $\Omega = (0, \ell)$, with $|\Omega| = \ell$. Therefore, the eigenvalues reflect the geometric information of Ω, in this 1D case, the volume of $\Omega =$ the length of $\Omega = \ell$.

In 1D, this line of work culminated in the work of Sturm and Liouville (1836-37), which also accounts *non-uniform* strings.

1 Sturm and Liouville's work

Given $\Omega = (a, b)$. Define

$$\begin{aligned} L_w^2[a,b] \quad &\triangleq \quad \Big\{ f \text{ supported on } [a,b] \,\Big|\, \int_a^b |f(x)|^2 w(x)\,\mathrm{d}x < \infty, \\ &\qquad \text{with } w(x) > 0 \text{ and } w \in C[a,b] \Big\}. \end{aligned} \tag{4}$$

equipped with the weighted inner product for all $f, g \in L_w^2[a, b]$ as

$$\langle f, g \rangle_w \triangleq \int_a^b f(x)\overline{g(x)} w(x)\,\mathrm{d}x.$$

Define $\mathcal{L} : L_w^2[a, b] \to L_w^2[a, b]$ such that for $f \in L_w^2[a, b]$,

$$\mathcal{L}f \triangleq (rf')' + pf \tag{5}$$

where $r \in C^1[a, b]$, $r > 0$ on $[a, b]$, and $p \in C[a, b]$ is real-valued.

We consider the Regular Sturm-Liouville Problem (RSLP):

$$\begin{cases} \mathcal{L}f + \lambda w f = 0 \\ \mathcal{B}_1 f = \mathcal{B}_2 f = 0 \end{cases} \tag{6}$$

where $\mathcal{B}_j f \triangleq \alpha_j f(a) + \alpha'_j f'(a) + \beta_j f(b) + \beta'_j f'(b)$, for $j = 1, 2$, with constants α_j, α'_j, β_j, and β'_j.

We say that the boundary conditions in (6) are *self-adjoint* (relative to $\mathcal{L}$) if

$$\left[r(f'\overline{g} - f\overline{g'})\right]_a^b = 0 \quad \text{for all } f, g \text{ satisfying } \mathcal{B}_j f = \mathcal{B}_j g = 0, j = 1, 2.$$

For any f, g belonging to a certain subspace of $L^2_w[a, b]$ and satisfying the self-adjoint boundary conditions, we can easily show that the differential operator defined in (5) satisfies $\langle \mathcal{L}f, g \rangle_w = \langle f, \mathcal{L}g \rangle_w$. Such an $\mathcal{L}$ is called a *self-adjoint operator*.

Note that for the vibration of the one-dimensional string with homogeneous Dirichlet boundary conditions, $r \equiv 1$, $p \equiv 0$, $\alpha_1 = 1$, $\alpha'_1 = 0$, $\beta_1 = 0$, $\beta'_1 = 0$, $\alpha_2 = 0$, $\alpha'_2 = 0$, $\beta_2 = 1$, and $\beta'_2 = 0$.

1.1 The Sturm-Liouville Theorem

For every RSLP (6), the following hold:

1. All eigenvalues are real.
2. Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to $\langle \cdot, \cdot \rangle_w$.
3. The eigenspace for any eigenvalue λ is at most 2-dimensional, and these two eigenfunctions can be chosen to be orthogonal.
4. ϕ_n, the n^{th} eigenfunction corresponding to the n^{th} distinct eigenvalue, has $n - 1$ zeros in (a, b).
5. For any $f \in C^2[a, b]$ satisfying $\mathcal{B}_1 f = \mathcal{B}_2 f = 0$ (but not necessarily $\mathcal{L}f + \lambda w f = 0$), the series $\sum_{n=1}^{\infty} \langle f, \phi_n \rangle_w \phi_n$ converges uniformly to f.

6. For any $f \in L^2_w[a,b]$, $\|f\|^2_{L^2_w([a,b])} = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle_w|^2$ (Parseval's equality).

In higher dimensions, we shall not delve into generalities of spatially varying coefficients, such as $r(x)$, $p(x)$ in RSLP (6). We shall stick with the simple Laplacian eigenvalue problem in $\mathbb{R}^d$, $d > 1$.

2 (Pre)History of the Laplacian Eigenvalue Problem in $\mathbb{R}^d$

2.1 The Lorentz Conjecture (from [2])

In late October of 1910, a Dutch physicist H. A. Lorentz delivered a series of five "Wolfskehl" lectures (via a donation of Mr. Wolfskehl, who intended to pay the prize for a person who solved "The Fermat Conjecture") titled *Old and New Problems of Physics* at Göttingen University in Germany. Referring to a Cambridge physicist J. H. Jeans's work in radiation theory, Lorentz said:

> "In an enclosure with a perfectly reflecting surface, there can form standing electromagnetic waves analogous to tones over an organ pipe: we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval $\mathrm{d}\nu$. To this end, he calculates the number of overtones which lie between frequencies ν and $\nu + \mathrm{d}\nu$, and multiplies this number by the energy which belongs to the frequency ν, and which according to a theorem of statistical mechanics, is the same for all frequencies."
>
> "It is here that there arises the mathematical problem to prove that *the number of sufficiently high overtones which lie between ν and $\nu + \mathrm{d}\nu$ is independent of the shape of the enclosure, and is simply proportional to its volume.* For many shapes for which calculations can be carried out, this theorem has been verified in a Leiden dissertation. There is no doubt that it holds in general even for multiply connected regions. Similar theorems for other vibrating structures, like membranes, air masses, etc., should also hold."

If we express the Lorentz conjecture in a vibrating membrane, it becomes of the following form:

$$\begin{aligned} N(\lambda) &= \textstyle\sum_{\lambda_n<\lambda} 1 \\ &= \#\{\lambda_n : \text{ Dirichlet-Laplacian eigenvalues (3)} \mid \lambda_n < \lambda\} \\ &\sim \frac{|\Omega|}{2\pi}\lambda. \end{aligned} \tag{7}$$

A mathematician D. Hilbert was attending these lectures and predicted as follows: "This theorem would not be proved in my life time." But, in fact, Hermann Weyl, a graduate student at that time, was also attending these lectures. Weyl proved this conjecture four months later in February of 1911.

2.2 Weyl's work

Let $\Omega \subset \mathbb{R}^d$ with $|\Omega| =$ volume of $\Omega = \int_\Omega \mathrm{d}\boldsymbol{x} < \infty$. Consider the vibration problem

$$\begin{cases} u_{tt} = \Delta u & \text{in } \Omega \\ u(\boldsymbol{x},t) = 0 & \text{on } \partial\Omega \\ u(\boldsymbol{x},0) = f(\boldsymbol{x}) & \text{in } \Omega \\ u_t(\boldsymbol{x},0) = g(\boldsymbol{x}) & \text{in } \Omega \end{cases} \tag{8}$$

Using the separation of variables $u(\boldsymbol{x},t) = X(\boldsymbol{x})T(t)$, we reach to

$$XT'' = (\Delta X)T, \quad \frac{T''}{T} = \frac{\Delta X}{X} = -\lambda$$

replacing X by u, we get the Dirichlet-Laplacian eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

Let $L = -\Delta$, where $\Delta = \dfrac{\partial^2}{\partial {x_1}^2} + \dfrac{\partial^2}{\partial {x_2}^2} + \cdots + \dfrac{\partial^2}{\partial {x_d}^2} = \nabla \cdot \nabla$

If $u \not\equiv 0$ satisfies (9), then it is called a eigenfunction and the corresponding λ is called the eigenvalue. Define $E_\lambda \triangleq \{u \in \mathcal{D}(L) \mid Lu = \lambda u\}$ the eigenspace corresponding to λ, with $\dim E_\lambda =$ multiplicity of λ.

In this problem, $\{\lambda\}$ consists of countably finite number of eigenvalues with finite multiplicity, i.e., we can order them as

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \longrightarrow \infty$$

Let $L\varphi_k = \lambda_k \varphi_k$, $k = 1, 2, \ldots$. And let $f \in C_0(\overline{\Omega})$, i.e., $f \in C(\overline{\Omega})$ and $f = 0$ on $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$. In fact it is ok to assume $f \in L^2(\Omega)$.
Then $f = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \varphi_k$. This is called an *eigenfunction expansion* of f, because $\langle \varphi_k, \varphi_\ell \rangle = \delta_{k\ell}$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis (ONB for short) of $L^2(\Omega)$, here $\langle f, g \rangle \triangleq \int_\Omega f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}$.

Expanding the initial conditions into the eigenbasis, we get

$$f(\boldsymbol{x}) = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \varphi_k, \quad g(\boldsymbol{x}) = \sum_{k=1}^{\infty} \langle g, \varphi_k \rangle \varphi_k$$

Then, we get the solution to the vibration problem

$$u(\boldsymbol{x}, t) = \sum_{k=1}^{\infty} \left\{ \langle f, \varphi_k \rangle \cos \sqrt{\lambda_k} t + \frac{\langle g, \varphi_k \rangle}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right\} \varphi_k(\boldsymbol{x})$$

So, the key was the Laplacian eigenvalue problem (9).

Weyl's Theorem

$$\lambda_k \sim \left(\frac{k}{C_d |\Omega|} \right)^{\frac{2}{d}} \quad \text{as } k \to \infty \tag{10}$$

where $C_d \triangleq (2\sqrt{\pi})^{-d} \Gamma\left(\frac{d}{2}\right)$.

Equivalently,

$$N(\lambda) \sim C_d |\Omega| \lambda^{d/2} \tag{11}$$

where $N(\lambda) = \#\{k \in \mathbb{N} \mid \lambda_k \leq \lambda\}$. This equivalence is clear since $N(\lambda_k) = k$, so $k \sim C_d |\Omega| \lambda_k^{d/2}$.

Weyl's Conjecture

$$N(\lambda) = C_d |\Omega|_d \lambda^{\frac{d}{2}} - \frac{C_{d-1}}{4} |\partial\Omega|_{d-1} \lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}}). \tag{12}$$

where $|\Omega|_d$: volume in $\mathbb{R}^d$ and $|\partial\Omega|_{d-1}$: volume in $\mathbb{R}^{d-1}$ or the area in $\mathbb{R}^d$.
This conjecture has not been completely solved yet and started the field known as "spectral geometry".

2.3 Can we hear the shape of a drum?

In 1966, Mark Kac (Rockefeller Univ.) asked "Can we hear the shape of a drum?" [2]. In other words, "how much can we know about the shape (geometric information) of Ω from the Laplacian eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$?"

Kac proceeded to show that for all bounded $\Omega \subset \mathbb{R}^2$,

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + o(t^{-\frac{1}{2}}) \ \text{ as } t \downarrow 0 \tag{13}$$

also

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1-r}{3} + o(1) \ \text{ as } t \downarrow 0, \tag{14}$$

if Ω has r holes and Ω and holes are polygons.

In 1967, McKean and Singer generalized Kac's result to $\Omega \subset \mathbb{R}^d$. For more about this work and the related work up to 1987, see [3].

References

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[2] M. KAC, "Can one hear the shape of a drum?" *Amer. Math. Monthly*, vol. 73, no. 4, part 2, pp. 1-23, 1966.

[3] M. H. PROTTER, "Can one hear the shape of a drum? Revisited" *SIAM Review*, vol. 29, no. 2, pp. 185-197, 1987.

[4] W. A. STRAUSS, *Partial Differential Equations: An Introduction*, Chap. 10-11, John Wiley & Sons, 1992.

[5] K. YOSIDA, *Lectures on Differential and Integral Equations*, Chap. 2, Wiley-Interscience, 1960. Republished by Dover in 1991.

[6] N. YOUNG, *An Introduction to Hilbert Space*, Chap. 9, 10, 11, Cambridge Univ. Press, 1988.