HITCHHIKER'S GUIDE TO LIE ALGEBRAS

These notes are intended to give you all the basics on Lie algebras that you need to know. Good references where you can find a detailed treatment of Lie algebras are the books by Fulton and Harris [1] and Humphreys [2]. For an easy orientation I added specific references to [1] to most of the definitions and statements.

1. Basic definitions

Definition 1. (see [1] p. 108)
An algebra A over a field $\mathbf{k}$ is a vector space with a product $[\cdot,\cdot] : A \times A \to A$ that is bilinear.
A Lie algebra $\mathfrak{g}$ is an algebra with the additional properties that

1. $[a, a] = 0$ for all $a \in \mathfrak{g}$,
2. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for all $a, b, c \in \mathfrak{g}$.

The identity in 2 is called Jacobi identity.

Example 2. (see [1] p. 111)

- The set of all $n \times n$ matrices $\mathfrak{gl}_n$ is a Lie algebra where $[a, b] = ab - ba$ is the commutator.
- The set of all traceless $n \times n$ matrices $\mathfrak{sl}_n$ with the same bracket is a Lie algebra.

Definition 3. (see [1] p. 109)
A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a map

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

such that $[a, b](v) = a(b(v)) - b(a(v))$ is preserved.

Since $\mathfrak{g}$ itself is a vector space a special representation is obtained by taking $V = \mathfrak{g}$.

Definition 4. The adjoint representation is

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

with $(\mathrm{ad}a)(b) = [a, b]$.

2. Classification of Lie algebras

One of the major achievements of Lie theory is the classification of all Lie algebras. It turns out that the classification of Lie algebras reduces to the classification of semisimple Lie algebras (see [1] chapter 9 for details). A semisimple Lie algebra is a Lie algebra without nonzero abelian ideals. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if $[h, g] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$ and $g \in \mathfrak{g}$. An algebra A is abelian if $[a, a'] = 0$ for all $a, a' \in A$.

A simple Lie algebra is a Lie algebra with $\dim \mathfrak{g} > 1$ and no nontrivial ideals.

It turns out that semisimple Lie algebras are direct sums of simple Lie algebras. Hence the classification of Lie algebras reduces to the classification of simple Lie algebras.

Theorem 5. With five exceptions every simple Lie algebra is isomorphic to $\mathfrak{sl}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$ for some n.
Exceptions: $\mathfrak{g}_2$, $\mathfrak{f}_2$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

Here $\mathfrak{so}_n$ is the set of nondegenerate symmetric $n \times n$ matrices and $\mathfrak{sp}_{2n}$ is the set of nondegenerate skewsymmetric $2n \times 2n$ matrices (see [1] chapters 16 and 18).

3. Simple Lie algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra. Here we discuss the main definitions/facts about semisimple Lie algebras (see [1] chapter 14).

1. Let $\mathfrak{h} \subset \mathfrak{g}$ be the maximal abelian subalgebra acting diagonally. This is called the **Cartan subalgebra** of $\mathfrak{g}$.

   Example 6. The Cartan subalgebra of $\mathfrak{sl}_n$ consists of all diagonal $n \times n$ traceless matrices. A basis of $\mathfrak{sl}_2$ is given by

   $$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

   with relations

   $$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = h. \tag{1}$$

   Hence $\mathfrak{h} = \mathbb{C}h$.

2. Let V be a finite-dimensional representation of $\mathfrak{g}$. Then it decomposes into eigenspaces

   $$V = \bigoplus_{\lambda} V_\lambda,$$

   where $v \in V_\lambda$ is an eigenvector of $\mathfrak{h}$, that is,

   $$h(v) = \lambda(h) \cdot v$$

   for all $h \in \mathfrak{h}$. Here $\lambda \in \mathfrak{h}^*$ is a linear functional. λ is called a **weight** and V_λ is called a **weight space**.
3. Act with $\mathfrak{h}$ on $\mathfrak{g}$ by the adjoint representation and decompose into weight spaces. This yields the **Cartan decomposition**

   $$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

   The set $R \subset \mathfrak{h}^*$ is the set of weights of the adjoint representation. The elements $\alpha \in R$ are called **roots**. $\mathfrak{h}$ preserves $\mathfrak{g}_\alpha$

   $$\mathrm{ad}(h)(g) = \alpha(h) \cdot g$$

   for all $g \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$. This may also be rewritten as

   $$[h,g] = \alpha(h) \cdot g.$$

   Example 7. By (1) the roots of $\mathfrak{sl}_2$ are $R = \{\pm 2\}$. The roots for $\mathfrak{sl}_n$ are given by $R = \{L_i - L_j | 1 \leq i,j \leq n, i \neq j\}$ where L_i picks out the i-th entry of the diagonal of an $n \times n$ traceless matrix.

4. To determine how $g \in \mathfrak{g}_\alpha$ acts let's do the following computation. Let $h \in \mathfrak{h}, g \in \mathfrak{g}_\alpha$ and $v \in V_\lambda$. Then
$$\begin{aligned} h(g(v)) &= g(h(v)) + [h,g](v) = g(\lambda(h)\cdot v) + (\alpha(h)\cdot g)(v) \\ &= (\alpha(h)+\lambda(h))g(v). \end{aligned}$$
This shows that
$$g_\alpha : V_\lambda \to V_{\lambda+\alpha}.$$
Example 8. (See [1] p. 149) For $\mathfrak{sl}_2$ we get for $v \in V_\lambda$
$$\begin{aligned} h(e(v)) &= (\lambda+2)e(v) \\ h(f(v)) &= (\lambda-2)f(v). \end{aligned}$$
For finite-dimensional representations there must be some n such that $eV_n = 0$. One may easily check that $ef^m v = m(n-m+1)f^{m-1}v$ for $v \in V_n$. By the finite-dimensionality we must have $0 = ef^m(v)$ for some m. Hence $m = n+1$. This implies that n is an integer and V is an $(n+1)$-dimensional representation with eigenvalues $n, n-2, \ldots, -n$.

5. It turns out that if $\alpha \in R$ then also $-\alpha \in R$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{h}$. The subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sl}_2$. Pick a basis $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.
6. By the results for $\mathfrak{sl}_2$, h_α must have integer eigenvalues. Hence all weights $\lambda \in \mathfrak{h}^*$ must satisfy $\lambda(h_\alpha) \in \mathbb{Z}$. The **weight lattice** is defined as
$$P = \{\lambda \in \mathfrak{h}^* | \lambda(h_\alpha) \in \mathbb{Z}, \forall \alpha \in R\}.$$
Define a pairing $\langle \cdot,\cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ by $\langle h_\alpha, \lambda \rangle = \lambda(h_\alpha)$.
7. Symmetry of the weights: For $\alpha \in R$ define a hyperplane
$$\Omega_\alpha = \{\lambda \in \mathfrak{h}^* | \langle h_\alpha, \lambda \rangle = 0\}.$$
Then
$$\omega_\alpha(\lambda) = \lambda - 2\frac{\lambda(h_\alpha)}{\alpha(h_\alpha)}\alpha$$
defines a reflection on the hyperplane Ω_α. The group W generated by the ω_α is called the **Weyl group**. It can be shown that the set of weights of any representation of $\mathfrak{g}$ is invariant under W.
8. Choose a direction in $\mathfrak{h}^*$ by specifying a real linear functional l and decomposing $R = R^+ \cup R^-$ where $R^+ = \{\alpha \in R | l(\alpha) > 0\}$. The elements in R^+ are called the positive roots and the elements in R^- are called the negative roots. $\alpha \in R^+$ is **simple** if it cannot be written as the sum of two positive roots. Denote the simple roots by α_i with $i \in I = \{1, 2, \ldots, n\}$.

Example 9. For a suitable l the positive roots of $\mathfrak{sl}_n$ are $\{L_i - L_j | i < j\}$ and the simple roots are $\{L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n\}$.

9. Let V be a representation of $\mathfrak{g}$. Then $v \in V$ is a **highest weight vector** if $e_\alpha v = 0$ for all $\alpha \in R^+$ and v is an eigenvector of $\mathfrak{h}$. The weight of v is called a **dominant weight**. Each dominant weight λ can be written as $\sum_{i \in I} a_i \Lambda_i$ with nonnegative integers a_i. The Λ_i are the **fundamental weights** defined by
$$\langle h_{\alpha_j}, \Lambda_i \rangle = \delta_{ij}.$$

Hence $P = \sum_{i \in I} \mathbb{Z}\Lambda_i$. There is a bilinear symmetric form $(\cdot,\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{R}$ such that $\langle h_i, \lambda \rangle = 2\frac{(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$. The **Cartan matrix** $A = (a_{ij})$ of $\mathfrak{g}$ is defined as $a_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ (see [1] p. 334).

Theorem 10. (see [1] p. 203) Let $\mathfrak{g}$ be a semisimple Lie algebra.

1. Every finite-dimensional representation V of $\mathfrak{g}$ possesses a highest weight vector.
2. Every irreducible representation possesses a unique highest weight vector (up to scalars).

4. Classification of simple Lie algebras

Here is the list of all classical simple Lie algebras (see [1] p. 324 and 326).

		simple roots	Dynkin diagram	
A_n	$\mathfrak{sl}_{n+1}$	$L_1 - L_2, \dots, L_n - L_{n+1}$		$n \geq 1$
B_n	$\mathfrak{so}_{2n+1}$	$L_1 - L_2, \dots, L_{n-1} - L_n, L_n$		$n \geq 2$
C_n	$\mathfrak{sp}_{2n}$	$L_1 - L_2, \dots, L_{n-1} - L_n, 2L_n$		$n \geq 3$
D_n	$\mathfrak{so}_{2n}$	$L_1 - L_2, \dots, L_{n-1} - L_n, L_{n-1} + L_n$		$n \geq 4$

References

[1] W. Fulton and J. Harris, Representation Theory, Springer-Verlag, New York Berlin Heidelberg, 1991.

[2] J. E. Humphreys, Introduction to Lie algebras and representation theory, Springer-Verlag, New York, 1972,1981.