MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lectures 9+10: Laplacian Eigenvalue Problems for General Domains – II. Computation of Eigenvalues

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The basic references for this lecture are the texts by Strauss [2, Sec. 11.3], and Courant and Hilbert [3, Sec. VI.1, VI.2]. For more advanced treatments, see [4, Chap. 4] and [5, Chap. 1].

The Dirichlet-Laplacian eigenvalues, in particular, λ_1 (the smallest), is quite important in many applications. To compute them we will use the *Rayleigh quotient*.

For any trial function $w \in C_0^2(\Omega)$, with $w \not\equiv 0$, we have

$$\lambda_1 \leq \frac{\|\nabla w\|^2}{\|w\|^2}.$$

If $w = \varphi_1$, we get exactly λ_1, but we do not know φ_1 at this point. So, we should be satisfied with a moderately clever choice of w that might provide a relatively good approximation.

Example 0.1. In Lecture 8, we already discussed the approximate minimum of

(MP):

$$m = \min_{\substack{w \in C_0^2(\Omega) \\ w \not\equiv 0}} \frac{\displaystyle\int_0^1 \left(w'\right)^2 \, \mathrm{d}x}{\displaystyle\int_0^1 w^2 \, \mathrm{d}x}, \quad w \in C_0^2(0,1).$$

We choose the trial function $w = \sqrt{30}x(1-x)$, which gives an estimate $\hat{\lambda}_1 = 10$ where the true eigenvalue is $\lambda_1 = \pi^2 \approx 9.8696$.

In this case we have done well, but in general, it is too difficult to come up with a good trial function to produce a good approximation of λ_1 or λ_j.

1 Rayleigh-Ritz Approximation (RRA)

Let $w_1, \cdots, w_n \in C_0^2(\Omega)$, $w_j \not\equiv 0$ be arbitrary trial functions. Let $A = (a_{jk})$, $B = (b_{jk})$, where a_{jk}, b_{jk} are defined as:

$$\begin{aligned} a_{jk} &\triangleq \langle \nabla w_j, \nabla w_k \rangle = \int_\Omega \nabla w_j \cdot \nabla w_k \, \mathrm{d}\boldsymbol{x}, \\ b_{jk} &\triangleq \langle w_j, w_k \rangle = \int_\Omega w_j w_k \, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{1}$$

for $j, k = 1, \ldots, n$. Thus, A and B are $n \times n$ symmetric matrices. Then

The roots of the polynomial equation $\det(A - \lambda B)$ are approximation to the first n eigenvalues $\lambda_1, \cdots, \lambda_n$.

Before presenting an informal proof of this, let us consider the following example.

Example 1.1. Consider the *r*adial vibrations of a circular membrane $\Omega = \{(x, y) \mid x^2 + y^2 < 1\}$. Then the Dirichlet-Laplacian (DL) problem is $-\Delta u = \lambda u$, which can be written in the polar coordinates (r, θ) as

$$(-r u_r)_r = \lambda r u \qquad (0 < r < 1), \quad u = 0 \text{ at } r = 1.$$

Here we apply the fact that radial function depends only on the parameter r, i.e., $u(x, y) = u(r)$.

Then the Rayleigh quotient is given by

$$Q = \frac{\displaystyle\iint |\nabla w|^2 \, \mathrm{d}\boldsymbol{x}}{\displaystyle\iint w^2 \, \mathrm{d}\boldsymbol{x}} = \frac{\displaystyle 2\pi \int_0^1 r w_r^2 \, \mathrm{d}r}{\displaystyle 2\pi \int_0^1 r w_r^2 \, \mathrm{d}r}. \tag{2}$$

Now we ask "what trial functions should we use?" The trial functions are required to satisfy the boundary conditions

$$w(0): \text{ finite}, \qquad w_r(0) = 0 = w(1).$$

A simple choice of a pair of them is $1 - r^2$ and $(1 - r^2)^2$. In this case, we can compute the entries for matrices A and B in (1) as follows

$$A = \begin{pmatrix} 2\pi & 4\pi/3 \\ 4\pi/3 & 4\pi/3 \end{pmatrix}, \qquad B = \begin{pmatrix} \pi/3 & \pi/4 \\ \pi/4 & \pi/5 \end{pmatrix}.$$

Hence

$$\begin{aligned} \det(A - \lambda B) &= \det \left[\begin{pmatrix} 2\pi - \lambda\pi/3 & 4\pi/3 - \lambda\pi/4 \\ 4\pi/3 - \lambda\pi/4 & 4\pi/3 - \lambda\pi/5 \end{pmatrix} \right] \\ &= \pi^2 \left\{ \left(2 - \frac{\lambda}{3}\right)\left(\frac{4}{3} - \frac{\lambda}{5}\right) - \left(\frac{4}{3} - \frac{\lambda}{4}\right)^2 \right\}. \end{aligned}$$

Solving the characteristic equation $\det(A - \lambda B) = 0$ gives us the estimates

$$\hat{\lambda}_1 = \frac{64 - 8\sqrt{34}}{3} \simeq 5.7841, \qquad \hat{\lambda}_2 = \frac{64 + 8\sqrt{34}}{3} \simeq 36.8825 \tag{3}$$

On the otherhand, the true eigenvalues are given by

$$\lambda_1 = j_{0,1}^2 = 5.783, \qquad \lambda_2 = j_{0,2}^2 = 30.4705 \tag{4}$$

where $j_{0,k}$ denote the kth zero of the Bessel function of the first kind of order 0. It becomes clear that our estimate for the first eigenvalue is good, but the second has significant error. To improve, we need to use either better trial functions or possibly use three trial functions.

1.1 Informal derivation of the Rayleigh-Ritz Approximation (RRA)

Let $\{w_j\}_{j=1}^n \subset C_0^2(\Omega)$ be arbitrary trial functions and linearly independent. As an approximation to the true minimum problem $(\mathrm{MP})_n$ (see Lecture 8), let us impose an additional condition for w:

$$w(\boldsymbol{x}) = \sum_{k=1}^{n} c_k w_k(\boldsymbol{x}), \qquad c_k : \text{ some constant} \tag{5}$$

So we only seek w of the form (5). Therefore, in general $\hat{\lambda}_1 \geq \lambda_1$, because we impose more constraints in $(\mathrm{MP})_n$. If we are incredibly smart, so that $w(\boldsymbol{x})$ would be an eigenfunction, then we have $-\Delta w = \lambda w$, and

$$\begin{aligned} \langle \nabla w, \nabla w_j \rangle &= \int_\Omega \nabla w \cdot \nabla w_j \; \mathrm{d}\boldsymbol{x} \\ &\overset{(*)}{=} -\int_\Omega \Delta w \cdot w_j \; \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w_j \; \mathrm{d}\boldsymbol{x} \\ &= \lambda \langle w, w_j \rangle , \end{aligned} \tag{6}$$

where we used Green's first identity in (*). Therefore, recalling the definitions of a_{jk} and b_{jk}, we get

$$\left\langle \sum_k c_k \nabla w_k, \nabla w_j \right\rangle = \lambda \left\langle \sum_k c_k w_k, w_j \right\rangle \Rightarrow \sum_k c_k a_{jk} = \lambda \sum_k c_k b_{jk}. \tag{7}$$

So, we can write them in a matrix form

$$A\boldsymbol{c} = \lambda B\boldsymbol{c}, \qquad \boldsymbol{c} \neq \boldsymbol{0}.$$

which leads to $\det(A - \lambda B) = 0$.

If we are not so smart, which is usually the case, we can still use this determinant as our approximation method. This leads to the *Minimax Principle*. See [5, Sec. 1.3] for more details.

2 Minimax Principle

In reality, what we really want is an exact formula to compute the eigenvalues instead of an approximation.

Let $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$ be the roots of $\det(A - \lambda B) = 0$. From numerical linear algebra (see Sec. 2.1), it is easy to see that

$$\lambda_n^* = \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\boldsymbol{c}^T A \boldsymbol{c}}{\boldsymbol{c}^T B \boldsymbol{c}} = \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\langle A\boldsymbol{c}, \boldsymbol{c} \rangle}{\langle B\boldsymbol{c}, \boldsymbol{c} \rangle}$$

Thus, we have

$$\begin{aligned} \lambda_n^* &= \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\sum_{j,k} a_{jk} c_j c_k}{\sum_{j,k} b_{jk} c_j c_k} \\ &= \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\left\langle \nabla(\sum_j c_j w_j), \nabla(\sum_k c_k w_k) \right\rangle}{\left\langle \sum_j c_j w_j, \sum_k c_k w_k \right\rangle} \\ &= \max_{\substack{\boldsymbol{c} \neq \boldsymbol{0} \\ w \in \text{span}\{w_1, \ldots, w_n\}}} \frac{||\nabla w||^2}{||w||^2} \end{aligned} \tag{8}$$

with $w \in \text{span}\{w_1, \cdots w_n\}$. Hence we have from the $(\text{MP})_n$

$$\lambda_n \leq \lambda_n^*$$

.

Thus this allows us to take the minimum on the RHS to get

$$\lambda_n = \min \lambda_n^*, \tag{9}$$

where the minimization is taken over all possible choices of $\{w_1, \ldots, w_n\} \subset C_0^2(\Omega)$.

Theorem 2.1 (Minimax Principle)**.** *Let $\{w_j\}_{j=1}^n \subset C_0^2(\Omega)$ be an arbitrary set of n trial functions. Define λ_n^* by* (8)*, Then the nth eigenvalue is*

$$\lambda_n = \min \lambda_n^*$$

where $\min$ *is taken over all possible choices of the n trial functions $\{w_j\}_{j=1}^n \subset C_0^2(\Omega)$.*

Before presenting the proof, consider the following intermission which plays a role in the proof.

2.1 Intermission: Rayleigh Quotient & Linear Algebra

This intermission is taken from [1, Lect. 27].

Let $A \in \mathbb{R}^{m \times m}$ be symmetric and $x \in \mathbb{R}^m$. Then we know A has real eigenvalues $\lambda_1, \ldots, \lambda_m$ and real orthogonal eigenvectors $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m$.

Define

$$r(\boldsymbol{x}) \triangleq \frac{\boldsymbol{x}^T A \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} \tag{10}$$

Clearly if $\boldsymbol{x} = \boldsymbol{\varphi}_j$, then $r(\boldsymbol{\varphi}_j) = \lambda_j$. Of course, we do not know $\boldsymbol{\varphi}_j$ a priori. So a motivating question is: "Given $\boldsymbol{x}$, what scalar α acts most like an eigenvalue for $\boldsymbol{x}$?"

The answer to this question leads to the following least square problem

$$\min_{\alpha \in \mathbb{R}} \|A\boldsymbol{x} - \alpha \boldsymbol{x}\|_2. \tag{11}$$

The answer to this least square problem is $\hat{\alpha} = r(\boldsymbol{x})$. The reason is as follows:

$$\begin{aligned} \|A\boldsymbol{x} - \alpha\boldsymbol{x}\|_2^2 &= \langle (A - \alpha I)\boldsymbol{x}, (A - \alpha I)\boldsymbol{x} \rangle \\ &= \boldsymbol{x}^T (A - \alpha I)^T (A - \alpha I)\boldsymbol{x} \\ &= \boldsymbol{x}^T A^2 \boldsymbol{x} - 2\alpha \boldsymbol{x}^T A \boldsymbol{x} + \alpha^2 \boldsymbol{x}^T \boldsymbol{x} \end{aligned}$$

By $\dfrac{\partial}{\partial \alpha}\|A\boldsymbol{x} - \alpha\boldsymbol{x}\|_2^2 = 0$, we get $2\alpha\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T A\boldsymbol{x} = 0$. So

$$\alpha = \frac{\boldsymbol{x}^T A \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} \tag{12}$$

is the solution of the least square problem (11).

Thus, if $\boldsymbol{x}$ is close to one of the eigenvectors, then the α in (12) should be close to the corresponding eigenvalue. Also notice that

$$\nabla r(\boldsymbol{x}) = \frac{2}{\boldsymbol{x}^T \boldsymbol{x}}(A\boldsymbol{x} - r(\boldsymbol{x})\boldsymbol{x}). \tag{13}$$

So if $\boldsymbol{x} = \boldsymbol{\varphi}_j$, then $\nabla r(\boldsymbol{\varphi}_j) = 0$. Conversely if $\nabla r(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \neq \mathbf{0}$, then $\boldsymbol{x}$ is an eigenvector of A and $r(\boldsymbol{x})$ is the corresponding eigenvalue. Thus the key point is that $\{\boldsymbol{\varphi}_j\}$ are the stationary points of $r(\boldsymbol{x})$. Further we can show that

$$r(\boldsymbol{x}) - r(\boldsymbol{\varphi}_j) = O(\|\boldsymbol{x} - \boldsymbol{\varphi}_j\|^2) \quad \text{as } \boldsymbol{x} \to \boldsymbol{\varphi}_j \tag{14}$$

This property is the basis for the method of power iterations. See [1, Lecture 27] and [6, Sec. 8.2].

2.2 Proof of Theorem 2.1

Proof. First let's fix $\{w_1, \cdots w_n\} \subset C_0^2(\Omega)$. Then choose $c_1, c_2, \cdots c_n$ (not all zeros) such that

$$w(x) \triangleq \sum_{k=1}^{n} c_k w_k(x) \quad \perp \quad \{\varphi_j, \cdots, \varphi_{n-1}\},$$

i.e., $\langle w, \varphi_k \rangle = \sum_{j=1}^{n} c_j \langle w_j, \varphi_k \rangle = 0, \;\; k = 1, \ldots, n-1.$

It is clear now that we have n unknowns and $n-1$ equations. Therefore it is possible to find $\{c_j\}_{j=1}^n$ not all zero. By the minimum principle, we have

$$\lambda_n \leq \frac{\|\nabla w\|^2}{\|w\|^2}. \tag{15}$$

On the other hand, the maximum in (8) is taken over all possible $\{c_j\}$. So

$$\frac{\|\nabla w\|^2}{\|w\|^2} \leq \lambda_n^*. \tag{16}$$

Therefore, we have

$$\lambda_n \leq \frac{\|\nabla w\|^2}{\|w\|^2} \leq \lambda_n^*. \tag{17}$$

Now (17) must be true for all possible $\{w_1, \cdots w_n\} \subset C_0^2(\Omega)$, which allows us to take minimum on the RHS, i.e.,

$$\lambda_n \leq \min \lambda_n^* \tag{18}$$

where the minimum is taken over all possible w_j. Yet we still need to show that $\lambda_n = \min \lambda_n^*$ is attainable.

Choose $w_j = \varphi_j$ for $j = 1, \cdots n$ where $\|\varphi_j\| = 1$. With this choice we have

$$\lambda_n^* = \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\|\nabla(\sum_j c_j \varphi_j)\|^2}{\|\sum_j c_j \varphi_j\|^2}. \tag{19}$$

Using Green's first identity (G1) results in

$$\langle \nabla\varphi_j, \nabla\varphi_k \rangle = \langle -\Delta\varphi_j, \varphi_k \rangle = \lambda_j \delta_{jk}.$$

Therefore, we have

$$\lambda_n^* = \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\sum_j \lambda_j c_j^2}{\sum_j c_j^2} \leq \frac{\lambda_n \sum_j c_j^2}{\sum_j c_j^2} = \lambda_n \tag{20}$$

so that $\lambda_n^* \leq \lambda_n$. Therefore $\lambda_n = \min \lambda_n^*$ is attainable. $\square$

References

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