MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 3: Problems of Spectral Geometry (Summary)

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In this lecture, we will review the famous problems of spectral geometry some of which are still open. The early history of spectral geometry can be found in [1] and [2].

1 On the First Few Eigenvalues

To get more details about the problems in this section, please read [3], [4] and references therein.

Let $\{\lambda_n\}$ denote the Dirichlet-Laplacian eigenvalues that satisfy the following equations:

$$\begin{cases} -\Delta\varphi_n = \lambda_n\varphi_n & \text{in } \Omega \\ \varphi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Furthermore, let $\{\nu_n\}$ denote the Neumann-Laplacian eigenvalues that satisfy the following equations:

$$\begin{cases} -\Delta\psi_n = \nu_n\psi_n & \text{in } \Omega \\ \dfrac{\partial\psi_n}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

As we show in the next lecture, all the eigenvalues above are nonnegative and can be ordered in the increasing order. More precisely, we have: $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and $0 = \nu_1 < \nu_1 \leq \nu_2 \leq \cdots$.

1.1 The Rayleigh Conjecture (1877)

The Rayleigh Conjecture was proven independently by Faber & Krahn in 1923 and claims the following:

Let $\mathcal{B}_1^d$ be the unit ball in $\mathbb{R}^d$, and consider a domain $\Omega \subset \mathbb{R}^d$ with $|\Omega| < \infty$. Furthermore, assume that $|\Omega| = |\mathcal{B}_1^d|$. Then

$$\lambda_1(\Omega) \geq \lambda_1(\mathcal{B}_1^d), \quad \text{with equality if and only if } \Omega \text{ is congruent to } \mathcal{B}_1^d.$$

NOTE: See Appendix for more about the unit ball and sphere in $\mathbb{R}^d$.

1.2 The Payne-Póyla-Weinberger conjecture (1956)

This conjecture was proven by Ashbaugh-Benguria in 1991. Under the same assumption of the Rayleigh conjecture, it states the following:

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\mathcal{B}_1^d)}{\lambda_1(\mathcal{B}_1^d)}, \quad \text{with equality if and only if } \Omega \text{ is congruent to } \mathcal{B}_1^d.$$

A more general statement is the following:

$$\frac{\lambda_{m+1}(\Omega)}{\lambda_m(\Omega)} < \frac{\lambda_2(\mathcal{B}_1^d)}{\lambda_1(\mathcal{B}_1^d)} \quad \text{for } m \in \mathbb{N}.$$

The cases with $m = 1, 2, 3$ were proven by Ashbaugh-Benguria in 1991-93. For $m \geq 4$ the problem is still open.

1.3 The Payne conjecture (1955)

The Payne conjecture was proven by L. Friedlander in 1991 and states as follows: Assume that $\partial\Omega$ has C^1 smoothness (i.e., no corners), then

$$\nu_{k+1} \leq \lambda_k, \quad k = 1, 2, \cdots$$

where ν_k and λ_k are the kth Neumann (2) and Dirichlet (1) Laplacian eigenvalues, respectively.

1.4 Another Payne-Póyla-Weinberger conjecture

This conjecture is still open. Assume $\Omega \subset \mathbb{R}^2$ and $|\Omega| = |\mathcal{B}_1^2|$, then

$$\frac{\lambda_2(\Omega) + \lambda_3(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\mathcal{B}_1^2) + \lambda_3(\mathcal{B}_1^2)}{\lambda_1(\mathcal{B}_1^2)}.$$

This can be generalized to $\Omega \subset \mathbb{R}^d$ as follows:

$$\frac{\lambda_2(\Omega) + \cdots + \lambda_{d+1}(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\mathcal{B}_1^d) + \cdots + \lambda_{d+1}(\mathcal{B}_1^d)}{\lambda_1(\mathcal{B}_1^d)}.$$

1.5 Yet another open problem

Find the optimal upper bound for

$$\frac{\lambda_3(\Omega)}{\lambda_1(\Omega)}, \quad \text{with } \Omega \subset \mathbb{R}^d, \;\; |\Omega| < \infty,$$

i.e., find

$$\arg \sup_{\substack{\Omega \subset \mathbb{R}^d \\ |\Omega| < \infty}} \frac{\lambda_3(\Omega)}{\lambda_1(\Omega)}$$

So far, for $d = 2$, the largest λ_3/λ_1 found equals $35/11 \approx 3.2$ where Ω is a rectangle with sides $\sqrt{8}$ and $\sqrt{3}$. The best upper bound found so far is approximately 3.831.

2 Isospectral Problems

For the review on isospectral problems, see e.g., [5].

Definition 2.1. Two domains $\Omega, \Omega' \subset \mathbb{R}^d$, with $|\Omega| < \infty, |\Omega'| < \infty$, are said to be **isospectral** if $\lambda_k(\Omega) = \lambda_k(\Omega')$ holds for any $k \in \mathbb{N}$ or $\nu_k(\Omega) = \nu_k(\Omega')$ holds for any $k \in \mathbb{N}$.

Question: Given $\Omega \subset \mathbb{R}^d$ and $|\Omega| < \infty$,

$$\text{Is } \{\Omega' \subset \mathbb{R}^d : |\Omega'| < \infty \text{ and isospectral to } \Omega\} \quad \text{a compact set?}$$

For $d = 2$, this is true. But for $d \geq 3$, this is an open problem.

Examples of isospectral domains:
In $\mathbb{R}^d$, these were given for $d \geq 4$ by Urakawa (1982), and for $d = 2, 3$, examples were given by Gordon-Webb-Wolpert (1991-92). In both cases Ω, Ω' have corners (or polygons). The problem is open for a smooth domain.

3 Appendix

Some useful materials are given below.

- Let $\mathcal{B}_1^d$ be the unit ball in $\mathbb{R}^d$. Then the volume of the unit ball is given by $|\mathcal{B}_1^d| = \frac{2\pi^{d/2}}{d\Gamma(d/2)}$. And the surface area of the unit ball is given by $|S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. For the derivations, see [6, Sec. 0B].

- The Gamma Function $\Gamma(x)$ is defined as:

$$\Gamma(x) \triangleq \int_0^\infty s^{x-1} \mathrm{e}^{-s} \, \mathrm{d}s, \qquad 0 < x < \infty.$$

- Some properties of the Gamma Function:

  1. $\Gamma(1) = 1$
  2. $\Gamma(x+1) = x\Gamma(x)$
  3. $\Gamma(n+1) = n!$, $n = 0, 1, 2, \cdots$
  4. $\Gamma(1/2) = \sqrt{\pi}$

  For the review of Gamma Function, see e.g., [7, Appendix 3]

References

[1] M. KAC: "Can one hear the shape of a drum?" *Amer. Math. Monthly*, vol. 73, no. 4, part 2, pp. 1-23, 1966.

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[4] I. CHAVEL: *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

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[6] G. B. FOLLAND: *Introduction to Partial Differential Equations*, 2nd Ed., Princeton Univ. Press, 1995.

[7] G. B. FOLLAND: *Fourier Analysis and Its Applications*, Brooks/Cole Publishing Company, 1992.