Homework 1

due Wednesday January 18, 2006 in class

1. (Biggs 10.3 # 3) Show that, if p is a prime and m is a positive integer, then an integer x in the range $1 \leq x \leq p^m$ is *not* coprime to p^m if and only if it is a multiple of p. Deduce that $\Phi(p^m) = p^m - p^{m-1}$.

2. (Biggs 10.7 # 2) In the usual set of dominos each domino may be represented by the symbol $[x|y]$, where x and y are members of the set $\{0, 1, 2, 3, 4, 5, 6\}$. The numbers x and y may be equal. Explain why the total number of dominos is 28 rather than 49.

3. (Biggs 10.7 # 8) Calculate the total number of permutations of σ of $\mathbb{N}_6$ which satisfy $\sigma^2 = \text{id}$, $\sigma \neq \text{id}$.

4. (Biggs 10.7 # 9) Let α and β be the permutations of $\mathbb{N}_9$ whose representations in cycle notation are

$$\alpha = (1237)(49)(58)(6)$$
$$\beta = (135)(246)(789)$$

Write down the cycle notations for $\alpha\beta$, $\beta\alpha$, α^2, β^2, α^{-1}, β^{-1}.

5. (Biggs 10.7 # 10) The rooms of the house shown in Fig. 10.1 of Biggs' book are to be painted in such a way that rooms with an interconnecting door have different colors. If there are n colors available, how many different color schemes are possible?

6. (Biggs 10.7 # 18) A pack of 52 cards is divided into two equal parts and then "interlaced", so that if the original order was $1, 2, 3, 4, \ldots$, the new order is $1, 27, 2, 28, \ldots$. How many times must this shuffle be repeated before the cards are once again in the original order?

7. (Biggs 11.3 # 2) Calculate the coefficient of

(1) x^5 in $(1 + x)^{11}$;
(2) a^2b^8 in $(a + b)^{10}$;
(3) a^6b^6 in $(a^2 + b^3)^5$;
(4) x^3 in $(3 + 4x)^6$.