MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 6: Diffusions on and Vibrations of a Membrane in 2D/3D–III. 3D Ball

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In this lecture, we consider diffusions on and vibrations of a membrane in 3D ball. Basic references for this lecture are [1, Sec. 10.3], [2, Sec. V. 8, V. 9.1, VII. 5] and [3, Sec. 6.3].

Consider a ball Ω of radius a in $\mathbb{R}^3$, i.e., $\Omega = B_a^3(0) = \{(x, y, z) \mid x^2 + y^2 + z^2 < a^2\}$, $\partial\Omega = S_a^2$. The wave equation for $u(\boldsymbol{x}, t)$ on this domain is as follows:

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\boldsymbol{x}, 0) = f(\boldsymbol{x}),\ u_t(\boldsymbol{x}, 0) = g(\boldsymbol{x}) & \text{in } \Omega \end{cases} \tag{1}$$

By using the method of separation of variables, i.e., setting $u(\boldsymbol{x}, t) = T(t) \cdot v(\boldsymbol{x})$, we get Dirichlet-Laplacian eigenvalue problem (see Lecture 4 for more details):

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Now consider the spherical (or polar) coordinates in $\mathbb{R}^3$, which introduce three parameters (r, ϕ, θ) as shown in Fig. 1, satisfying the followings:

$$\begin{cases} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{cases}$$

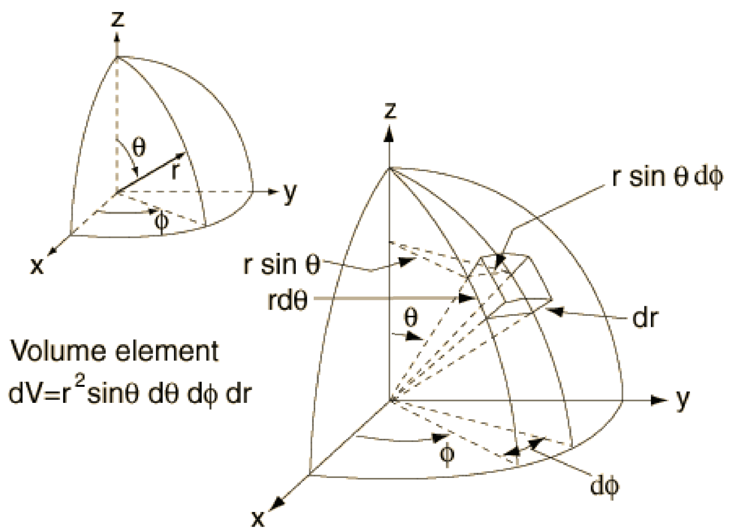

Figure 1: Spherical Coordinates in $\mathbb{R}^3$.

where $0 \le r < a$, $-\pi \le \phi < \pi$, $0 \le \theta \le \pi$.

Therefore, we can write

$$\begin{aligned} 0 &= \Delta v + \lambda v \\ &= v_{rr} + \tfrac{2}{r} v_r + \tfrac{1}{r^2}\left(\tfrac{1}{\sin^2\theta} v_{\phi\phi} + \tfrac{1}{\sin\theta}\big(v_\theta \sin\theta\big)_\theta \right) \end{aligned} \tag{3}$$

We do the following separation of variables $v = R(r) \cdot Y(\theta, \varphi)$ and obtain:

$$0 = \lambda r^2 + \frac{r^2 R_{rr} + 2rR_r}{R} + \frac{\frac{1}{\sin^2\theta} Y_{\phi\phi} + \frac{1}{\sin\theta}\big(Y_\theta \sin\theta\big)_\theta}{Y}$$

Notice that the first two terms in the right-hand side depend on r only and so does the last term in the right-hand side on θ and ϕ only. This means both of them must be constants whose sum is equal to zero, i.e.,

$$\lambda r^2 + \frac{r^2 R_{rr} + 2rR_r}{R} = +\gamma, \qquad \frac{\frac{1}{\sin^2\theta} Y_{\phi\phi} + \frac{1}{\sin\theta}\big(Y_\theta \sin\theta\big)_\theta}{Y} = -\gamma.$$

where γ is a constant.

Then we have

$$R_{rr} + \frac{2}{r} R_r + \big(\lambda - \frac{\gamma}{r^2}\big) R = 0, \quad \text{with } R(0) \text{ is finite and } R(a) = 0. \tag{4}$$

and

$$\frac{1}{\sin^2\theta} Y_{\phi\phi} + \frac{1}{\sin\theta}\big(Y_\theta \sin\theta\big)_\theta + \gamma Y = 0. \tag{5}$$

(4) is similar to Bessel's equation, except the coefficient of R_r term $\frac{2}{r}$, instead of $\frac{1}{r}$.

So, let us use the change of dependent variable:

$$w(r) \triangleq \sqrt{r} R(r), \ \text{ which gives us } R = \frac{1}{\sqrt{r}} w, \quad R_r = -\frac{1}{2} r^{-3/2} w + r^{-1/2} w_r.$$

Then (4)⇒

$$w_{rr} + \frac{1}{r} w_r + \Big(\lambda - \frac{\gamma + \frac{1}{4}}{r^2}\Big) w = 0, \quad \text{with } w(0) \text{ is finite and } w(a) = 0. \tag{6}$$

This is now Bessel's differential equation with $n = \sqrt{\gamma + \frac{1}{4}}$. So,

$$w(r) = J_{\sqrt{\gamma + \frac{1}{4}}}(\sqrt{\lambda} r) \ \text{ or its constant multiple.}$$

⇒

$$R(r) = \frac{J_{\sqrt{\gamma + \frac{1}{4}}}(\sqrt{\lambda} r)}{\sqrt{r}}.$$

$R(a) = 0$ forces us to have $J_{\sqrt{\gamma + \frac{1}{4}}}(\sqrt{\lambda} a) = 0$. We will analyze this later after we proceed a bit more on Y. (By doing so, we get the possible values of γ.)

Notice that the boundary conditions for Y are: $Y(\theta, \phi + 2\pi) = Y(\theta, \phi)$, i.e., Y is 2π periodic in ϕ. And also $Y(0, \phi)$, $Y(\pi, \phi)$ are finite. Such Y satisfying (5) and the above boundary conditions are called *spherical harmonics* (see [4, Sec. 2H], [5, Sec. IV.2], and [6]). Also, see [7], [8], and [9] for computational aspect of spherical harmonics.

To solve (5) with these boundary conditions, we do one step of separation variables as $Y(\theta, \phi) = p(\theta) q(\phi)$ to get

$$\frac{q_{\phi\phi}}{q} + \frac{\sin\theta (p_\theta \sin\theta)_\theta}{p} + \gamma \sin^2\theta = 0.$$

Notice that the first term in the left-hand side depends on ϕ only, so does the last two terms in the left-hand side on θ only. So, both of them must be constant:

$$\frac{q_{\phi\phi}}{q} = -\alpha, \qquad \frac{\sin\theta(p_\theta \sin\theta)_\theta}{p} + \gamma \sin^2\theta = \alpha.$$

where α is a constant.

Now let us solve them one by one.

First:

$$q_{\phi\phi} + \alpha q = 0, \qquad \text{with } q \text{ is } 2\pi \text{ periodic in } \phi,$$
$$\Rightarrow q(\phi) = A\cos m\phi + B\sin m\phi, \qquad \alpha = m^2.$$

Second:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{dp}{d\theta}\Big) + \Big(\gamma - \frac{m^2}{\sin^2\theta}\Big)p = 0. \tag{7}$$

Here we have boundary conditions that $p(0)$ and $p(\pi)$ are finite.

Let us introduce a new variable $s = \cos\theta$, i.e., $\sin^2\theta = 1 - s^2$. Then (7) becomes

$$\frac{d}{ds}\Big[(1-s^2)\frac{dp}{ds}\Big] + \Big(\gamma - \frac{m^2}{1-s^2}\Big)p = 0, \quad -1 \le s \le 1. \tag{8}$$

where $p(\pm 1)$ are finite. This is the so-called *associated Legendre equation*.

Note that $m = 0$ case corresponds to the usual Legendre equation whose solutions are the Legendre polynomial $P_n(s)$, where $\gamma = \ell(\ell+1)$ and $\ell \in \mathbb{N}$. The equation (8) can be solved via the power series as in Bessel's equation. We need to omit the details of computation, which can be found in [3, Sec. 6.3] and [1].

Now let $P(s)$ be a general solution for the Legendre equation with general γ, $\big((1-s^2)P'\big)' + \gamma P = 0$. Then the solution to (8) can be written as $p(s) = (1-s^2)^{\frac{m}{2}} P^{(m)}(s)$. If $\gamma = \ell(\ell+1)$, with $\ell \ge m$ and $\ell \in \mathbb{N}$, then $p(s)$ can be written as

$$p(s) = P_\ell^m(s) \triangleq \frac{(-1)^m}{2^\ell \ell!}(1-s^2)^{\frac{m}{2}}\frac{d^{\ell+m}}{ds^{\ell+m}}(s^2-1)^\ell$$

Here $P_\ell^m(s)$ is called *the associated Legendre function*, which is merely a polynomial in s with multiplication of a power of $\sqrt{1-s^2}$. Also notice that $\sqrt{\gamma+\frac{1}{4}} = \sqrt{\ell^2+\ell+\frac{1}{4}} = \sqrt{\left(\ell+\frac{1}{2}\right)^2} = \ell+\frac{1}{2}$.

Finally, putting everything together, we have

$$\begin{aligned} v(r,\theta,\phi) &= R(r)p(\theta)q(\phi) \\ &= \frac{J_{\ell+\frac{1}{2}}(\sqrt{\lambda}r)}{\sqrt{r}} P_\ell^m(\cos\theta)(A\cos m\phi + B\sin m\phi) \end{aligned}$$

By replacing $\cos$, $\sin$ by complex exponentials, we can also write a basic solution as

$$\begin{cases} v_{\ell m j}(r,\theta,\phi) &= \frac{J_{\ell+\frac{1}{2}}(\sqrt{\lambda_{\ell j}}r)}{\sqrt{r}} P_\ell^{|m|}(\cos\theta)\mathrm{e}^{im\phi}, \\ J_{\ell+\frac{1}{2}}(\sqrt{\lambda_{\ell j}}a) &= 0, \qquad \lambda_{\ell 1} < \lambda_{\ell 2} < \cdots, \text{ for each } \ell. \end{cases}$$

where $\ell = 0, 1, \ldots, \infty$, $m = -\ell, \ldots, 0, \ldots, \ell$, and $j = 1, 2, \ldots, \infty$. Therefore, for each (ℓ, j), there exist $2\ell+1$ eigenfunctions, i.e., $\lambda_{\ell j}$ has $2\ell+1$ multiplicity.

Notice that $v_{\ell m j}$ are orthogonal, i.e.,

$$\begin{aligned} \langle v_{\ell m j}, v_{\ell' m' j'}\rangle &= \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^a v_{\ell m j}\, v_{\ell' m' j'}\, r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= c\,\delta_{\ell\ell'}\delta_{mm'}\delta_{jj'} \end{aligned}$$

Finally, we have

$$\begin{cases} u(\boldsymbol{x},t) &= \displaystyle\sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=-\ell}^{\ell} (A_{\ell m j}\cos\sqrt{\lambda_{\ell j}}ct + B_{\ell m j}\sin\sqrt{\lambda_{\ell j}}ct)\cdot v_{\ell m j}, \\ u(\boldsymbol{x},0) &= f(\boldsymbol{x}) = \displaystyle\sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m j}v_{\ell m j}, \\ u_t(\boldsymbol{x},0) &= g(\boldsymbol{x}) = \displaystyle\sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=-\ell}^{\ell} c\sqrt{\lambda_{\ell j}}B_{\ell m j}v_{\ell m j}. \end{cases}$$

$$\Rightarrow A_{\ell m j} = \frac{\langle f, v_{\ell m j}\rangle}{\|v_{\ell m j}\|_2^2}, \qquad c\sqrt{\lambda_{\ell j}}B_{\ell m j} = \frac{\langle g, v_{\ell m j}\rangle}{\|v_{\ell m j}\|_2^2}.$$

$A_{\ell mj}$, $B_{\ell mj}$ are complex numbers in general and more simplification happens by $\mathrm{e}^{im\phi} \longrightarrow (\cos m\phi, \sin m\phi)$.

Example: Let $f(\boldsymbol{x}) \equiv 0$ and $g(\boldsymbol{x}) = g(r)$.
Then $A_{\ell mj} = 0, m = 0, \ell = 0$. We also know that $P_0^0(s) = P_0(s) = 1$. Therefore

$$\begin{aligned} u(\boldsymbol{x}, t) &= \sum_{j=1}^{\infty} B_j \sin(\sqrt{\lambda_{0j}} ct) v_{00j} \\ &= \sum_{j=1}^{\infty} B_j \sin(\sqrt{\lambda_{0j}} ct) \cdot \frac{J_{1/2}(\sqrt{\lambda_{0j}} r)}{\sqrt{r}} \end{aligned}$$

with $B_j = \dfrac{1}{\sqrt{\lambda_{0j}} c} \displaystyle\int_0^a r^2 \frac{J_{1/2}(\sqrt{\lambda_{0j}} r)}{\sqrt{r}} g(r)\,\mathrm{d}r \Bigg/ \int_0^a \frac{J_{1/2}^2(\sqrt{\lambda_{0j}} r)}{r} r^2\,\mathrm{d}r.$

Amazingly, in this case $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ (see Appendix). Thus

$$u(\boldsymbol{x}, t) = \sqrt{\frac{2}{\pi}} \sum_{j=1}^{\infty} B_j \sin(\sqrt{\lambda_{0j}} ct) \cdot \frac{\sin(\sqrt{\lambda_{0j}} r)}{\sqrt{\lambda_{0j}} r},$$

where $\sqrt{\lambda_{0j}} a = j\pi, \quad j = 1, 2, \cdots$, by the Dirichlet boundary condition.

Therefore, $u(\boldsymbol{x}, t) = \sqrt{\dfrac{2}{\pi}} \displaystyle\sum_{j=1}^{\infty} B_j \sin\left(\frac{c\pi j t}{a}\right) \cdot \frac{\sin\left(\dfrac{j\pi r}{a}\right)}{\dfrac{j\pi r}{a}}.$

We get the fundamental frequency $\sqrt{\lambda_{01}} c = \dfrac{\pi}{a} c$, which is the same as that of the 1D string of length a.

Appendix

To derive the formula $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$, we need the series definition of the Bessel function of first kind of order α as follows

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

Let $\alpha = 1/2$. Since for the gamma function $\Gamma(z)$, we have

$$\begin{cases} \Gamma(z+1) = z\Gamma(z) \\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{cases}$$

$$\Rightarrow \Gamma\left(\frac{1}{2} + m + 1\right) = \frac{1}{2} \cdot \left(\frac{1}{2} + 1\right) \cdot \cdots \cdot \left(\frac{1}{2} + m\right).$$

We have

$$\begin{aligned} J_{1/2}(x) &= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(\frac{1}{2} + m + 1\right)} \left(\frac{x}{2}\right)^{2m+1} \\ &= \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}, \end{aligned}$$

from which we get $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$.

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