MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 5: Diffusions on and Vibrations of a Membrane in 2D/3D–II 2D Disk

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1 Vibrations of a 2D Drumhead

The basic references for this lecture are the texts by Strauss [1, Sec. 10.2], and Courant and Hilbert [2, Sec. V.5.5].

1.1 Dirichlet Boundary Conditions

Given a disk domain $\Omega = \{(x, y) \,|\, x^2 + y^2 < a^2\} \subset \mathbb{R}^2$, consider the following Dirichlet boundary condition (BC) problem

$$\begin{array}{lll} \text{(DE):} & u_{tt} = c^2 \Delta u = c^2(u_{xx} + u_{yy}) & \text{in } \Omega \\ \text{(BC):} & u = 0 & \text{on } \partial\Omega \\ \text{(IC):} & u(\boldsymbol{x}, 0) = f(\boldsymbol{x}),\ u_t(\boldsymbol{x}, 0) = g(\boldsymbol{x}). & \end{array} \tag{1}$$

Recall in polar coordinates that $(x, y) = (r\cos\theta, r\sin\theta)$ for $0 \le r < a$ and $-\pi \le \theta < \pi$. In 2D the Laplace operator can be expressed as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

As before, assume our solution admits a separation of variables

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t).$$

This results in

$$\frac{T''}{c^2 T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda.$$

Let $\frac{\Theta''}{\Theta} = -\gamma$, where γ is a constant. We obtain the following three ODEs:

(i) $T'' + c^2\lambda T = 0$;

(ii) $\Theta'' + \gamma\Theta = 0$;

(iii) $R'' + \frac{1}{r}R' + (\lambda - \frac{\gamma}{r^2})R = 0$.

We begin by solving (ii) first (we will save (i) for last because it involves the IC). Note that we have the periodic BC $\Theta(\theta + \pi) = \Theta(\theta), \forall\theta$. The characteristic equation for (ii) is $p^2 + \gamma = 0$ or $p = \pm\sqrt{-\gamma}$. This will generate a feasible solution only when $\gamma > 0$. Thus,

$$\Theta(\theta) = A\cos\sqrt{\gamma}\theta + B\sin\sqrt{\gamma}\theta$$

which implies that $\sqrt{\gamma} = n \in \mathbb{N}$ due to the 2π-periodicity. Finally, we arrive at

$$\Theta(\theta) = \begin{cases} \frac{1}{2}A_0, & n = 0; \\ A_n\cos n\theta + B_n\sin n\theta, & n \in \mathbb{N}. \end{cases}$$

where A_0, A_n and B_n are appropriate constants.

Next, we solve (iii) for $0 \le r < a$. We realistically impose that at the origin $R(0)$ be finite. Also, the Dirichlet BC requires $R(a) = 0$. From the last lecture we know that the Dirichlet-Laplacian eigenvalues are positive, i.e., $\lambda > 0$.

Now let us use the change of variable: $\rho = \sqrt{\lambda}\,r$, which results in

$$R_r = R_\rho\frac{\mathrm{d}\rho}{\mathrm{d}r} = \sqrt{\lambda}R_\rho, \qquad R_{rr} = \lambda R_{\rho\rho}.$$

Now, (iii) can be rewritten as

$$R_{\rho\rho} + \frac{1}{\rho} R_\rho + (1 - \frac{n^2}{\rho^2}) R = 0 \tag{2}$$

This is **Bessel's differential equation of order n**. For more about Bessel functions see [4], [5], [6, Ch. 5], [7, Ch. 8], [8, Ch. 5,6].

1.2 Solutions of Bessel's Differential Equation

At $\rho = 0$ the coefficients of the R_ρ and R terms in (2) blow up. So instead, multiply first by ρ^2:

$$\rho^2 R_{\rho\rho} + \rho R_\rho + (\rho - n^2) R = 0.$$

Now at $\rho = 0$ the coefficients of the R_ρ and R do not blow up.

This $\rho = 0$ is called a *singular point*. However, it is not really a "bad" singular point since the denominators of the coefficients of R_ρ and R in (2) approach zero *polynomially* as $\rho \to 0$. In other words, the pole does not have *exponential* growth.

More precisely, if

$$\lim_{\rho \to 0} \rho \cdot \frac{\text{coef}(R_\rho)}{\text{coef}(R_{\rho\rho})} < \infty$$

and

$$\lim_{\rho \to 0} \rho^2 \cdot \frac{\text{coef}(R)}{\text{coef}(R_{\rho\rho})} < \infty,$$

then such a singular point is called a *regular singular point*. In this case we seek a solution in the form of the following series (see [9, Part I, Sec. 5.2], [3, Ch. VII] for more about the method of series solution for ODEs, as well as regular singular points)

$$R(\rho) = \rho^\alpha \sum_{k=0}^{\infty} a_k \rho^k, \qquad a_0 \neq 0$$

for some α and $\{a_k\}_{k=0}^{\infty}$ to be determined. By plugging this into (2) we get

$$\rho^\alpha \sum_{k=0}^{\infty} \Big[(\alpha + k)(\alpha + k - 1) a_k \rho^{k-2} + (\alpha + k) a_k \rho^{k-2} + a_k \rho^k - n^2 a_k \rho^{k-2} \Big] = 0.$$

Now rewriting $\sum_{k=0}^{\infty} a_k \rho^k = \sum_{k=2}^{\infty} a_{k-2}\rho^{k-2}$ we have

$$\begin{array}{ll} [\alpha(\alpha-1)+\alpha-n^2]a_0 = 0, & \text{when } k=0; \\ [(\alpha+1)\alpha+\alpha+1-n^2]a_1 = 0, & \text{when } k=1; \\ [(\alpha+k)(\alpha+k-1)+\alpha+k-n^2]a_k + a_{k-2} = 0, & \text{when } k\geq 2. \end{array}$$

So for $a_0 \neq 0$, from the Case $k=0$, we have that $\alpha^2 - n^2 = 0$, i.e., $\alpha = \pm n$.

Consider $\alpha = n$. Substituting in the Case $k=1$ above, we get

$$[n^2+n+n+1-n^2]a_1 = 0 \;\Rightarrow\; [2n+1]a_1 = 0.$$

Thus, $a_1 = 0$. Note that when $k \geq 2$ above we see that the square bracket factor simplifies to $[(\alpha+k)^2 - n^2]$. Therefore,

$$a_k = \frac{-a_{k-2}}{(n+k)^2 - n^2}, \qquad k = 2, 3, \ldots.$$

Since $a_1 = 0$, we know that $a_k = 0$ whenever k is odd. Now choose $a_0 = 1/(2^n n!)$. Then we get

$$\begin{aligned} R(\rho) &= \frac{\rho^n}{2^n n!}\Big[1 - \frac{\rho^2}{2^2(n+1)} + \frac{\rho^4}{2!2^4(n+1)(n+2)} - \cdots\Big] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\Big(\frac{\rho}{2}\Big)^{n+2k} \\ &\equiv J_n(\rho) \end{aligned}$$

where $J_n(\cdot)$ is defined as the *Bessel function of the first kind of order* n, $n = 0, 1, 2, \cdots$.

Asymptotically, $J_n(\rho) \sim \sqrt{\frac{2}{\pi\rho}}\cos(\rho - \frac{\pi}{4} - \frac{n\pi}{2}) + O(\rho^{-3/2})$ as $\rho \to \infty$. Here, the $\sqrt{\frac{2}{\pi\rho}}$ factor represents a decay.

1.3 The Eigenfunction Expansion

Now we know $R(\rho) = \kappa J_n(\rho)$, where κ is an arbitrary constant. With $\rho = \sqrt{\lambda}r$, we have the solution of the form (here we just consider the stationary part of the solution)

$$\underbrace{J_n(\sqrt{\lambda}r)}_{radial}\underbrace{(A_n\cos n\theta + B_n\sin n\theta)}_{angular}$$

which separates the radial and angular components.

The Dirichlet BC dictates at $r = a$ that the radial component $J_n(\sqrt{\lambda}a) = 0$. Thus $\sqrt{\lambda}a$ is a zero of $J_n(x)$.

Note that each $J_n(x)$ has an infinite number of positive zeros, and that they are *not* regularly or periodically spaced. This can be seen more clearly in Figure 1. For a table of the zeros of Bessel functions, see [4] and [10, Ch. 9].

In general, let $j_{n,m}$ be the mth positive zero of $J_n(x)$ for $n = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$. Then the zeros obey the following orders:

$$0 < j_{n,1} < j_{n,2} < \cdots \qquad \text{and} \qquad j_{n,m} < j_{n+1,m} < j_{n,m+1},$$

where in our scenario $j_{n,m} = \sqrt{\lambda_{n,m}}\, a$, i.e., $\lambda_{n,m} = \left(\frac{j_{n,m}}{a}\right)^2$.

Finally, we have the solution modulo IC as

$$\begin{aligned} u(r,\theta,t) &= \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0,m}}\, r)\Big(C_{0,m}\cos(\sqrt{\lambda_{0,m}}ct) + D_{0,m}\sin(\sqrt{\lambda_{0,m}}ct)\Big) \\ &\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{n,m}}\, r)\Big(A_{n,m}\cos(n\theta) + B_{n,m}\sin(n\theta)\Big) \\ &\qquad \cdot\Big(C_{n,m}\cos(\sqrt{\lambda_{n,m}}ct) + D_{n,m}\sin(\sqrt{\lambda_{n,m}}ct)\Big) \end{aligned} \tag{3}$$

Note that the factors $J_0(\sqrt{\lambda_{0,m}}\, r)$ and $J_n(\sqrt{\lambda_{n,m}}\, r)\Big(A_{n,m}\cos(n\theta) + B_{n,m}\sin(n\theta)\Big)$ represent eigenfunctions of the Laplacian operator.

In order to determine the coefficients $A_{n,m}, B_{n,m}$ for $n, m = 1, 2, \ldots$, and $C_{n,m}, D_{n,m}$ for $n = 0, 1, \ldots,\ m = 1, 2, \ldots$ we use the ICs: $u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), u_t(\boldsymbol{x}, 0) = g(\boldsymbol{x})$.

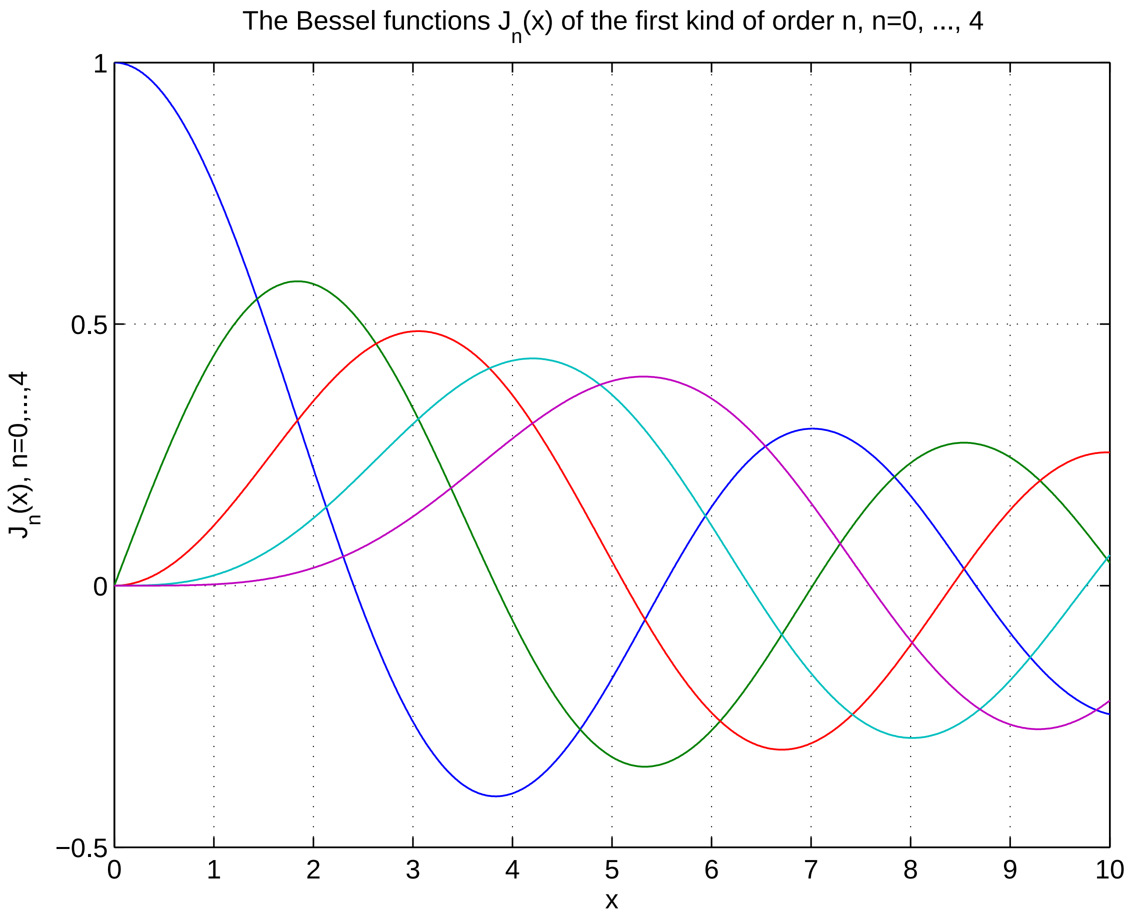

Figure 1: Plots of the Bessel functions $J_n(x)$ of the first kind of order $n = 0, \cdots, 4$.

Write $\sqrt{\lambda_{n,m}} = \beta_{n,m} = j_{n,m}/a$ for simplicity. Then at $t = 0$, (3) reduces to

$$
\begin{aligned}
u(r,\theta,0) &= f(r,\theta) \\
&= \sum_{m=1}^{\infty} C_{0,m} J_0(\beta_{0,m}\, r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} J_n(\beta_{n,m}\, r) \\
&\qquad \cdot \Big(A_{n,m}\cos(n\theta) + B_{n,m}\sin(n\theta)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
u_t(r,\theta,0) &= g(r,\theta) \\
&= \sum_{m=1}^{\infty} c\beta_{0,m} D_{0,m} J_0(\beta_{0,m}\, r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\beta_{n,m} D_{n,m} J_n(\beta_{n,m}\, r) \\
&\qquad \cdot\Big(A_{n,m}\cos(n\theta) + B_{n,m}\sin(n\theta)\Big).
\end{aligned}
$$

Let us exploit the orthogonality of $\varphi_{n,m,l}(r,\theta) = J_n(\beta_{n,m} r)\binom{\cos}{\sin}(m\theta)$, $l = 0$ for $\cos(m,\theta)$ and $l = 1$ for $\sin(m,\theta)$.

Define the inner product for any $p, q \in L^2(\Omega)$ as

$$
\begin{aligned}
\langle p, q\rangle &:= \int_\Omega p(\boldsymbol{x})\, \overline{q(\boldsymbol{x})}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y} \\
&= \int_{-\pi}^{\pi}\int_0^a p(r,\theta)\, \overline{q(r,\theta)}\, r\ \mathrm{d}r\ \mathrm{d}\theta
\end{aligned}
$$

where the functional forms of p and q are adjusted with respect to the respective coordinate system (i.e., Cartesian, polar, etc.).

For fixed m it is easy to see that

$$
\int_0^a J_n(\beta_{n,m}\, r) J_{n'}(\beta_{n',m}\, r)\, r \mathrm{d}r = \kappa\delta_{n,n'}
$$

for some normalization constant κ. Here, δ is the Kronecker delta, and different n's correspond to different "frequencies." Also, for fixed n (i.e., for a fixed "frequency") we have that

$$
\int_0^a J_n(\beta_{n,m}\, r) J_n(\beta_{n,m'}\, r)\, r \mathrm{d}r = \tilde{\kappa}\delta_{m,m'}
$$

for some normalization constant $\tilde{\kappa}$. Furthermore, $\cos n\theta$ and $\sin n\theta$ are mutually perpendicular for $-\pi \leq \theta < \pi$. In summary, we have

$$
\langle \varphi_{n,m,l}, \varphi_{n,m,l'}\rangle = \text{const} \cdot \delta_{n,n'}\, \delta_{m,m'}\, \delta_{l,l'}.
$$

So, we can do the Fourier series expansion of $f(r,\theta)$ and $g(r,\theta)$, and find the matching coefficients $\{C_{o,m}, C_{n,m}A_{n,m}, C_{n,m}B_{n,m}, D_{0,m}, D_{n,m}A_{n,m}, D_{n,m}B_{n,m}\}$ as

$$\begin{aligned}
C_{0,m} &= \frac{1}{2\pi a_{0,m}} \int_{-\pi}^{\pi} \int_{0}^{a} f(r,\theta) J_0(\beta_{0,m} r)\, r \ \mathrm{d}r \ \mathrm{d}\theta \\
C_{n,m}A_{n,m} &= \frac{1}{\pi a_{n,m}} \int_{-\pi}^{\pi} \int_{0}^{a} f(r,\theta) J_n(\beta_{n,m} r) \cos n\theta\, r \ \mathrm{d}r \ \mathrm{d}\theta \\
C_{n,m}B_{n,m} &= \frac{1}{\pi a_{n,m}} \int_{-\pi}^{\pi} \int_{0}^{a} f(r,\theta) J_n(\beta_{n,m} r) \sin n\theta\, r \ \mathrm{d}r \ \mathrm{d}\theta
\end{aligned}$$

and

$$\begin{aligned}
c\beta_{0,m}D_{0,m} &= \frac{1}{2\pi a_{0,m}} \int_{-\pi}^{\pi} \int_{0}^{a} g(r,\theta) J_0(\beta_{0,m} r)\, r \ \mathrm{d}r \ \mathrm{d}\theta \\
c\beta_{n,m}D_{n,m}A_{n,m} &= \frac{1}{\pi a_{n,m}} \int_{-\pi}^{\pi} \int_{0}^{a} g(r,\theta) J_n(\beta_{n,m} r) \cos n\theta\, r \ \mathrm{d}r \ \mathrm{d}\theta \\
c\beta_{n,m}D_{n,m}B_{n,m} &= \frac{1}{\pi a_{n,m}} \int_{-\pi}^{\pi} \int_{0}^{a} g(r,\theta) J_n(\beta_{n,m} r) \sin n\theta\, r \ \mathrm{d}r \ \mathrm{d}\theta,
\end{aligned}$$

where the normalization constant $a_{n,m}$ is defined as

$$\begin{aligned}
a_{n,m} &\triangleq \int_{0}^{a} \Big[J_n(\beta_{n,m} r)\Big]^2 r \ \mathrm{d}r \ \mathrm{d}\theta \\
&= \frac{1}{2}\, a^2 \Big[J_n'(\beta_{n,m} a)\Big]^2 \\
&= \frac{1}{2}\, a^2 \Big[J_{n\pm 1}(\beta_{n,m} a)\Big]^2.
\end{aligned}$$

Example: Beat this drum with a stick in the middle (i.e., at $r = 0$) at $t = 0$, with ICs $u(r, \theta, 0) = f(r, \theta) = 0$, and $u_t(r, \theta, 0) = g(r, \theta) = g(r)$. Since $f(r, \theta) = 0$, it must be that $C_{n,m} = 0, \forall n, m$. Moreover, in examining $g(r)$ we see that $n = 0$ is the only remaining term, i.e., $D_{n,m}A_{n,m} = D_{n,m}B_{n,m} = 0$, for $n \geq 1$. So we have

$$u(r, \theta, t) = u(r, t) = \sum_{m=1}^{\infty} D_{0,m} J_0(\beta_{0,m}\, r) \sin(\beta_{0,m} ct)$$

where

$$
\begin{aligned}
D_{0,m} &= \frac{1}{c\beta_{0,m}a_{0,m}}\int_0^a g(r)J_0(\beta_{0,m}r)\,r\mathrm{d}r \\
&= \frac{\int_0^a g(r)J_0(\beta_{0,m}r)\,r\mathrm{d}r}{\frac{1}{2}\,a^2c\beta_{0,m}\Big[J_1(\beta_{0,m}a)\Big]^2}.
\end{aligned}
$$

Therefore, the lowest "note" (or "frequency") we hear is the fundamental frequency corresponding to $\beta_{0,1}c = \frac{1}{a}j_{0,1}c \approx \frac{2.40483c}{a}$. Compare this with a 1-D string of length a (with density ρ and under tension $\mathcal{T}$). There, the lowest note would be $\frac{\pi c}{a} \approx \frac{3.141592c}{a}$, where $c = \sqrt{\frac{\mathcal{T}}{\rho}}$. Thus, if the c parameter is the same in both of the 1-D and 2-D cases, we see that the 2-D drum can support a lower note. In the next lecture we will see, curiously, under the same conditions that the lowest note of a 3-D ball is actually equal to that of the 1-D string! For the science of real drums and percussion, see [11].

2 Neumann BC

Denote $\lambda^{(N)}$ as Neumann-Laplacian eigenvalues, and $\lambda^{(D)}$ as Dirichlet-Laplacian eigenvalues. Let

$$
\psi(r,\theta) = \psi_n(r,\theta) = J_n(\sqrt{\lambda^{(N)}}r)\Big(A_n\cos(n\theta) + B_n\sin(n\theta)\Big).
$$

The Neumann BC specifies

$$
\left.\frac{\partial\psi_n}{\partial r}\right|_{\partial\Omega} = \frac{\partial\psi_n}{\partial r}(a,0) = 0
$$

which implies

$$
\sqrt{\lambda^{(N)}}J_n'(\sqrt{\lambda^{(N)}}a)\Big(A_n\cos(n\theta) + B_n\sin(n\theta)\Big) = 0.
$$

Therefore, either $J_n'(\sqrt{\lambda^{(N)}}a) = 0, n = 0, 1, 2, \ldots$, or $\lambda^{(N)} = 0$.

2.1 Recurrence relations of Bessel Functions

The Bessel functions obey the following relationships (see e.g. [4, Sec. 3.2], [10, Sec. 9.1]):

$$
\begin{aligned}
J_{\nu-1}(x) + J_{\nu+1}(x) &= \frac{2\nu}{x} J_\nu(x) \\
J_{\nu-1}(x) - J_{\nu+1}(x) &= 2J'_\nu(x) \\
J'_\nu(x) &= J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) \\
J'_\nu(x) &= -J_{\nu+1}(x) + \frac{\nu}{x} J_\nu(x).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
J'_0(x) &= -J_1(x) \\
J'_1(x) &= J_0(x) - \frac{J_1(x)}{x} \\
&= \frac{1}{2}\Big(J_0(x) - J_2(x)\Big) \\
J'_2(x) &= \frac{1}{2}\Big(J_1(x) - J_3(x)\Big).
\end{aligned}
$$

Therefore, $\lambda^{(N)}_{0,m} = \lambda^{(D)}_{1,m}, m = 1, 2, \ldots$, and from Figure 1, we have

$$0 = \lambda^{(N)}_{0,1} \;<\; \lambda^{(N)}_{1,1} \;<\; \lambda^{(N)}_{2,1} \;<\; \lambda^{(N)}_{0,2} \;<\; \lambda^{(N)}_{3,1} \;<\; \cdots$$

Also, the Dirichlet BC Laplacian eigenvalue $\lambda^{(D)}_{0,1}$ lies in the interval

$$\lambda^{(D)}_{0,1} \in (\lambda^{(N)}_{1,1}, \lambda^{(N)}_{2,1})$$

and

$$\lambda^{(D)}_{1,2} = \lambda^{(N)}_{0,2}.$$

Thus, Neumann BCs have the potential in general to give lower fundamental frequencies.

In general, we can show that

$$\lambda^{(N)}_{n+1} \;\le\; \lambda^{(D)}_{n}, \quad n = 1, 2, \ldots$$

which is a special case of the Friedlander Theorem discussed in Lecture 3.

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