Handout for 18.735 on December 4, 2000

BIJECTION BETWEEN SEMISTANDARD TABLEAUX AND RIGGED CONFIGURATIONS

rig (15th century)

(1) to fit out (as a ship)
(2) clothe, dress
(3) to furnish with special gear - equip
(4) to put in condition or position for use - adjust, arrange, construct

rig (1851)

(1) manipulate or control usually by deceptive or dishonest means
(2) to fix in advance for a desired result

1. Rigged configurations

Let λ and μ be partitions (the following definitions also work when μ is a composition).

A (λ,μ)-**configuration** is a sequence of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \ldots)$ such that

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j.$$

For a partition ρ we denote by $m_n(\rho)$ the number of parts of ρ of size n. Also, given a (λ,μ)-configuration ν define the **vacancy number**

$$P_n^{(k)}(\nu) = Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)})$$

for all $n, k \geq 1$ where $\nu^{(0)} = \mu$ and $Q_n(\rho) = \rho_1^t \cdots + \rho_n^t$ is the sum of the first n columns of the partition ρ.

We say that a (λ,μ)-configuration is **admissible** if $P_n^{(k)}(\nu) \geq 0$ for all $n, k \geq 1$. The set of all admissible (λ,μ)-configurations is denoted by $\mathrm{C}(\lambda,\mu)$.

A partition can be viewed as a multiset of positive integers. A **rigged partition** is a multiset of pairs (n,x) where $n \in \mathbb{Z}_{>0}$ and $x \in \mathbb{Z}_{\geq 0}$. The pair (n,x) is called a **string**, n is the **length** of the string and x is the **rigging** or **quantum number**.

A (λ,μ)-**rigged configuration** is a sequence of rigged partitions

$$(\nu, J) = ((\nu,J)^{(1)}, (\nu,J)^{(2)}, \ldots)$$

where $\nu \in \mathrm{C}(\lambda,\mu)$ and

$$0 \leq x \leq P_n^{(k)}(\nu) \qquad \text{for all } (n,x) \in (\nu,J)^{(k)}.$$

The set of all (λ,μ)-rigged configurations is denoted by $\mathrm{RC}(\lambda,\mu)$.

A **singular string** of a rigged configuration (ν,J) is a string $(n,x) \in (\nu,J)^{(k)}$ such that its rigging attains its maximal value, that is, $x = P_n^{(k)}(\nu)$.

2. Bijection between tableaux and rigged configurations

Motivated by Bethe Ansatz calculations, Kirillov and Reshetikhin [1] defined an algorithm for a bijection between semistandard tableaux and rigged configurations. The bijection $\phi : \mathrm{SST}(\lambda,\mu) \to \mathrm{RC}(\lambda,\mu)$ is defined recursively by the following algorithm:

(1) $\phi(\emptyset) = \emptyset$

(2) Let $T \in \mathrm{SST}(\lambda,\mu)$ and $\overline{T}$ obtained from T by removing the rightmost largest letter. Suppose that $(\overline{\nu},\overline{J}) = \phi(\overline{T})$ is already known. Then $(\nu,J) = \phi(T)$ is obtained from $(\overline{\nu},\overline{J})$ as follows:

- Suppose the box in $T\backslash\overline{T}$ is in row r. Successively select singular strings in $(\overline{\nu},\overline{J})^{(r-1)}, (\overline{\nu},\overline{J})^{(r-2)}, \dots, (\overline{\nu},\overline{J})^{(1)}$ of length $\ell^{(r-1)}, \ell^{(r-2)}, \dots, \ell^{(1)}$, respectively, such that all $\ell^{(j)}$ are largest subject to the constraint
$$\ell^{(r-1)} \geq \ell^{(r-2)} \geq \cdots \geq \ell^{(1)} \geq 0.$$
If no singular string exists in $(\overline{\nu},\overline{J})^{(j)}$ set $\ell^{(j)} = 0$.
- Add a box to each of the selected strings in $(\overline{\nu},\overline{J})$ and make them singular again. If $\ell^{(j)} = 0$ add a new part of length 1 to $(\overline{\nu},\overline{J})^{(j)}$.

It is not completely trivial to see that the above algorithm indeed defines a bijection. This was proven in [2] (the results in that paper are in fact more general and prove that there is a bijection between Littlewood–Richardson tableaux and rigged configurations; the definitions of section 1 need to be slightly amended to get the more general result).

Theorem 1 ([2])**.** The map $\phi : \mathrm{SST}(\lambda,\mu) \to \mathrm{RC}(\lambda,\mu)$ is a bijection and $\mathrm{co}(t) = \mathrm{cc}(\phi(t))$ for all $t \in \mathrm{SST}(\lambda,\mu)$ where
$$\begin{aligned} \mathrm{cc}(\nu,J) &= \mathrm{cc}(\nu) + |J| \\ \mathrm{cc}(\nu) &= \sum_{k,i\geq 1} \alpha_i^{(k)}(\alpha_i^{(k)} - \alpha_i^{(k+1)}) \end{aligned}$$
and $\alpha_i^{(k)}$ is the i-th column of $\nu^{(k)}$ and $|J|$ denotes the sum of all riggings in (ν,J).

References

[1] A. N. Kirillov, N. Y. Reshetikhin, *The Bethe Ansatz and the combinatorics of Young tableaux*, J. Soviet. Math. **41** (1988) 925–955.

[2] A. N. Kirillov, A. Schilling, M. Shimozono, *A bijection between Littlewood–Richardson tableaux and rigged configurations*, preprint math.QA/9901037.