Homework 2: due Wednesday, May 5 2010

Problem 1: On the connection between (in)coherence parameter μ and restricted isometry constant δ_s: Show that $\delta_1 = 0$, $\delta_2 = \mu$ and $\delta_s \leq (s-1)\mu$.

Problem 2: Assume that A satisfies the RIP of order s with RIC δ_s. Using the polarization identity

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|_2^2 - \|x-y\|_2^2), \qquad x, y \in \mathbb{R}^n.$$

prove the inequality

$$|\langle Ax, Ay\rangle| \leq \delta_{s+t}\|x\|_2\|y\|_2$$

for all $x, y \in \mathbb{R}^n$ supported on disjoint subsets $S, T \in \{1, \ldots, n\}$ with $|S| \leq s, |T| \leq t$.

Problem 3: Programing Exercise (you may want to use the software package CVX for this and the next exercise). Let A be a Gaussian random matrix of dimension 100×400. Determine via numerical simulations the range of s for which Basis Pursuit is able to recover successfully a randomly generated s-sparse vector x from $y = Ax$. Here, "randomly generated" means that the s non-zero locations are chosen randomly from $\{1, \ldots, 400\}$ according to the uniform distribution, and the amplitudes of x are randomly chosen from the normal distribution. In this example, let us agree that a *successful recovery* means that the relative error $\|x - x^*\|_2/\|x\|_2$ between the true solution x and reconstructed solution x^* is less than 10^{-3}. A good way to illustrate your findings is to produce a graph that shows how the successfull revocery rate changes as you increase s.

(Since x is not an arbitrary s-sparse vector, but randomly generated, your results will only allow conclusions for *most* sparse vectors and not for *all* sparse vectors.)

Problem 4: Programing Exercise: Assume the physical constraints of the problem force/allow you to sample a signal by applying an 128×512 random partial Fourier matrix (generated by extracting 128 randomly chosen rows according to the uniform distribution from the 512×512 DFT matrix), call this matrix A. Let $b = Ax$ be the measured vector of size 128×1. You want to recover the original vector x, where x itself is not sparse in the standard basis. But it is known that x is s-sparse in the Hadamard basis H with $s = 10$. (you can construct a Hadamard matrix in Matlab with `hadamard`).

(a) Try to recover x via

$$\min \|z\|_1 \qquad \text{subject to } Bz = y$$

where $B = AH$ and $x = Hz$. What happens? How do you interpret the results?
(b) Now make the following modification. Construct a 512×512 diagonal matrix D whose entries are randomly chosen from $\{\pm 1\}$ and try to recover x via solving

$$\min \|z\|_1 \qquad \text{subject to } ADHz = y.$$

What happens now? Explain heuristically what might be behind the different outcomes. Is this modification a valid approach (in terms of the given constraints with respect to measurement matrix and sparsity of x)?