MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lectures 8+9: Laplacian Eigenvalue Problems for General Domains – I. Eigenvalues as Minima of the Potential Energy

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In this lecture, we will consider Laplacian eigenvalue problems for general domains. Since the explicit formulas only exist for special domains (e.g., rectangles, disks, balls, etc.), what can we say about $\{(\lambda_n, \varphi_n)\}$ for a domain Ω of general shape?

The basic references for this lecture are the texts by Strauss [6, Sec. 11.1-11.2], and Courant and Hilbert [1, Sec. VI.1]. For the details and the survey up to the recent results, consult [3].

1 The Eigenvalues as the Minima of the Potential Energy

Consider the following Dirichlet-Laplacian (DL) Problem, where Ω is an open domain with general shape, $|\Omega| < \infty$, and $\partial\Omega$ is piecewise smoooth.

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In this lecture, we list

$$0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_n \le \cdots ,$$

where each eigenvalue is repeated according to its multiplicity.

Now consider the following minimization problem (MP):

$$m = \min_{\substack{w \in C_0^2(\Omega) \\ w \not\equiv 0}} \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \right\} \qquad \text{(MP)}$$

where $C_0^2(\Omega) = \{\, w \in C^2(\Omega) \mid w = 0 \ \text{ on } \partial\Omega\}$. The term $\dfrac{\|\nabla w\|^2}{\|w\|^2}$ is called the *Rayleigh quotient*. And $\frac{1}{2}\|\nabla w\|^2 = \frac{1}{2}\int_\Omega |\nabla w|^2 \, \mathrm{d}\boldsymbol{x}$ is the *potential energy* or *roughness* of $w(\boldsymbol{x})$ in Ω.

Notice that if $u(\boldsymbol{x})$ is solution for (MP), then so is $a \cdot u(\boldsymbol{x})$, where a is an arbitrary nonzero constant.

Theorem 1.1. *Let*

$$\lambda_1 = m = \min_{\substack{w \in C_0^2(\Omega) \\ w \not\equiv 0}} \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \right\} \ \textit{and}$$

$$\varphi_1 = \arg\min_{\substack{w \in C_0^2(\Omega) \\ w \not\equiv 0}} \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \right\}.$$

then $-\Delta\varphi_1 = \lambda_1\varphi_1$.

In other words, "the first eigenvalue is the minimum of the potential energy, and the first eigenfunction is the ground state (state of the lowest energy)."

Proof. From now on, we will call a function from $C_0^2(\Omega)$ a *trial function*.

Let u be the solution of (MP) with minimum value $m \ge 0$. Then, for any trial function $w \in C_0^2(\Omega)$, we have

$$m = \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}\boldsymbol{x}}{\int_\Omega |u|^2 \, \mathrm{d}\boldsymbol{x}} \le \frac{\int_\Omega |\nabla w|^2 \, \mathrm{d}\boldsymbol{x}}{\int_\Omega |w|^2 \, \mathrm{d}\boldsymbol{x}}.$$

Let $v \in C_0^2(\Omega)$ be any other trial function such that $w(\boldsymbol{x}) = u(\boldsymbol{x}) + \varepsilon v(\boldsymbol{x})$, where ε is any real constant.

Then define

$$f(\varepsilon) \triangleq \frac{\int_\Omega |\nabla(u+\varepsilon v)|^2 \ \mathrm{d}\boldsymbol{x}}{\int_\Omega |u+\varepsilon v|^2 \ \mathrm{d}\boldsymbol{x}},$$

which has a minimum at $\varepsilon = 0$, i.e., $f^{'}(0) = 0$.

Expanding $f(\varepsilon)$ in ε yields

$$f(\varepsilon) = \frac{\int_\Omega \left(|\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2\right) \ \mathrm{d}\boldsymbol{x}}{\int_\Omega \left(u^2 + 2\varepsilon uv + \varepsilon^2 v^2\right) \ \mathrm{d}\boldsymbol{x}}.$$

Using the quotient rule for differentiation and substituting $\varepsilon = 0$, we obtain that

$$0 = f^{'}(0) = \frac{\left(\int_\Omega 2\nabla u \cdot \nabla v \ \mathrm{d}\boldsymbol{x}\right)\left(\int_\Omega u^2 \ \mathrm{d}\boldsymbol{x}\right) - \left(\int_\Omega |\nabla u|^2 \ \mathrm{d}\boldsymbol{x}\right)\left(\int_\Omega 2uv \ \mathrm{d}\boldsymbol{x}\right)}{\left(\int_\Omega u^2 \ \mathrm{d}\boldsymbol{x}\right)^2}.$$

A simple algebraic manipulation produces

$$\int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}\boldsymbol{x} = \left(\frac{\int_\Omega |\nabla u|^2 \ \mathrm{d}\boldsymbol{x}}{\int_\Omega u^2 \ \mathrm{d}\boldsymbol{x}}\right)\int_\Omega uv \ \mathrm{d}\boldsymbol{x} = m\int_\Omega uv \ \mathrm{d}\boldsymbol{x}.$$

Also, by Green's first identity (G1[1]), we may write

$$\int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}\boldsymbol{x} + \int_\Omega v \Delta u \ \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \ \mathrm{d}S = 0.$$

The last equality follows from the fact that $v \in C_0^2(\Omega)$, i.e., $v|_{\partial\Omega} = 0$. Therefore,

$$\int_\Omega (\Delta u + mu)\, v \ \mathrm{d}\boldsymbol{x} = 0.$$

This is true for any $v \in C_0^2(\Omega)$. Therefore, we must have $\Delta u + mu = 0$, i.e., m and u are the eigenpair for the Dirichlet-Laplacian problem (1).

We still need to show m is actually the smallest eigenvalue, i.e., $m = \lambda_1$, $u = \varphi_1$.

[1] See Lecture 4 for some details.

Let $-\Delta v_j = \lambda_j v_j$, where λ_j is any eigenvalue of the Dirichlet-Laplacian problem (1). Then, by definition,

$$m \le \frac{\int_\Omega |\nabla v_j|^2 \,\mathrm{d}\boldsymbol{x}}{\int_\Omega v_j^2 \,\mathrm{d}\boldsymbol{x}} = -\frac{\int_\Omega v_j \Delta v_j \,\mathrm{d}\boldsymbol{x}}{\int_\Omega v_j^2 \,\mathrm{d}\boldsymbol{x}} = \frac{\int_\Omega \lambda_j v_j^2 \,\mathrm{d}\boldsymbol{x}}{\int_\Omega v_j^2 \,\mathrm{d}\boldsymbol{x}} = \lambda_j.$$

The first equality follows from (G1). So, $m \le \lambda_j,\ \forall j$, and m is an eigenvalue of (1). So $m = \lambda_1$ and $u = \varphi_1$. □

In this proof, we apply the idea of calculus of variations. Classical but excellent general references on calculus of variations are [1, Chap. IV], [4, Part II], and [5, Chap. II]. Finally, an excellent treatment of calculus of variations related to PDEs is [8, Chap. 8].

Example 1.2. Find $m = \min\limits_{\substack{w \in C_0^2(\Omega) \\ w \not\equiv 0}} \dfrac{\int_0^1 \left(w'\right)^2 \,\mathrm{d}x}{\int_0^1 w^2 \,\mathrm{d}x}, \quad w \in C_0^2(0,1).$

Answer. $m = \pi^2$, since the solution of this (MP) is $w(x) = \sin \pi x = \varphi_1$ and $\lambda_1 = \pi^2$.

Interesting to Note: We can easily pick a function $w \in C_0^2(0,1)$ to get an approximate value of m. For example, we can choose $w(x) = ax(1-x)$, a is an arbitrary constant. Let us compare the true solution and this w. Here we choose the functions with unit L^2 norm. See Figure 1.

$$\begin{aligned} &\sqrt{2}\sin \pi x && \Rightarrow \int_0^1 ((\sqrt{2}\sin \pi x)')^2 \,\mathrm{d}x = \pi^2 \approx 9.8696, \\ &\sqrt{30}x(1-x) && \Rightarrow \int_0^1 ((\sqrt{30}x(1-x))')^2 \,\mathrm{d}x = 10. \end{aligned} \tag{2}$$

2 The Other Eigenvalues

Theorem 2.1 (Minimum Principle for the nth Eigenvalue)**.** *Suppose that* $\{(\lambda_j, \varphi_j)\}_{j=1}^{n-1}$ *are already known. Then*

$$\lambda_n = \min_{\substack{w \in C_0^2(\Omega), \\ w \not\equiv 0, \\ \langle w, \varphi_j \rangle = 0,\, \forall j \in \{1,\ldots,n-1\}}} \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \right\}. \qquad (\mathrm{MP})_n$$

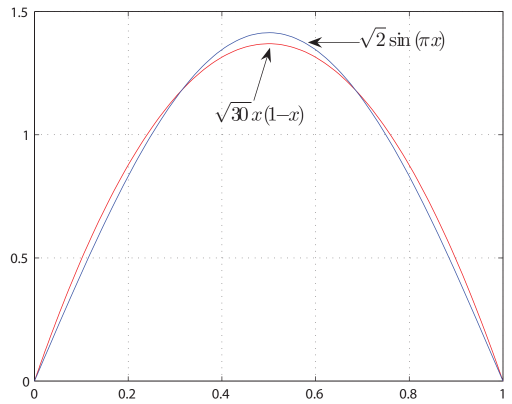

Figure 1: Plots of (MP) solution $\sqrt{2}\sin \pi x$ and a trial function $\sqrt{30}x(1-x)$.

assuming that the minimum exists. Furthermore, the minimizing function is $\varphi_n(\boldsymbol{x})$, i.e. the nth eigenfunction.

Note that this theorem implies $\lambda_{n-1} \le \lambda_n$, $\forall n \ge 2$.

Proof. By assumption, there exists $u(\boldsymbol{x})$ that is a solution to (MP)$_n$. Let m^* be the minimum value of (MP)$_n$. So $u|_{\partial\Omega} = 0$, and $u \perp \varphi_1, \ldots, \varphi_{n-1}$.

As in the proof of the previous theorem, let $w(\boldsymbol{x}) = u(\boldsymbol{x}) + \varepsilon v(\boldsymbol{x})$, where w and v satisfy the conditions for (MP)$_n$. Then, exactly as before, we have

$$\int_\Omega (\Delta u + m^* u)\, v \; \mathrm{d}\boldsymbol{x} = 0, \tag{3}$$

for any $v \in C_0^2(\Omega)$ with $v \perp \varphi_1, \ldots, \varphi_{n-1}$.

Now consider, for $j = 1, \ldots, n-1$,

$$\begin{aligned}\int_\Omega (\Delta u + m^* u)\, \varphi_j \; \mathrm{d}\boldsymbol{x} &\overset{(a)}{=} \int_\Omega u \left(\Delta \varphi_j + m^* \varphi_j\right) \; \mathrm{d}\boldsymbol{x} \\ &\overset{(b)}{=} (m^* - \lambda_j) \int_\Omega u \varphi_j \; \mathrm{d}\boldsymbol{x} \\ &\overset{(c)}{=} 0.\end{aligned} \tag{4}$$

where (a) is derived by Green's second identity (G2[1]), (b) is from the fact that $\Delta\varphi_j = -\lambda_j\varphi_j$, and (c) is derived by the fact $u \perp \varphi_j$.

Now let $h(\boldsymbol{x})$ be an arbitrary trial function and set

$$v(\boldsymbol{x}) = h(\boldsymbol{x}) - \sum_{k=1}^{n-1} c_k\varphi_k(\boldsymbol{x}), \qquad c_k = \frac{\langle h, \varphi_k\rangle}{\langle \varphi_k, \varphi_k\rangle}. \tag{5}$$

Then $\langle v, \varphi_j\rangle = 0$ for $j = 1, \ldots, n-1$.

Since $h, \varphi_j \in C_0^2(\Omega)$, $\forall j \in \{1, \ldots, n-1\}$, (3) is valid for v defined in (5).

From (3) and (4),

$$\int_\Omega (\Delta u + m^* u)\left(v + \sum_{k=1}^{n-1} c_k\varphi_k\right)\, \mathrm{d}\boldsymbol{x} = \int_\Omega (\Delta u + m^* u)\, h\, \mathrm{d}\boldsymbol{x} = 0, \qquad \forall h \in C_0^2(\Omega).$$

This implies that $-\Delta u = m^* u$. Similarly to the previous theorem with induction, we can show that $m^* = \lambda_n$, $u = \varphi_n$. □

Remark 2.2. The *existence* of the minima (MP) and $(\text{MP})_n$ is a delicate mathematical issue that we have avoided, which led to the theory of Sobolev spaces. In fact, there are domains D with rough boundaries for which (MP) does not have any solution at all. For further information, see [7], [8, Chap. 5], [9, Chap. 6] and [10, Chap. 7-8]. Also [11] is the paper that put the end to the confusion of the two different definitions of the Sobolev spaces.

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[1]See Lecture 4 for some details.

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