MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 20: Introduction to Spectral Graph Theory–III. Graph Cut and Cheeger Constants, Isospectral Graphs, and Discrete Laplacian Eigenvalue Problems

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1 The Cheeger Ratio and the Cheeger Constants of a Graph

This lecture is based primarily on material in [17] and [3, Sec. 2.1-2.3].

Definition 1.1. Given a graph $G = (V, E)$. Let $S \subset V$ be a subset of vertices. Then $\partial S \triangleq \{e = (x, y) \in E \,|\, x \in S,\ y \notin S\}$,

Example 1.2. In Figure 1, S is the set of dark nodes, and the dotted lines form ∂S. $\overline{S} \triangleq V \backslash S$. Also, $\mathrm{vol}(S) = m(S) \triangleq \sum_{x \in S} m(x)$.

Definition 1.3. The **Cheeger ratio** for $S \subset V$ is defined as

$$h(S) \triangleq \frac{\#(\partial S)}{\min(\mathrm{vol}(S), \mathrm{vol}(\overline{S}))} = \frac{|\partial S|}{\min(\mathrm{vol}(S), \mathrm{vol}(\overline{S}))},$$

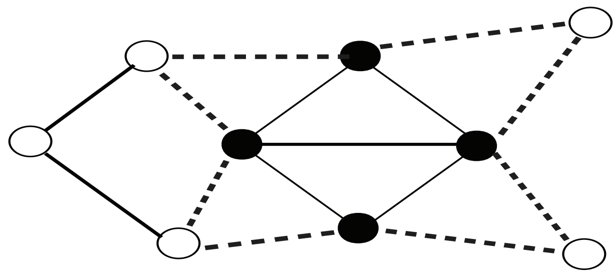


Figure 1: S consists of the dark nodes. The dotted lines form ∂S.

Example 1.4. In Figure 1, vol$(S) = 4+5+4+5 = 18$ and vol$(\overline{S}) = 3+2+3+2+2 = 12$. $\#(\partial S) = 8$. So the Cheeger ratio is $h(S) = 8/12 = 2/3$.

The Cheeger ratio tells about the quality of the **cut** of V into $S \cup \overline{S} = V$. In Figure 2, a graph that nearly separates into two separate graphs is shown. Note that this graph is well balanced, i.e., vol$(S) \approx$ vol$(\overline{S})$; and there exist few connections between S and $\overline{S}$, i.e., $|\partial S|$ is small. In such case, this cut will give us small $h(S)$.

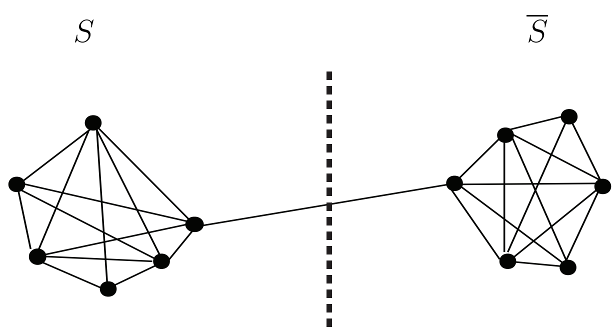


Figure 2: The cut into S and $\overline{S}$ generates a small Cheeger ratio.

Definition 1.5. The **Cheeger constants** $h_P(G)$ and $h_A(G)$ are defined as

$$h_P(G) \triangleq \inf_{S \subset V,\ S \neq \emptyset} h(S),$$

$$h_A(G) \triangleq \inf_{S \subset V,\ S \neq \emptyset} \frac{\#(\partial S)}{\min\{\#(S), \#(\overline{S})\}} = \inf_{S \subset V,\ S \neq \emptyset} \frac{|\partial S|}{\min(|S|, |\overline{S}|)}.$$

The following theorem provides the upper bounds for $h_P(G)$ and $h_A(G)$.

Theorem 1.6.

1. *Dodzink and Kendall (1986) [5]:* $h_P(G) \leq \sqrt{2\lambda_2^{(P)}}$, *i.e.,* $\lambda_2^{(P)} \geq \dfrac{h_P^2(G)}{2}$;
2. *Mohar (1987) [12]:* $h_A(G) \lneqq \sqrt{\lambda_2^{(A)}(2m_\infty(G) - \lambda_2^{(A)})}$ *for* $p = \#(V) \geq 4$;
3. *Tan (2003) [15]:* $h_P(G) \leq \sqrt{\lambda_2^{(P)}(2 - \lambda_2^{(P)})}$ *, for* $p \geq 4$. *Equality holds if and only if* $G = K_{1,p-1}$.

These inequalities are used to evaluate the lower bounds of $\lambda_2^{(P)}$, $\lambda_2^{(A)}$.

For the upper bounds of the eigenvalues, we have

Theorem 1.7 (Urakawa 1999 [16])**.**

For $2 \leq j \leq \lfloor \frac{diam(G)}{2} \rfloor$,

$$\lambda_j^{(A)} \leq m_\infty(G) - 2\sqrt{m_\infty(G) - 1}\cos\left(\frac{\pi}{\frac{diam(G)}{2j} + 1}\right);$$

$$\lambda_j^{(P)} \leq 1 - \frac{2\sqrt{m_\infty(G) - 1}}{m_\infty(G)}\cos\left(\frac{\pi}{\frac{diam(G)}{2j} + 1}\right).$$

Note that there also exists the simpler upper bound:

$$\lambda_2^{(P)} \leq 2h_P(G).$$

Let us prove this inequality.

Proof. Using the minimum principle (MP_2), we have

$$\begin{aligned}
\lambda_2^{(P)} &= \inf_{f\in C_0(V),\ \langle f,1\rangle=0,\ f\not\equiv 0} \frac{\langle\, \mathrm{d}f,\, \mathrm{d}f\rangle_0}{\|f\|^2} \\
&= \inf_{f\in C_0(V),\ \langle f,1\rangle=0,\ f\not\equiv 0} \frac{\langle \Delta_A f,\, \mathrm{d}f\rangle_0}{\|f\|^2} \\
&= \inf_{f\in C_0(V),\ \langle f,1\rangle=0,\ f\not\equiv 0} \frac{\displaystyle\sum_{x\sim y}(f(x)-f(y))^2}{\displaystyle\sum_{x\in V} m(x) f^2(x)},
\end{aligned}$$

where we have used the following definitions:

$$\begin{aligned}
\langle f, g\rangle_0 &\triangleq \sum_{u\in V} f(u)g(u), \text{ and} \\
\|f\|^2 &\triangleq \sum_{u\in V} m(u) f^2(u).
\end{aligned}$$

Now suppose S achieves the Cheeger constant $h_P(G)$. Set

$$f(x) \triangleq \begin{cases} \frac{1}{\mathrm{vol}(S)} & \text{if} \quad x\in S, \\ -\frac{1}{\mathrm{vol}(\overline{S})} & \text{if} \quad x\notin S. \end{cases}$$

Then

$$\begin{aligned}
\langle f, 1\rangle &= \sum_{x\in V} f(x)\cdot 1\cdot m(x) \\
&= \sum_{x\in S} \frac{1}{\mathrm{vol}(S)} m(x) + \sum_{x\in \overline{S}} \left(-\frac{1}{\mathrm{vol}(\overline{S})}\right) m(x) \\
&= \frac{\mathrm{vol}(S)}{\mathrm{vol}(S)} - \frac{\mathrm{vol}(\overline{S})}{\mathrm{vol}(\overline{S})} \\
&= 0.
\end{aligned}$$

We also have

$$\begin{aligned}\|f\|^2 &= \sum_{x\in V} f^2(x)m(x)\\ &= \sum_{x\in S}\frac{m(x)}{\text{vol}(S)^2} + \sum_{x\in \overline{S}}\frac{m(x)}{\text{vol}(\overline{S})^2}\\ &= \frac{1}{\text{vol}(S)} + \frac{1}{\text{vol}(\overline{S})}.\end{aligned}$$

Moreover,

$$\begin{aligned}\langle\, \mathrm{d}f,\, \mathrm{d}f\rangle_0 &= \langle \Delta_A f, f\rangle_0\\ &= \sum_{x\sim y}(f(x)-f(y))^2\\ &= \sum_{\substack{x\sim y\\ x,y\in S}}(f(x)-f(y))^2 + \sum_{\substack{x\sim y\\ x\in S, y\in\overline{S}}}(f(x)-f(y))^2 + \sum_{\substack{x\sim y\\ x,y\in \overline{S}}}(f(x)-f(y))^2\\ &= 0 + |\partial S|\left(\frac{1}{\text{vol}(S)} + \frac{1}{\text{vol}(\overline{S})}\right)^2 + 0.\end{aligned}$$

Now, using the inequality $\frac{a+b}{ab} \le \frac{2}{\min(a,b)}$ for $a, b > 0$, we find

$$\begin{aligned}\lambda_2^{(P)} &\le \frac{|\partial S|\cdot\left(\frac{1}{\text{vol}(S)} + \frac{1}{\text{vol}(\overline{S})}\right)^2}{\frac{1}{\text{vol}(S)} + \frac{1}{\text{vol}(\overline{S})}}\\ &= |\partial S|\left(\frac{1}{\text{vol}(S)} + \frac{1}{\text{vol}(\overline{S})}\right)\\ &\le \frac{2|\partial S|}{\min(\text{vol}(S), \text{vol}(\overline{S}))}\\ &= 2h_P(G).\end{aligned}$$

The last equality is derived by the assumption that S achieves $h_P(G)$.

□

2 Isospectrality

Definition 2.1. Let $G = (V, E)$. Define Spec(G, A), Spec(G, Δ_A) and Spec(G, Δ_P) as the sets of spectra (eigenvalues) of A, Δ_A and Δ_P of G, respectively. For given G_1 and G_2, if Spec$(G_1, A) =$ Spec(G_2, A), then G_1 and G_2 are said to be **cospectral**. If Spec$(G_1, \Delta_A) =$ Spec(G_2, Δ_A) or Spec$(G_1, \Delta_P) =$ Spec(G_2, Δ_P), then G_1 and G_2 are said to be **isospectral**.

For cospectrality and isospectrality of graphs, see [6], [1], [14] and [7]. For more on the isospectral inequalities, see [9], [10] and chapter 2. of [3, Chap. 2].

Example 2.2. (Fisher 1966 [6], Baker 1966 [1]). Figure 3 shows two cospectral graphs, however they are not isomorphic.

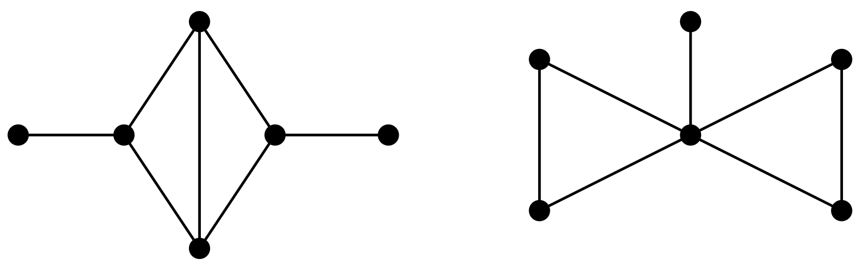


Figure 3: Cospectral but not isomorphic graphs

Example 2.3. (Fuji-Katsuda 1999 [7], Tan 1998 [14]). Figure 4 is an example of two Δ_A-isospectral graphs.

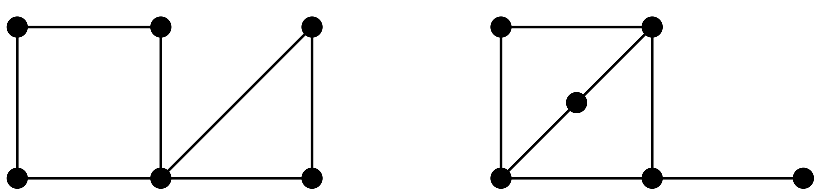


Figure 4: Δ_A-isospectral graphs.

Example 2.4. (Fuji-Katsuda 1999 [7], Tan 1998 [14]). Figure 5 shows an example of two Δ_P-isospectral graphs.

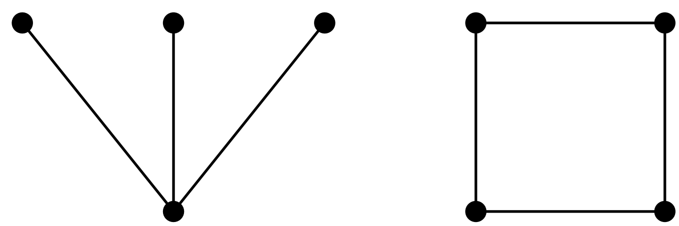


Figure 5: Δ_P-isospectral graphs.

3 Discrete Laplacian Eigenvalue Problems

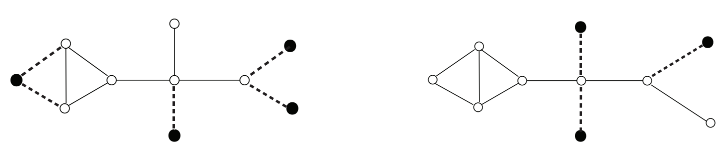
Definition 3.1. $G = (V, E)$ is said to have **boundary** $\partial G = (\partial V, \partial E)$ if the following two conditions are met:

1. $$\begin{cases} V = \overset{o}{V} \bigcup \partial V & \overset{o}{V} \bigcap \partial V = \emptyset; \\ E = \overset{o}{E} \bigcup \partial E & \overset{o}{E} \bigcap \partial E = \emptyset. \end{cases}$$

2. For each $e = (x, y) \in E$, $x, y \in V$,

$$e \in \overset{o}{E} \iff x, y \in V;$$
$$e \in \partial E \iff x \in \overset{o}{V},\ y \in \partial V \text{ or } x \in \partial V,\ y \in \overset{o}{V}\ .$$

Example 3.2. Note that there exists some arbitrariness of ∂G. In Figure 6, the open nodes are in $\overset{o}{V}$, the closed nodes in ∂V, the solid lines in E, and the dashed lines in ∂E.



(a) (b)

Figure 6: The arbitrariness of ∂G.

Consider the discrete Laplacian eigenvalue problem, defined as

$$\begin{cases} \Delta_P u = \mu^{(P)} u & \text{in } \overset{o}{V}, \\ u = 0 & \text{on } \partial V. \end{cases} \quad \text{Dirichlet-Laplacian } \left(\text{D-L}^{(P)}\right);$$

$$\begin{cases} \Delta_P u = \nu^{(P)} u & \text{in } \overset{o}{V}, \\ du = 0 & \text{on } \partial E. \end{cases} \quad \text{Neumann-Laplacian } \left(\text{N-L}^{(P)}\right).$$

Of course, we can define the discrete Laplacian eigenvalue problem with ΔA. We can order these eigenvalues as

$$D - L^{(A)} : \mu_1^{(A)} \le \mu_2^{(A)} \le \ldots \le \mu_k^{(A)}, \quad k = \#(\overset{o}{V}).$$

$$D - L^{(P)} : \mu_1^{(P)} \le \mu_2^{(P)} \le \ldots \le \mu_k^{(P)}, \quad k = \#(\overset{o}{V}).$$

We have the following theorem.

Theorem 3.3. *$\mu_1^{(P)} > 0$, and $\mu_1^{(A)} > 0$, both with multiplicity 1. And there exist $\varphi_1^{(P)}(x) > 0$ and $\varphi_1^{(A)}(x) > 0$ for all $x \in \overset{o}{V}$.*

Proof. The proof is essentially the same as the continuum case, i.e., use the discrete Green's identity with the boundary condition. For example, let $\varphi_1^{(P)}$ be an eigenfunction for $\mu_1^{(P)}$ with $\|\varphi_1^{(P)}\| = 1$, then using the discrete Green's identity,

$$\mu_1^{(P)} = \left\langle \Delta\varphi_1^{(P)}, \varphi_1^{(P)} \right\rangle = \left\langle \mathrm{d}\varphi_1^{(P)}, \mathrm{d}\varphi_1^{(P)} \right\rangle = \| \mathrm{d}\varphi_1^{(P)} \|^2 \ge 0.$$

If $\mathrm{d}\varphi_1^{(P)} \equiv 0$, then $\varphi_1^{(P)} \equiv$ const on V. But $\varphi_1^{(P)} = 0$ on ∂V forces us to have $\varphi_1^{(P)} \equiv 0$. So $\mu_1^{(P)} \neq 0$. Therefore, $\mu_1^{(P)} > 0$.

□

Theorem 3.4 (The discrete Faber-Krahn inequality)**.** *see [9], [10].*

If $\#(\overset{o}{E} \cup \partial E) = n$, then

$$\mu_1^{(P)}(L_n) \le \mu_1^{(P)}(G)$$

where equality holds if and only if $G = L_n$. L_n a graph is shown in Figure 7. There are n nodes. Only the last to the right belongs to ∂V, the rest belong to $\overset{o}{V}$, and the same is true for the edges.

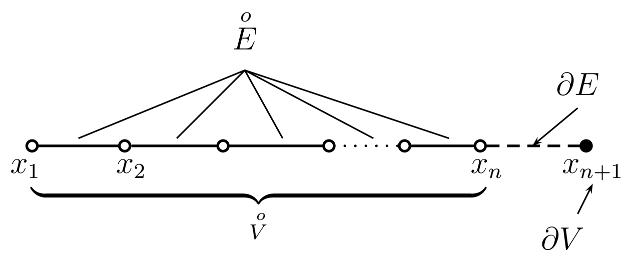


Figure 7: Graph L_n.

References

[1] G. A. BAKER: "Drum shapes and isospectral graphs", *J. Math. Phys.*, vol. 7, pp.2238-2242, 1966.

[2] M. BELKIN AND P. NIYOGI: "Laplacian eigenmaps for dimensionality reduction and data representation", *Neural Comp.*, vol. 15, no. 6, pp. 1373-1396, 2003.

[3] F. R. K. CHUNG: *Spectral Graph Theory*, AMS, 1997.

[4] *Applied and Computational Harmonic Analysis, Special Issue on Diffusion Maps and Wavelets*, (D. Donoho and C. Chui, Eds.), vol. 21, no. 1, pp. 1-144, 2006.

[5] J. DODZIUK AND W. S. KENDALL: "Combinatorial Laplacians and isoperimetric inequality", *From Local Times to Global Geometry, Control and Physics* (K. D. Elworthy, ed.), Pitman Res. Notes Math. Ser., vol. 150, pp. 68-74, Longman Sci. Tech., 1986.

[6] M. E. FISHER: "On hearing the shape of a drum", *J. Comb. Theory*, vol. 1, pp. 105-125, 1966.

[7] H. FUJII AND A. KATSUDA: "Isospectral graphs and isoperimetric constants", *Discrete Math.*, vol. 207, no. 1-3, pp. 33-52, 1999.

[8] L. GRADY AND E. L. SCHWARTZ: "Isoperimetric graph partitioning for image segmentation", *IEEE Trans. on Pattern Anal. Machine Intell.*, vol. 28, no. 3, pp. 469-475, 2006.

[9] A. KATSUDA AND H. URAKAWA: "The first eigenvalue of the discrete Dirichlet problem for a graph", *J. Combin. Math. Combin. Comput.* vol. 27, pp. 217-225, 1998.

[10] A. KATSUDA AND H. URAKAWA: "The Faber-Krahn type isoperimetric inequalities for a graph", *Tohoku Math. J.* (2) vol. 51, no. 2, pp. 267-281, 1999.

[11] R. KANNAN, S. VEMPALA, AND A. VETTA: "On clusterings: good, bad and spectral", *J. Assoc. Comput. Mach.* vol. 51 no. 3, pp. 497-515, 2004.

[12] B. MOHAR: "Isoperimetric numbers of graphs", *J. Comb. Theory*, Ser. B, vol. 47, no. 3, pp. 274-291, 1989.

[13] J. SHI AND J. MALIK: "Normalized cuts and image segmentation", *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 22, no. 8, pp. 888-905, 2000.

[14] J. TAN: "On isospectral graphs", *Interdisciplinary Information Sciences*, vol. 4, no. 2, pp. 117-124, 1998.

[15] J. TAN: "On Cheeger inequalities of a graph", *Discrete Math.*, vol. 269, no. 1-3, pp. 315-323, 2003.

[16] H. URAKAWA: "Eigenvalue comparison theorems of the discrete Laplacians for a graph", *Geom. Dedicata*, vol. 74 no. 1, pp. 95-112, 1999.

[17] H. URAKAWA: "Spectral geometry and graph theory", *Ouyou Suuri* (Bulletin of the Japan Society of Industrial and Applied Mathematics), vol. 12, no. 1, pp.29-45, 2002.

[18] Y. WEISS: "Segmentation using eigenvectors: a unifying view", in *Proc. the Seventh IEEE International Conference on Computer Vision*, vol. 2, pp. 975-982, 1999.