Homework

due November 4, 2004

This homework is due on November 4 at the beginning of class. It is expected that you work on this homework set on your own and that your solutions reflect your own understanding of the material. Please do not discuss your solutions with other students in the class.

Problem 1. Give a combinatorial proof of the following q-analog of the convolution formula for binomial coefficients
$$\begin{bmatrix} m+n \\ k \end{bmatrix} = \sum_{i+j=k} q^{(m-i)j} \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix}.$$

Problem 2.

(1) Prove the q-binomial theorem
$$\prod_{i=0}^{m-1}(1+xq^i) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} q^{\binom{j}{2}} x^j,$$
where m is a nonnegative integer and x is an indeterminate.

(2) Deduce from this
$$\prod_{i=1}^{s}(1+x^{-1}q^i)\prod_{i=0}^{t-1}(1+xq^i) = \sum_{j=-s}^{t} \begin{bmatrix} s+t \\ s+j \end{bmatrix} q^{\binom{j}{2}} x^j.$$

(3) By letting s and t tend to infinity, prove Jacobi's triple product identity:
$$\sum_{j\in\mathbb{Z}}(-1)^j q^{\binom{j}{2}} x^j = \prod_{i\geq 0}(1-xq^i)(1-x^{-1}q^{i+1})(1-q^{i+1}).$$

Problem 3. Use the inclusion-exclusion principle to show that the number of surjections from an n-set to a k-set is
$$\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n.$$

Problem 4. The famous first Rogers–Ramanujan identity is given by
$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+1})(1-q^{5j+4})},$$

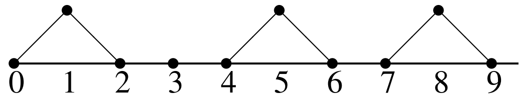

FIGURE 1. A path of length 9

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n > 0$ and $(q)_0 = 1$.

(1) Show that the right-hand side is the generating function of partitions with parts congruent 1 or 4 modulo 5.

(2) Consider $L+1$ points on a line labeled by $i = 0, 1, 2, \dots, L$. Assign to each point a height variable σ_i which takes on the values 0 or 1. In addition the height variables satisfy the restrictions $\sigma_0 = \sigma_L = 0$ and $\sigma_i \sigma_{i+1} = 0$. An allowed configuration of height variables for a given length L is called a *path* of length L. One can illustrate a path graphically by drawing all points (i, σ_i) and connecting adjacent points by straight lines. An example for a path with $L = 9$ is given in figure 1. The condition $\sigma_i \sigma_{i+1} = 0$ requires that the paths consist of a certain number of non-overlapping triangles. To each path p one may assign an energy $E(p)$ by summing up the positions of the peaks, that is

$$E(p) = \sum_{j=1}^{L} j\sigma_j.$$

Define the function

$$F(L) = \sum_p q^{E(p)}$$

where the sum is over all paths of length L. Show that $F(L)$ satisfies the initial condition $F(0) = F(1) = 1$ and recursion relation

$$F(L) = F(L-1) + q^{L-1}F(L-2).$$

(3) Prove that

$$F(L) = \sum_{n=0}^{\infty} q^{n^2} \begin{bmatrix} L-n \\ n \end{bmatrix}.$$

(4) Show that $\lim_{L\to\infty} F(L)$ yields the left-hand side of the Rogers–Ramanujan identity.

(5) Use the path interpretation to show that the left-hand side of the Rogers–Ramanujan identity is the generating function of partitions for which the difference between any two parts is at least two.

(6) Conclude that the Rogers–Ramanujan identity implies that the number of partitions of an integer N into parts in which the difference between any two parts is at least 2 is the same as the number of partitions of N into parts congruent to 1 or 4 modulo 5.
Verify this statement for $N = 6$.