EXISTENCE AND UNIQUENESS OF CRYSTAL BASES

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra over a field K as defined in class. In general integrable $U_q(\mathfrak{g})$-modules are not completely reducible. The following family however has this property:

Definition 1. Let $\mathcal{O}_{\text{int}}$ be the category of integrable $U_q(\mathfrak{g})$-modules M such that for all $u \in M$ there exists an $\ell \geq 1$ such that $e_{i_1} \cdots e_{i_\ell} u = 0$ for all $i_1, \ldots, i_\ell \in I$.

Denote by $P_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i$ the the set of dominant integral weights of $\mathfrak{g}$ and let $\Lambda \in P_+$. Define $V(\Lambda)$ to be the $U_q(\mathfrak{g})$-module generated by u_Λ satisfying

$$\begin{aligned} q^h u_\Lambda &= q^{\langle h, \Lambda \rangle} u_\Lambda \\ e_i u_\Lambda &= 0 \qquad \forall i \in I \\ f_i^{1+\langle h_i, \Lambda \rangle} u_\Lambda &= 0. \end{aligned}$$

Lusztig [2] proved the following theorem for fields K of characteristic 0.

Theorem 2. Let $\Lambda \in P_+$.

1. $V(\Lambda)_\Lambda = \{u \in V(\Lambda) | e_i u = 0 \quad \forall i \in I\} = K u_\Lambda \neq 0$.
2. $V(\Lambda)$ is an irreducible integrable $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\text{int}}$.
3. All $U_q(\mathfrak{g})$-modules in $\mathcal{O}_{\text{int}}$ are completely reducible.
4. Any irreducible $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\text{int}}$ is isomorphic to $V(\Lambda)$ for some $\Lambda \in P_+$.

In ref. [1] Kashiwara proved the existence and uniqueness of crystals as summarized in the next two theorems.

Theorem 3. Existence:

1. $V(\Lambda)$ has a unique crystal base $(L(\Lambda), B(\Lambda))$ such that $B(\Lambda)_\Lambda = \{u_\Lambda\}$.
2. $B(\Lambda) = \{\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_\ell}^{a_\ell} u_\Lambda | \ell \geq 0, i_1, \ldots i_\ell \in I, a_1, \ldots a_\ell > 0\} \backslash \{0\}$.

Theorem 4. Let $M \in \mathcal{O}_{\text{int}}$ and let (L, B) be a crystal base of M. Then there exists a $U_q(\mathfrak{g})$-linear isomorphism

$$M \to \bigoplus_j V(\Lambda^{(j)})$$

by which (L, B) is isomorphic to $\bigoplus_j (L(\Lambda^{(j)}), B(\Lambda^{(j)}))$.

The proofs of these theorems are rather involved and we refer here to [1] for the details.

References

[1] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991) 465–516.

[2] G. Lusztig, *On quantum groups*, J. Algebra **131** (1990) 466–475.