**Sum and Product Game**

## 1. Introduction

The sum and product game starts as an almost paradoxical puzzle. There are three agents: Chooser and two honest perfectly logical players: Sum and Product.

Chooser chooses a pair of integers between two and one hundred and tells their sum to Sum and their product to Product. The players then alternate announcing whether they know what the pair is. After thirteen turns of announcing No one player announces Yes. What is the pair? The answer is eighty and eightyfour.

The key observation is that the information that one player does not know the pair can be helpful. The analysis boils down to studying the bipartite graph with $S$ nodes for the possible sum values, $P$ nodes for the possible product values and an edge for each pair of numbers. Thus there are edges $P10 - S7 - P12$ arising from the pairs $\{2, 5\}$ and $\{3, 4\}$ since the first pair has product ten and sum seven while the second also has sum seven but product twelve. As the game progresses with players saying No or Yes an observer can keep track of a game state which is a subgraph of the above with impossible nodes removed. The starting state is the entire graph. When a player announces No all of their nodes which are leaves are deleted (pruned) and when they announce Yes their nonleaf nodes are deleted. The longest possible sequence of alternating No steps before a Yes is the number of pruning steps which leave a leaf-free graph. This graph is finite and direct computation shows that after thirteen prunings there is a single leaf product node $P6720$ remaining and after Product announces Yes and nonleaf product nodes are removed the state graph is just the edge $S164 - P6720$ arising from the pair $\{80, 84\}$.

The game becomes interesting when the maximum number one hundred is replaced by $n$. The analysis is exactly as above with starting graph $G(n)$ which becomes leaf-free after $p(n)$ pruning steps. From above $p(100) = 13$. The fundamental sum-product question is:

**Conjecture:** $p(n)$ is unbounded.

## 2. approaches

2.1. **Pseudorandom.** One approach is to consider the graph process $G(n)$ as pseudorandom and try to guess how $p(n)$ behaves but this remains to be explored.

2.2. **Kites.** More directly the question amounts to looking for witness subgraphs of $G(n)$. A kite in $G$ is a subgraph of $G$ which is the union of a cycle and a path (tail) with one end connected to a vertex of the cycle for which no tail node is in any cycle of $G$. As long as $n$ is at least ten $G(n)$ has a cycle and $p(n)$ is just the largest tail length of a kite in $G(n)$.

**Lemma:** If $v > 10$ and $Sv$ is in a tail of a kite in $G(n)$ then

(1) $v < 2n - 2\sqrt{2n+7} + 5$.
(2) $v > 2n - 20n^{\frac{3}{4}}$ if also $v > 30$ and $n > 10000$.

The lower bound is quite rough and $Sv$ is in a length four cycle (diamond) for smaller $v$. If $v$ is above the upper bound then $Sv$ is the center of a star component of $G(n)$ and this is sharper.

As an example of the upper bound consider $n = 9$ so $13 = 2n - 2\sqrt{2n+7}+5$. Here $S18-P81$, $S17-P72$, $P63-S16-P64$, $P54-S15-P56$ and $S14$ connected to $P45$, $P48$ and $P49$ are star components of $G(9)$ since for each of the $Ps$ listed above $s$ factors uniquely into a pair of numbers between 2 and 9 (eg $48 = 8 \cdot 6$) so the lemma holds. Also $S13 - P36 - S12$ is contained in $G(9)$ since $4 \cdot 9 = 36 = 6 \cdot 6$ so $S13$ is not the center of a star component of $G(9)$.

2.3. **Varieties.** While it is immediate that the diamonds are the positive integer points of a quadratic variety in eight variables (the four number pairs) with four relations, a linear change of coordinates rewrites them as the positive integer and half integer points in the maximal torus of a singular affine toric fourfold. The correct momentum map associated to a branched cover takes a diamond $Pv - Sx - Pu - Sy - Pv$ in $G(n)$ to $\left(\sqrt{x+y}, \sqrt{|x-y|}, \sqrt{x^2-4u}, \sqrt{x^2-4v}\right)$. The projection to the first two coordinates falls in a first quadrant triangle:

**Lemma:**(Pan Lin) For a diamond in $G(n)$ as above
$\sqrt{x+y} + \sqrt{|x-y|} \leq 2\sqrt{n}$ and $\sqrt{x+y} - \sqrt{|x-y|} < \sqrt{30} - 2$.

The image of this momentum map appears to be fairly dense in a polyhedron though less so in the regime most relevant to the pruning conjecture where $\sqrt{x+y} \sim 2\sqrt{n}$ and $\sqrt{|x-y|} \sim 0$.