$\underline{\underline{\text{WTS}}}$ $\ker \hat{\Phi}_A \ni \ell \iff \sum \ell_i m_i = 0$, i.e., $\sum_i \ell_i m_{ij} = 0$ $\quad m_i = (m_{i1} \dots m_{in})$

- $A = \{m_1 \dots m_s\} \subseteq M \cong \mathbb{Z}^n$; M = char. latt of T, some n-torus.
- $\Phi_A : T \to \mathbb{C}^s$

  $t \mapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t))$

  clearly a gp hom.
  Actually since $\chi^{m_i}(t) \in \mathbb{C}^*$ we can think of Φ_A as a map $T \to (\mathbb{C}^*)^s = T'$

  - <u>PROP 1.1.1 (a)</u>. T, T' tori, $\Phi : T \to T'$ gp morphism.

    $\Rightarrow$ $\operatorname{im}\Phi \cong (\mathbb{C}^*)^k$ and $\operatorname{im}\Phi \leq T'$ closed.

i.e., we have a triangle

$$\begin{tikzcd} T \arrow[r, "\Phi_A"] \arrow[dr, hook] & (\mathbb{C}^*)^s \subseteq \mathbb{C}^s \\ & \operatorname{im}\Phi_A = T'' \arrow[u, hook] \end{tikzcd}$$

i.e., $\hat{\Phi}_A(\ell) = \ell \circ \Phi_A : T \to (\mathbb{C}^*)^s \to \mathbb{C}^*$!!!

Apply $\operatorname{Hom}(-, \mathbb{C}^*)$: (contravariant functor!)

$$\begin{tikzcd} M = \operatorname{Hom}(T, \mathbb{C}^*) & \operatorname{Hom}((\mathbb{C}^*)^s, \mathbb{C}^*) = \mathbb{Z}^s \arrow[l, "\hat{\Phi}_A := \operatorname{Hom}(\Phi_A, \mathbb{C}^*)"'] \arrow[d] \\ & \operatorname{Hom}(T'', \mathbb{C}^*) =: M'' \arrow[ul] \end{tikzcd}$$

So $\ell \in \ker \hat{\Phi}_A$

[where ℓ refers to $\chi^\ell : (\mathbb{C}^*)^s \to \mathbb{C}^*$ $\quad$ ← Here we're using a remark that all characters arise this way.

(I'm abusing notation, sorry) $\quad (y_1, \dots, y_s) \longmapsto \prod y_i^{\ell_i}$]

where

$$\hat{\Phi}_A(\ell) : (t_1 \dots t_n) \overset{\Phi_A}{\longmapsto} \left(\prod t_j^{m_{1j}}, \dots, \prod t_j^{m_{sj}}\right) \overset{\ell = \chi^\ell}{\longmapsto} \prod t_j^{m_{1j}\ell_1} \cdots \prod t_j^{m_{sj}\ell_s}$$

$$= \prod_{j=1}^{n} t_j^{\sum_{i=1}^{s} \ell_i m_{ij}}$$

Therefore, need exponents to be zero to get $1 \in \mathbb{C}^*$.