Lectures on Ordinary Differential Equations[1]

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Contents

Notation

Symbol	Definition of Symbol
$\mathbf{R}$	field of real numbers
$\mathbf{R}^n$	the n-dimensional vector space over the reals
$\mathbf{C}$	field of complex numbers
$\dot{x}$	the derivative dx/dt
$\ddot{x}$	the second derivative d^2x/dt^2
:=	equals by definition
ODE	ordinary differential equation
KE	kinetic energy
PE	potential energy
det	determinant
δ_{ij}	the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise
$\mathcal{L}$	the Laplace tranform operator

Chapter 1

The Mathematical Pendulum

Many interesing ordinary differential equations (ODEs) arise from applications. One reason for understanding these applications in a mathematics class is that you can combine your physical intuition with your mathematical intuition in the same problem. Usually the result is an improvement of both. One such application is the motion of a pendulum, i.e. a ball of mass m suspended from an ideal rigid rod that is fixed at one end. The problem is to describe the motion of the mass point in a constant gravitational field. Since this is a mathematics class we will not normally be interested in deriving the ODE from physical priniciples; rather, we will simply write down various differential equations and claim that they are "interesting." However, to give you the flavor of such derivations (which you will see repeatedly in your science and engingeering courses), we will derive from Newton's equations the differential equation that describes the time evolution of the angle of deflection of the pendulum.

Let

$$\begin{aligned} \ell &= \text{length of the rod measured, say, in meters,} \\ m &= \text{mass of the ball measured, say, in kilograms,} \\ g &= \text{accerleration due to gravity} = 9.8\, m/s^2. \end{aligned}$$

The motion of the pendulum is confined to a plane (this is an assumption on how the rod is attached to the pivot point), which we take to be the $x-y$ plane. We treat the ball as a "mass point" and observe there are two forces acting on this ball: the force due to gravity, mg, which acts vertically downward and the tension $\vec{T}$ in the rod (acting in the direction indicated in figure). Newton's equations for the motion of a point $\vec{x}$ in a plane are vector equations[1]

$$\vec{F} = m\vec{a}$$

where $\vec{F}$ is the sum of the forces acting on the the point and $\vec{a}$ is the acceleration of the point, i.e.

$$\vec{a} = \frac{d^2\vec{x}}{dt^2}.$$

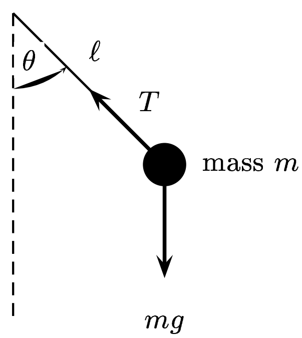
In $x - y$ coordinates Newton's equations become two equations

$$F_x = m\frac{d^2x}{dt^2}, \quad F_y = m\frac{d^2y}{dt^2},$$

where F_x and F_y are the x and y components, respectively, of the force $\vec{F}$. From the figure (note definition of the angle θ) we see, upon resolving $\vec{T}$ into its x and y components, that

$$F_x = -T\sin\theta, \quad F_y = T\cos\theta - mg.$$

(T is the magnitude of the vector $\vec{T}$.)



Substituting these expressions for the forces into Newton's equations, we obtain the differential equations

$$-T\sin\theta = m\frac{d^2x}{dt^2}, \tag{1.1}$$

$$T\cos\theta - mg = m\frac{d^2y}{dt^2}. \tag{1.2}$$

From the figure we see that

$$x = \ell\sin\theta, \; y = \ell - \ell\cos\theta. \tag{1.3}$$

[1] In your applied courses vectors are usually denoted with arrows above them. We adopt this notation when discussing certain applications; but in later chapters we will drop the arrows and state where the quantity lives, e.g. $x \in \mathbf{R}^2$.

(The origin of the $x-y$ plane is chosen so that at $x=y=0$, the pendulum is at the bottom.) Differentiating[2] (1.3) with respect to t, and then again, gives

$$\begin{aligned}
\dot{x} &= \ell\cos\theta\,\dot{\theta}, \\
\ddot{x} &= \ell\cos\theta\,\ddot{\theta} - \ell\sin\theta\,(\dot{\theta})^2, \qquad (1.4)\\
\dot{y} &= \ell\sin\theta\,\dot{\theta}, \\
\ddot{y} &= \ell\sin\theta\,\ddot{\theta} + \ell\cos\theta\,(\dot{\theta})^2. \qquad (1.5)
\end{aligned}$$

Substitute (1.4) in (1.1) and (1.5) in (1.2) to obtain

$$\begin{aligned}
-T\sin\theta &= m\ell\cos\theta\,\ddot{\theta} - m\ell\sin\theta\,(\dot{\theta})^2, \qquad (1.6)\\
T\cos\theta - mg &= m\ell\sin\theta\,\ddot{\theta} + m\ell\cos\theta\,(\dot{\theta})^2. \qquad (1.7)
\end{aligned}$$

Now multiply (1.6) by $\cos\theta$, (1.7) by $\sin\theta$, and add the two resulting equations to obtain

$$-mg\sin\theta = m\ell\ddot{\theta},$$

or

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.} \qquad (1.8)$$

Remarks

- The ODE (1.8) is called a second-order equation because the highest derivative appearing in the equation is a second derivative.
- The ODE is nonlinear because of the term $\sin\theta$ (this is not a linear function of the unknown quantity θ).
- A solution to this ODE is a function $\theta=\theta(t)$ such that when it is substituted into the ODE, the ODE is satisfied for all t.
- Observe that the mass m dropped out of the final equation. This says the motion will be independent of the mass of the ball.
- The derivation was constructed so that the tension, $\vec{T}$, was eliminated from the equations. We could do this because we started with two unknowns, T and θ, and two equations. We manipulated the equations so that in the end we had one equation for the unknown $\theta=\theta(t)$.

[2] We use the dot notation for time derivatives, e.g. $\dot{x}=dx/dt$, $\ddot{x}=d^2x/dt^2$.

- We have not discussed how the pendulum is initially started. This is very important and such conditions are called the *initial conditions.*

We will return to this ODE later in the course. At this point we note that if we were interested in only small deflections from the origin (this means we would have to start out near the origin), there is an obvious approximation to make. Recall from calculus the Taylor expansion of $\sin\theta$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots .$$

For small θ this leads to the approximation $\sin\theta \approx \theta$. Using this small deflection approximation in (1.8) leads to the ODE

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\,\theta = 0.} \qquad (1.9)$$

We will see that (1.9) is mathematically simpler than (1.8). The reason for this is that (1.9) is a linear ODE. It is linear because the unknown quantity, θ, and its derivatives appear only to the first or zeroth power.

Chapter 2

First Order Equations

2.1 Linear First Order Equations

2.1.1 Introduction

The simplest differential equation is one you already know from calculus; namely,

$$\frac{dy}{dx} = f(x). \tag{2.1}$$

To find a solution to this equation means one finds a function $y = y(x)$ such that its derivative, dy/dx, is equal to $f(x)$. The fundamental theorem of calculus tells us that all solutions to this equation are of the form

$$y(x) = y_0 + \int_{x_0}^{x} f(s)\, ds.$$

Observe

- $y(x_0) = y_0$ and y_0 is arbitrary. That is, there is a one-parameter family of solutions; $y = y(x; y_0)$ to (2.1). The solution is unique once we specify the initial condition $y(x_0) = y_0$. This is the solution to the initial value problem. That is, we have found a function that satisfies both the ODE and the initial value condition.
- Every calculus students knows that differentiation is easier than integration. Observe that solving a differential equation is like integration—you must find a function such that when it and its derivatives are substituted into the equation the equation is identically satisfied. Thus we sometimes say we "integrate" a differential equation. In the above case it is exactly integration as you understand it from calculus. This also suggests that solving differential equations can be expected to be difficult.

A simple generalization of (2.1) is to replace the right-hand side by a function that depends upon both x and y

$$\frac{dy}{dx} = f(x,y).$$

Some examples are $f(x,y) = xy^2$, $f(x,y) = y$, and the case (2.1). The simplest choice in terms of the y dependence is for $f(x,y)$ to depend linearly on y. Thus we are led to study

$$\frac{dy}{dx} = g(x) - p(x)y,$$

where $g(x)$ and $p(x)$ are functions of x. We leave them unspecified. (We have put the minus sign into our equation to conform with the standard notation.) The conventional way to write this equation is

$$\boxed{\frac{dy}{dx} + p(x)y = g(x).} \tag{2.2}$$

It's possible to give an algorithm to solve this ODE for more or less general choices of $p(x)$ and $g(x)$. We say more or less since one has to put some restrictions on p and g—that they are continuous will suffice. It should be stressed at the outset that this ability to find an explicit algorithm to solve an ODE is the exception—most ODEs encountered will not be so easily solved.

2.1.2 Method of Integrating Factors

If (2.2) were of the form (2.1), then we could immediately write down a solution in terms of integrals. For (2.2) to be of the form (2.1) means the left-hand side is expressed as the derivative of our unknown quantity. We have some freedom in making this happen—for instance, we can multiply (2.2) by a function, call it $\mu(x)$, and ask whether the resulting equation can be put in form (2.1). Namely, is

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \frac{d}{dx}\left(\mu(x)y\right)\,? \tag{2.3}$$

Taking derivatives we ask can μ be chosen so that

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x) = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y$$

holds? This immediately simplifies to[1]

$$\mu(x)p(x) = \frac{d\mu}{dx},$$

[1] Notice y and its first derivative drop out. This is a good thing since we wouldn't want to express μ in terms of the unknown quantity y.

or

$$\frac{d}{dx} \log \mu(x) = p(x).$$

Integrating this last equation gives

$$\log \mu(x) = \int p(s)\, ds + c.$$

Taking the exponential of both sides (one can check later that there is no loss in generality if we set $c = 0$) gives

$$\mu(x) = \exp\left(\int^x p(s)\, ds\right). \tag{2.4}$$

Defining $\mu(x)$ by (2.4), the differential equation (2.3) is transformed to

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x).$$

This last equation is precisely of the form (2.1), so we can immediately conclude

$$\mu(x)y(x) = \int^x \mu(s)g(s)\, ds + c,$$

or

$$\boxed{y(x) = \frac{1}{\mu(x)} \int^x \mu(s)g(s)\, ds \; + \; \frac{c}{\mu(x)}\,,} \tag{2.5}$$

where $\mu(x)$, called the *integrating factor*, is defined by (2.4). The constant c will be determined from the initial condition $y(x_0) = y_0$.

2.1.3 Application to Mortgage Payments

Suppose an amount P, called the principal, is borrowed at an interest I (100I%) for a period of N years. One is to make monthly payments in the amount $D/12$ (D equals the amount paid in one year). The problem is to find D in terms of P, I and N. Let

$$y(t) = \text{amount owed at time } t \text{ (measured in years).}$$

We have the initial condition

$$y(0) = P \text{ (at time 0 the amount owed is } P).$$

We are given the additional information that the loan is to be paid off at the end of N years,

$$y(N) = 0.$$

We want to derive an ODE satisfied by y. Let Δt denote a small interval of time and Δy the change in the amount owed during the time interval Δt. This change is determined by

- Δy is increased by compounding at interest I; that is, Δy is increased by the amount $I y(t)\Delta t$.
- Δy is decreased by the amount paid back in the time interval Δt. If D denotes this constant rate of payback, then $D\Delta t$ is the amount paid back in the time interval Δt.

Thus we have

$$\Delta y = I y \Delta t - D\Delta t,$$

or

$$\frac{\Delta y}{\Delta t} = I y - D.$$

Letting $\Delta t \to 0$ we obtain the sought after ODE,

$$\frac{dy}{dt} = I y - D. \tag{2.6}$$

This ODE is of form (2.2) with $p = -I$ and $g = -D$. One immediately observes that this ODE is not exactly what we assumed above, i.e. D is not known to us. Let us go ahead and solve this equation for any constant D by the method of integrating factors. So we choose μ according to (2.4),

$$\begin{aligned}\mu(t) &:= \exp\left(\int^t p(s)\, ds\right)\\ &= \exp\left(-\int^t I\, ds\right)\\ &= \exp(-It).\end{aligned}$$

Applying (2.5) gives

$$\begin{aligned}y(t) &= \frac{1}{\mu(t)}\int^t \mu(s)g(s)\, ds + \frac{c}{\mu(t)}\\ &= e^{It}\int^t e^{-Is}(-D)\, ds + ce^{It}\\ &= -De^{It}\left(-\frac{1}{I}e^{-It}\right) + ce^{It}\\ &= \frac{D}{I} + ce^{It}.\end{aligned}$$

The constant c is fixed by requiring

$$y(0) = P,$$

that is

$$\frac{D}{I} + c = P.$$

Solving this for c gives $c = P - D/I$. Substituting this expression for c back into our solution $y(t)$ gives

$$y(t) = \frac{D}{I} - \left(\frac{D}{I} - P\right) e^{It}.$$

First observe that $y(t)$ *grows* if $D/I < P$. (This might be a good definition of loan sharking!) We have not yet determined D. To do so we use the condition that the loan is to be paid off at the end of N years, $y(N) = 0$. Substituting $t = N$ into our solution $y(t)$ and using this condition gives

$$0 = \frac{D}{I} - \left(\frac{D}{I} - P\right) e^{NI}.$$

Solving for D,

$$D = PI\,\frac{e^{NI}}{e^{NI} - 1}, \tag{2.7}$$

gives the sought after relation between D, P, I and N. For example, if $P = \$100,000$, $I = 0.06$ (6% interest) and the loan is for $N = 30$ years, then $D = \$7,188.20$ so the monthly payment is $D/12 = \$599.02$. Some years ago the mortgage rate was 12%. A quick calculation shows that the monthly payment on the same loan at this interest would have been \$1028.09.

We remark that this model is a continuous model—the rate of payback is at the continuous rate D. In fact, normally one pays back only monthly. Banks, therefore, might want to take this into account in their calculations. I've found from personal experience that the above model predicts the bank's calculations to within a few dollars.

Suppose we increase our monthly payments by, say, \$50. (We assume no prepayment penalty.) This \$50 goes then to paying off the principal. The problem then is how long does it take to pay off the loan? It is an exercise to show that the number of years is (D is the total payment in one year)

$$-\frac{1}{I}\log\left(1 - \frac{PI}{D}\right). \tag{2.8}$$

Another questions asks on a loan of N years at interest I how long does it take to pay off one-half of the principal? That is, we are asking for the time T when

$$y(T) = \frac{P}{2}.$$

It is an exercise to show that

$$T = \frac{1}{I} \log\left(\frac{1}{2}(e^{NI} + 1)\right). \tag{2.9}$$

For example, a 30 year loan at 9% is half paid off in the 23rd year. Notice that T does not depend upon the principal P.

2.2 Separation of Variables Applied to Mechanics

2.2.1 Energy Conservation

Consider the motion of a particle of mass m in one dimension, i.e. the motion is along a line. We suppose that the force acting at a point x, $F(x)$, is *conservative.* This means there exists a function $V(x)$, called the *potential energy*, such that

$$F(x) = -\frac{dV}{dx}.$$

(Tradition has it we put in a minus sign.) In one dimension this requires that F is only a function of x and not $\dot{x}$ ($= dx/dt$) which physically means there is no friction. In higher spatial dimensions the requirement that $\vec{F}$ is conservative is more stringent. The concept of *conservation of energy* is that

$$E = \text{Kinetic energy } + \text{ Potential energy}$$

does not change with time as the particle's position and velocity evolves according to Newton's equations. We now prove this fundamental fact. We recall from elementary physics that the kinetic energy (KE) is given by

$$\text{KE} = \frac{1}{2} m v^2, \;\; v = \text{velocity} = \dot{x}.$$

Thus the energy is

$$E = E(x, \dot{x}) = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + V(x).$$

To show that $E = E(x, \dot{x})$ does not change with t when $x = x(t)$ satisfies Newton's equations, we differentiate E with respect to t and show the result is zero:

$$\begin{aligned}\frac{dE}{dt} &= m\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dV}{dx}\frac{dx}{dt} \quad \text{(by the chain rule)} \\ &= \frac{dx}{dt}\left(m\frac{d^2x}{dt^2} + \frac{dV(x)}{dx}\right) \\ &= \frac{dx}{dt}\left(m\frac{d^2x}{dt^2} - F(x)\right).\end{aligned}$$

Now not any function $x = x(t)$ describes the motion of the particle—$x(t)$ must satisfy

$$F = m\frac{d^2x}{dt^2},$$

and we now get the desired result

$$\frac{dE}{dt} = 0.$$

This implies that E is constant on solutions to Newton's equations.

We now use energy conservation and what we know about separation of variables to solve the problem of the motion of a point particle in a potential $V(x)$. Now

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) \tag{2.10}$$

is a nonlinear first order differential equation. (We know it is nonlinear since the first derivative is squared.) We rewrite the above equation as

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\left(E - V(x)\right),$$

or

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V(x)\right)}.$$

(In what follows we take the $+$ sign, but in specific applications one must keep in mind the possibility that the $-$ sign is the correct choice of the square root.) This last equation is of the form in which we can separate variables. We do this to obtain

$$\frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}} = dt.$$

This can be integrated to

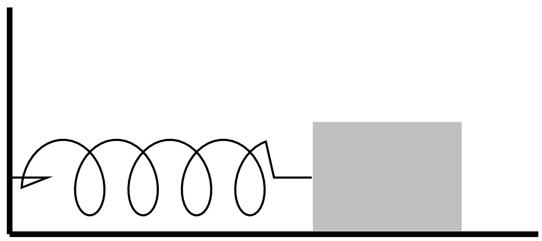
$$\boxed{\pm\int \frac{1}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}}\, dx \;=\; t - t_0.} \tag{2.11}$$

2.2.2 Hooke's Law

Consider a particle of mass m subject to the force

$$F = -kx,\ k > 0,\ \text{(Hooke's Law)}. \tag{2.12}$$

The minus sign (with $k > 0$) means the force is a restoring force—as in a spring. Indeed, to a good approximation the force a spring exerts on a particle is given by Hooke's Law. In this case $x = x(t)$ measures the displacement from the equilibrium position at time t; and the constant k is called the spring constant. Larger values of k correspond to a stiffer spring.



Newton's equations are in this case

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{2.13}$$

This is a second order linear differential equation, the subject of the next chapter. However, we can use the energy conservation principle to derive an associated nonlinear first order equation as we discussed above. To do this, we first determine the potential corresponding to Hooke's force law.

One easily checks that the potential equals

$$V(x) = \frac{1}{2}\,k\,x^2.$$

(This potential is called the *harmonic potential.*) Let's substitute this particular V into (2.11):

$$\int \frac{1}{\sqrt{2E/m - kx^2/m}}\,dx = t - t_0. \tag{2.14}$$

Recall the indefinite integral

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{|a|}\right) + c.$$

Using this in (2.14) we obtain

$$\begin{aligned} \int \frac{1}{\sqrt{2E/m - kx^2/m}}\,dx &= \frac{1}{\sqrt{k/m}} \int \frac{dx}{\sqrt{2E/k - x^2}} \\ &= \frac{1}{\sqrt{k/m}} \arcsin\left(\frac{x}{\sqrt{2E/k}}\right) + c. \end{aligned}$$

Thus (2.14) becomes[2]

$$\arcsin\left(\frac{x}{\sqrt{2E/k}}\right) = \sqrt{\frac{k}{m}}\,t + c.$$

Taking the sine of both sides of this equation gives

$$\frac{x}{\sqrt{2E/k}} = \sin\left(\sqrt{\frac{k}{m}}\,t + c\right),$$

or

$$x(t) = \sqrt{\frac{2E}{k}}\,\sin\left(\sqrt{\frac{k}{m}}\,t + c\right). \tag{2.15}$$

Observe that there are two constants appearing in (2.15), E and c. Suppose one initial condition is

$$x(0) = x_0.$$

Evaluating (2.15) at $t = 0$ gives

$$x_0 = \sqrt{\frac{2E}{k}}\,\sin(c). \tag{2.16}$$

Now use the sine addition formula,

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\,\cos\theta_2 + \sin\theta_2\,\cos\theta_1,$$

in (2.14):

$$\begin{aligned} x(t) &= \sqrt{\frac{2E}{k}}\left\{\sin\left(\sqrt{\frac{k}{m}}\,t\right)\cos c + \cos\left(\sqrt{\frac{k}{m}}\,t\right)\sin c\right\} \\ &= \sqrt{\frac{2E}{k}}\,\sin\left(\sqrt{\frac{k}{m}}\,t\right)\cos c + x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right) \end{aligned} \tag{2.17}$$

where we use (2.16) to get the last equality.

Now substitute $t = 0$ into the energy conservation equation,

$$E = \frac{1}{2}\,mv_0^2 + V(x_0) = \frac{1}{2}\,mv_0^2 + \frac{1}{2}\,k\,x_0^2.$$

(v_0 equals the velocity of the particle at time $t = 0$.) Substituting (2.16) in the right hand side of this equation gives

$$E = \frac{1}{2}\,mv_0^2 + \frac{1}{2}\,k\,\frac{2E}{k}\,\sin^2 c$$

[2]We use the same symbol c for yet another unknown constant.

or

$$E(1-\sin^2 c) = \frac{1}{2}mv_0^2.$$

Recalling the trig identity $\sin^2\theta + \cos^2\theta = 1$, this last equation can be written as

$$E\cos^2 c = \frac{1}{2}mv_0^2.$$

Solve this for v_0 to obtain the identity

$$v_0 = \sqrt{\frac{2E}{m}}\cos c.$$

We now use this in (2.17)

$$x(t) = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) + x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right).$$

To summarize, we have eliminated the two constants E and c in favor of the contants x_0 and v_0. As it must be, $x(0) = x_0$ and $\dot{x}(0) = v_0$. The last equation is more easily interpreted if we define

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{2.18}$$

Observe that ω_0 has the units of 1/time, i.e. frequency. Thus our final expression for the position $x = x(t)$ of a particle of mass m subject to Hooke's Law is

$$\boxed{x(t) = \frac{v_0}{\omega_0}\sin(\omega_0 t) + x_0\cos(\omega_0 t).} \tag{2.19}$$

Observe that this solution depends upon two arbitrary constants, x_0 and v_0.[3] In (2.5), the general solution depended only upon one constant. It is a general fact that the number of independent constants appearing in the general solution of a nth order[4] ODE is n.

[3] ω_0 is a contant too, but it is a parameter appearing in the differential equation that is fixed by the mass m and the spring constant k. Observe that we can rewrite (2.13) as

$$\ddot{x} + \omega_0^2 x = 0. \tag{2.20}$$

Dimensionally this equation is pleasing: $\ddot{x}$ has the dimensions of d/t^2 (d is distance and t is time) and so does $\omega_0^2\,x$ since ω_0 is a frequency. It is instructive to substitute (2.19) into (2.20) and verify directly that it is a solution. Please do so!

[4] The order of a scalar differential equation is equal to the order of the highest derivative appearing in the equation. Thus (2.2) is first order whereas (2.13) is second order.

Period of Mass-Spring System Satisfying Hooke's Law

The sine and cosine are periodic functions of period 2π, i.e.

$$\sin(\theta + 2\pi) = \sin\theta,\ \ \cos(\theta + 2\pi) = \cos\theta.$$

This implies that our solution $x = x(t)$ is periodic in time,

$$x(t + T) = x(t),$$

where the period T is

$$\boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}.} \tag{2.21}$$

2.2.3 Period of the Nonlinear Pendulum

In this section we use the method of separation of variables to derive an exact formula for the period of the pendulum. Recall that the ODE describing the time evolution of the angle of deflection, θ, is (1.8). This ODE is a second order equation and so the method of separation of variables does not apply to this equation. However, we will use energy conservation in a manner similar to the previous section on Hooke's Law.

To get some idea of what we should expect, first recall the approximation we derived for small deflection angles, (1.9). Comparing this differential equation with (2.13), we see that under the identification $x \to \theta$ and $\frac{k}{m} \to \frac{g}{\ell}$, the two equations are identical. Thus using the period derived in the last section, (2.21), we get as an approximation to the period of the pendulum

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\ell}{g}}. \tag{2.22}$$

An important feature of T_0 is that it does not depend upon the amplitude of the oscillation.[5] That is, suppose we have the initial conditions[6]

$$\theta(0) = \theta_0,\ \ \dot{\theta}(0) = 0, \tag{2.23}$$

[5] Of course, its validity is only for small oscillations.

[6] For simplicity we assume the initial angular velocity is zero, $\dot{\theta}(0) = 0$. This is the usual initial condition for a pendulum.

then T_0 does not depend upon θ_0. We now proceed to derive our formula for the period, T, of the pendulum.

We claim that the energy of the pendulum is given by

$$E = E(\theta, \dot{\theta}) = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell(1 - \cos\theta). \tag{2.24}$$

Proof of (2.24)

We begin with

$$\begin{aligned} E &= \text{Kinetic energy } + \text{ Potential energy} \\ &= \frac{1}{2} mv^2 + mgy. \end{aligned} \tag{2.25}$$

(This last equality uses the fact that the potential at height h in a constant gravitational force field is mgh. In the pendulum problem with our choice of coordinates $h = y$.) The x and y coordinates of the pendulum ball are, in terms of the angle of deflection θ, given by

$$x = \ell \sin\theta, \quad y = \ell(1 - \cos\theta).$$

Differentiating with respect to t gives

$$\dot{x} = \ell \cos\theta \, \dot{\theta}, \quad \dot{y} = \ell \sin\theta \, \dot{\theta},$$

from which it follows that the velocity is given by

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 \\ &= \ell^2 \dot{\theta}^2. \end{aligned}$$

Substituting these in (2.25) gives (2.24).

The energy conservation theorem states that for solutions $\theta(t)$ of (1.8), $E(\theta(t), \dot{\theta}(t))$ is independent of t. Thus we can evaluate E at $t = 0$ using the initial conditions (2.23) and know that for subsequent t the value of E remains unchanged,

$$\begin{aligned} E &= hf \, m\ell^2 \, \dot{\theta}(0)^2 + mg\ell \, (1 - \cos\theta(0)) \\ &= mg\ell(1 - \cos\theta_0). \end{aligned}$$

Using this (2.24) becomes

$$mg\ell(1 - \cos\theta_0) = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell(1 - \cos\theta),$$

which can be rewritten as

$$\frac{1}{2} m\ell^2 \dot{\theta}^2 = mg\ell(\cos\theta - \cos\theta_0).$$

Solving for $\dot{\theta}$,

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}\left(\cos\theta - \cos\theta_0\right)}\,,$$

followed by separating variables gives

$$\frac{d\theta}{\sqrt{\frac{2g}{\ell}\left(\cos\theta - \cos\theta_0\right)}} = dt. \tag{2.26}$$

We now integrate (2.26). The next step is a bit tricky—to choose the limits of integration in such a way that the integral on the right hand side of (2.26) is related to the period T. By the definition of the period, T is the time elapsed from $t = 0$ when $\theta = \theta_0$ to the time T when θ first returns to the point θ_0. By symmetry, $T/2$ is the time it takes the pendulum to go from θ_0 to $-\theta_0$. Thus if we integrate the left hand side of (2.26) from $-\theta_0$ to θ_0 the time elapsed is $T/2$. That is,

$$\frac{1}{2}T = \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{\ell}\left(\cos\theta - \cos\theta_0\right)}}\,.$$

Since the integrand is an even function of θ,

$$\boxed{T = 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{\ell}\left(\cos\theta - \cos\theta_0\right)}}\,.} \tag{2.27}$$

This is the sought after formula for the period of the pendulum. For small θ_0 we expect that T, as given by (2.27), should be approximately equal to T_0 (see (2.22)). It is instructive to see this precisely.

We now assume $|\theta_0| \ll 1$ so that the approximation

$$\cos\theta \approx 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4$$

is accurate for $|\theta| < \theta_0$. Using this approximation we see that

$$\begin{aligned} \cos\theta_0 - \cos\theta &\approx \frac{1}{2!}(\theta_0^2 - \theta^2) + \frac{1}{4!}(\theta_0^4 - \theta^4) \\ &= \frac{1}{2}(\theta_0^2 - \theta^2)\left(1 - \frac{1}{12}(\theta_0^2 + \theta^2)\right). \end{aligned}$$

From Taylor's formula we get the approximation, valid for $|x| \ll 1$,

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x.$$

Thus

$$\begin{aligned}
\frac{1}{\sqrt{\frac{2g}{\ell}\left(\cos\theta-\cos\theta_0\right)}} &\approx \sqrt{\frac{\ell}{g}}\frac{1}{\sqrt{\theta_0^2-\theta^2}}\frac{1}{\sqrt{1-\frac{1}{12}\left(\theta_0^2+\theta^2\right)}}\\
&\approx \sqrt{\frac{\ell}{g}}\frac{1}{\sqrt{\theta_0^2-\theta^2}}\left(1+\frac{1}{24}\left(\theta_0^2+\theta^2\right)\right).
\end{aligned}$$

Now substitute this approximate expression for the integrand appearing in (2.27) to find

$$\frac{T}{4}=\sqrt{\frac{\ell}{g}}\int_0^{\theta_0}\frac{1}{\sqrt{\theta_0^2-\theta^2}}\left(1+\frac{1}{24}\left(\theta_0^2+\theta^2\right)\right)+\text{higher order corrections.}$$

Make the change of variables $\theta=\theta_0 x$, then

$$\begin{aligned}
\int_0^{\theta_0}\frac{d\theta}{\sqrt{\theta_0^2-\theta^2}} &= \int_0^1\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{2},\\
\int_0^{\theta_0}\frac{\theta^2\,d\theta}{\sqrt{\theta_0^2-\theta^2}} &= \theta_0^2\int_0^1\frac{x^2\,dx}{\sqrt{1-x^2}}=\theta_0^2\frac{\pi}{4}.
\end{aligned}$$

Using these definite integrals we obtain

$$\begin{aligned}
\frac{T}{4} &= \sqrt{\frac{\ell}{g}}\left(\frac{\pi}{2}+\frac{1}{24}(\theta_0^2\frac{\pi}{2}+\theta_0^2\frac{\pi}{4})\right)\\
&= \sqrt{\frac{\ell}{g}}\frac{\pi}{2}\left(1+\frac{\theta_0^2}{16}\right)+\text{higher order terms.}
\end{aligned}$$

Recalling (2.22), we conclude

$$T=T_0\left(1+\frac{\theta_0^2}{16}+\cdots\right) \tag{2.28}$$

where the $\cdots$ represent the higher order correction terms coming from higher order terms in the expansion of the cosines. These higher order terms will involve higher powers of θ_0. It now follows from this last expression that

$$\lim_{\theta_0\to 0}T=T_0.$$

Observe that the first correction term to the linear result, T_0, depends upon the initial amplitude of oscillation θ_0.

Numerical Example

Suppose we have a pendulum of length $\ell = 1$ meter. The linear theory says that the period of the oscillation for such a pendulum is

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{1}{9.8}} = 2.0071 \text{ sec.}$$

If the amplitude of oscillation of the of the pendulum is $\theta_0 \approx 0.2$ (this corresponds to roughly a 20 cm deflection for the one meter pendulum), then (2.28) gives

$$T = T_0\left(1 + \frac{1}{16}(.2)^2\right) = 2.0121076 \text{ sec.}$$

One might think that these are so close that the correction is not needed. This might well be true if we were interested in only a few oscillations. What would be the difference in one week (1 week=604,800 sec)?

One might well ask how good an approximation is (2.28) to the exact result (2.27)? To answer this we have to evaluate numerically the integral appearing in (2.27). Evaluating (2.27) numerically (using say Mathematica's `NIntegrate`) is a bit tricky because the endpoint θ_0 is singular—an integrable singularity but it causes numerical integration routines some difficulty. Here's how you get around this problem. One isolates where the problem occurs—near θ_0—and takes care of this analytically. For $\varepsilon > 0$ and $\varepsilon \ll 1$ we decompose the integral into two integrals: one over the interval $(0, \theta_0 - \varepsilon)$ and the other one over the interval $(\theta_0 - \varepsilon, \theta_0)$. It's the integral over this second interval that we estimate analytically. Expanding the cosine function about the point θ_0, Taylor's formula gives

$$\cos\theta = \cos\theta_0 - \sin\theta_0\,(\theta - \theta_0) - \frac{\cos\theta_0}{2}(\theta - \theta_0)^2 + \cdots.$$

Thus

$$\cos\theta - \cos\theta_0 = \sin\theta_0\,(\theta - \theta_0)\left(1 - \frac{1}{2}\cot\theta_0\,(\theta - \theta_0)\right) + \cdots.$$

So

$$\begin{aligned} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}} &= \frac{1}{\sqrt{\sin\theta_0\,(\theta - \theta_0)}}\,\frac{1}{\sqrt{1 - \frac{1}{2}\cot\theta_0(\theta_0 - \theta)}} + \cdots \\ &= \frac{1}{\sqrt{\sin\theta_0\,(\theta_0 - \theta)}}\left(1 + \frac{1}{4}\cot\theta_0\,(\theta_0 - \theta)\right) + \cdots \end{aligned}$$

Thus

$$\begin{aligned} \int_{\theta_0-\varepsilon}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} &= \int_{\theta_0-\varepsilon}^{\theta_0} \frac{d\theta}{\sqrt{\sin\theta_0\,(\theta_0 - \theta)}}\left(1 + \frac{1}{4}\cot\theta_0\,(\theta - \theta_0)\right) d\theta + \cdots \\ &= \frac{1}{\sqrt{\sin\theta_0}}\left(\int_0^\varepsilon u^{-1/2}\,du + \frac{1}{4}\cot\theta_0\int_0^\varepsilon u^{1/2}\,du + \cdots\right) \quad (u := \theta_0 - \theta) \\ &= \frac{1}{\sqrt{\sin\theta_0}}\left(2\varepsilon^{1/2} + \frac{1}{6}\cot\theta_0\,\varepsilon^{3/2}\right) + \cdots. \end{aligned}$$

Choosing $\varepsilon = 10^{-2}$, the error we make in using the above expression is of order $\varepsilon^{5/2} = 10^{-5}$. Substituting $\theta_0 = 0.2$ and $\varepsilon = 10^{-2}$ into the above expression, we get the approximation

$$\int_{\theta_0-\varepsilon}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \approx 0.4506$$

where we estimate the error lies in fifth decimal place. Now any numerical integration routine can quickly evaluate the other integral:

$$\int_0^{\theta_0-\varepsilon} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \approx 1.7764$$

for $\theta_0 = 0.2$ and $\varepsilon = 10^{-2}$. Hence for $\theta_0 = 0.2$ we have

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \approx 0.4506 + 1.77664 = 2.2270$$

This implies

$$T \approx 2.0121.$$

Thus the first order approximation (2.28) is accurate to some four decimal places when $\theta_0 \leq 0.2$. (The reason for such good accuracy is that the correction term to (2.28) is of order θ_0^4.)

2.3 Exercises for Chapter 2

#1: Morgage Payment Problem

In the problem dealing with mortgate rates, prove (2.8) and (2.9).

#2. Application to Population Dynamics

In biological applications the population P of certain organisms at time t is sometimes assumed to obey the equation

$$\frac{dP}{dt} = aP\left(1 - \frac{P}{E}\right) \tag{2.29}$$

where a and E are positive constants.

1. Find the equilibrium solutions. (That is solutions that don't change with t.)

2. From (2.29) determine the regions of P where P is increasing (decreasing) as a function of t. Again using (2.29) find an expression for d^2P/dt^2 in terms of P and the constants a and E. From this expression find the regions of P where P is convex ($d^2P/dt^2 > 0$) and the regions where P is concave ($d^2P/dt^2 < 0$).

3. Using the method of separation of variables solve (2.29) for $P = P(t)$ assuming that at $t = 0$, $P = P_0 > 0$. Find

$$\lim_{t\to\infty} P(t)$$

Hint: To do the integration first use the identity

$$\frac{1}{P(1-P/E)} = \frac{1}{P} + \frac{1}{E-P}$$

4. Sketch P as a function of t for $0 < P_0 < E$ and for $E < P_0 < \infty$.

#3: Nonlinear Mass-Spring System

Consider a mass-spring system where $x = x(t)$ denotes the displacement of the mass m from its equilibrium position at time t. The linear spring (Hooke's Law) assumes the force exerted by the spring on the mass is given by (2.12). Suppose instead that the force F is given by

$$F = F(x) = -kx - \varepsilon\, x^3 \tag{2.30}$$

where ε is a small positive number.[7] The second term represents a nonlinear correction to Hooke's Law. Why is it reasonable to assume that the first correction term to Hooke's Law is of order x^3 and not x^2? (Hint: Why is it reasonable to assume $F(x)$ is an *odd* function of x?) Using the solution for the period of the pendulum as a guide, find an *exact* integral expression for the period T of this nonlinear mass-spring system assuming the initial conditions

$$x(0) = x_0,\ \frac{dx}{dt}(0) = 0.$$

Define

$$z = \frac{\varepsilon x_0^2}{2k}.$$

Show that z is dimensionless and that your expression for the period T can be written as

$$T = \frac{4}{\omega_0}\int_0^1 \frac{1}{\sqrt{1-u^2+z-zu^4}}\, du \tag{2.31}$$

[7]One could also consider $\varepsilon < 0$. The case $\varepsilon > 0$ is a called a *hard* spring and $\varepsilon < 0$ a *soft* spring.

where $\omega_0 = \sqrt{k/m}$. We now assume that $z \ll 1$. (This is the precise meaning of the parameter ε being small.) Taylor expand the function

$$\frac{1}{\sqrt{1-u^2+z-zu^4}}$$

in the variable z to first order. You should find

$$\frac{1}{\sqrt{1-u^2+z-zu^4}} = \frac{1}{\sqrt{1-u^2}} - \frac{1+u^2}{2\sqrt{1-u^2}}\, z + \mathrm{O}(z^2).$$

Now use this approximate expression in the integrand of (2.31), evaluate the definite integrals that arise, and show that the period T has the Taylor expansion

$$T = \frac{2\pi}{\omega_0}\left(1 - \frac{3}{4}z + \mathrm{O}(z^2)\right).$$

#4: Mass-Spring System with Friction

We reconsider the mass-spring system but now assume there is a frictional force present and this frictional force is proportional to the velocity of the particle. Thus the force acting on the particle comes from two terms: one due to the force exerted by the spring and the other due to the frictional force. Thus Newton's equations become

$$-kx - \beta\dot{x} = m\ddot{x} \tag{2.32}$$

where as before $x = x(t)$ is the displacement from the equilibrium position at time t. β and k are positive constants. Introduce the energy function

$$E = E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2, \tag{2.33}$$

and show that if $x = x(t)$ satisfies (2.32), then

$$\frac{dE}{dt} < 0.$$

What is the physical meaning of this last inequality?

#5: Motion in a Central Field

A (three-dimensional) force $\vec{F}$ is called a *central force* if the direction of $\vec{F}$ lies along the the direction of the position vector $\vec{r}$. This problem asks you to show that the motion of a particle in a central force, satisfying

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}, \tag{2.34}$$

lies in a plane.

1. Show that
$$\vec{M} := \vec{r} \times \vec{p} \;\text{ with }\; \vec{p} := m\vec{v} \tag{2.35}$$
is *constant* in t for $\vec{r} = \vec{r}(t)$ satisfying (2.34). (Here $\vec{v}$ is the velocity vector and $\vec{p}$ is the *momentum vector*.) The $\times$ in (2.35) is the vector cross product. Recall (and you may assume this result) from vector calculus that
$$\frac{d}{dt}(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}.$$
The vector $\vec{M}$ is called the *angular momentum* vector.

2. From the fact that $\vec{M}$ is a constant vector, show that the vector $\vec{r}(t)$ lies in a plane perpendicular to $\vec{M}$. Hint: Look at $\vec{r} \cdot \vec{M}$. Also you may find helpful the vector identity
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}).$$

#6: Motion in a Central Field (cont)

From the preceding problem we learned that the position vector $\vec{r}(t)$ for a particle moving in a central force lies in a plane. In this plane, let (r, θ) be the polar coordinates of the point $\vec{r}$, i.e.
$$x(t) = r(t)\cos\theta(t), \;\; y(t) = r(t)\sin\theta(t) \tag{2.36}$$

1. In components, Newton's equations can be written (why?)
$$F_x = f(r)\frac{x}{r} = m\ddot{x}, \;\; F_y = f(r)\frac{y}{r} = m\ddot{y} \tag{2.37}$$
where $f(r)$ is the magnitude of the force $\vec{F}$. By twice differentiating (2.36) with respect to t, derive formulas for $\ddot{x}$ and $\ddot{y}$ in terms of r, θ and their derivatives. Use these formulas in (2.37) to show that Newton's equations in polar coordinates (and for a central force) become
$$\frac{1}{m}f(r)\cos\theta = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta, \tag{2.38}$$
$$\frac{1}{m}f(r)\sin\theta = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta. \tag{2.39}$$
Multiply (2.38) by $\cos\theta$, (2.39) by $\sin\theta$, and add the resulting two equations to show that
$$\ddot{r} - r\dot{\theta}^2 = \frac{1}{m}f(r). \tag{2.40}$$

Now multiply (2.38) by $\sin\theta$, (2.39) by $\cos\theta$, and substract the resulting two equations to show that

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \tag{2.41}$$

Observe that the left hand side of (2.41) is equal to

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

Using this observation we then conclude (why?)

$$r^2\dot{\theta} = H \tag{2.42}$$

for some constant H. Use (2.42) to solve for $\dot{\theta}$, eliminate $\dot{\theta}$ in (2.40) to conclude that the polar coordinate function $r = r(t)$ satisfies

$$\ddot{r} = \frac{1}{m}f(r) + \frac{H^2}{r^3}. \tag{2.43}$$

2. Equation (2.43) is of the form that a second derivative of the unknown r is equal to some function of r. We can thus apply our general *energy method* to this equation. Let Φ be a function of r satisfying

$$\frac{1}{m}f(r) = -\frac{d\Phi}{dr},$$

and find an *effective potential* $V = V(r)$ such that (2.43) can be written as

$$\ddot{r} = -\frac{dV}{dr} \tag{2.44}$$

(Ans: $V(r) = \Phi(r) + \frac{H}{2r^2}$). Remark: The most famous choice for $f(r)$ is the inverse square law

$$f(r) = \frac{mMG_0}{r^2}$$

which describes the gravitational attraction of two particles of masses m and M. (G_0 is the universal gravitational constant.) In your physics courses, this case will be analyzed in great detail. The starting point is what we have done here.

Chapter 3

Second Order Linear Differential Equations

3.1 Theory of Second Order Equations

3.1.1 Vector Space of Solutions

First order linear differential equations are of the form

$$\frac{dy}{dx} + p(x)y = f(x). \tag{3.1}$$

Second order linear differential equations are linear differential equations whose highest derivative is second order:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x). \tag{3.2}$$

If $f(x) = 0$,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \tag{3.3}$$

the equation is called *homogeneous*. For the discussion here, we assume p and q are continuous functions on an interval $[a, b]$. There are many important examples where this condition fails and the points at which either p or q fail to be continuous are called *singular points*. Though the textbook discusses singular points, we will not take up this topic in this class. Here are some important examples where the continuity condition fails.

Legendre's Eqn

$$p(x) = -\frac{2x}{1-x^2}, \;\; q(x) = \frac{n(n+1)}{1-x^2}\,.$$

At the points $x = \pm 1$ both p and q fail to be continuous.

Bessel's Eqn

$$p(x) = \frac{1}{x}, \;\; q(x) = 1 - \frac{\nu^2}{x^2}\,.$$

At the point $x = 0$ both p and q fail to be continuous.

We saw that a solution to (3.1) was uniquely specified once we gave one initial condition,

$$y(x_0) = y_0.$$

In the case of second order equations we must give two initial conditions to specify uniquely a solution:

$$y(x_0) = y_0 \;\text{ and }\; y'(x_0) = y_1. \tag{3.4}$$

This is a basic theorem of the subject. It says that if p and q are continuous on some interval (a, b) and $a < x_0 < b$, then there exists an unique solution to (3.3) satisfying the initial conditions (3.4).[1] We will not prove this theorem in this class. As an example of the appearance to two constants in the general solution, recall that the solution of the harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0$$

contained x_0 and v_0.

Let $\mathcal{V}$ denote the set of all solutions to (3.3). The most important feature of $\mathcal{V}$ is that it is a *two-dimensional vector space*. That it is a vector space follows from the linearity of (3.3). (If y_1 and y_2 are solutions to (3.3), then so is $c_1 y_1 + c_2 y_2$ for all constants c_1 and c_2.) To prove that the dimension of $\mathcal{V}$ is two, we first introduce two special solutions. Let Y_1 and Y_2 be the unique solutions to (3.3) that satisfy the initial conditions

$$Y_1(0) = 1, \;\; Y_1'(0) = 0, \;\;\text{ and }\;\; Y_2(0) = 0, \;\; Y_2'(0) = 1,$$

respectively.

We claim that $\{Y_1, Y_2\}$ forms a basis for $\mathcal{V}$. To see this let $y(x)$ be any solution to (3.3).[2] Let $c_1 := y(0)$, $c_2 := y'(0)$ and

$$\Delta(x) := y(x) - c_1\, Y_1(x) - c_2\, Y_2(x).$$

Since y, Y_1 and Y_2 are solutions to (3.3), so too is Δ. ($\mathcal{V}$ is a vector space.) Observe

$$\Delta(0) = 0 \;\text{ and }\; \Delta'(0) = 0. \tag{3.5}$$

[1] See Theorem 3.2.1 in the textbook, pg. 131.

[2] We assume for convenience that $x = 0$ lies in the interval (a, b).

Now the function $y_0(x) :\equiv 0$ satisfies (3.3) and the initial conditions (3.5). Since solutions are unique, it follows that $\Delta(x) \equiv y_0 \equiv 0$. That is,

$$y = c_1 Y_1 + c_2 Y_2.$$

To summarize, we've shown every solution to (3.3) is a linear combination of Y_1 and Y_2. That Y_1 and Y_2 are linearly independent follows from their initial values: Suppose

$$c_1 Y_1(x) + c_2 Y_2(x) = 0.$$

Evaluate this at $x = 0$, use the initial conditions to see that $c_1 = 0$. Take the derivative of this equation, evaluate the resulting equation at $x = 0$ to see that $c_2 = 0$. Thus, Y_1 and Y_2 are linearly independent. We conclude, therefore, that $\{Y_1, Y_2\}$ is a basis and $\dim\mathcal{V} = 2$.

3.1.2 Wronskians

Given two solutions y_1 and y_2 of (3.3) it is useful to find a simple condition that tests whether they form a basis of $\mathcal{V}$. Let φ be the solution of (3.3) satisfying $\varphi(x_0) = \varphi_0$ and $\varphi'(x_0) = \varphi_1$. We ask are there constants c_1 and c_2 such that

$$\varphi(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x? A necessary and sufficient condition that such constants exist at $x = x_0$ is that the equations

$$\begin{aligned} \varphi_0 &= c_1\, y_1(x_0) + c_2\, y_2(x_0), \\ \varphi_1 &= c_1\, y'(x_0) + c_2\, y_2'(x_0), \end{aligned}$$

have a unique solution $\{c_1, c_2\}$. From linear algebra we know this holds if and only if the determinant

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0.$$

We define the Wronskian of two solutions y_1 and y_2 of (3.3) to be

$$W(y_1, y_2; x) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x). \end{vmatrix} = y_1(x) y_2'(x) - y_1'(x) y_2(x). \tag{3.6}$$

From what we have said so far one would have to check that $W(y_1, y_2; x) \neq 0$ for all x to conclude $\{y_1, y_2\}$ forms a basis.

We now derive a formula for the Wronskian that will make the check necessary at only one point. Since y_1 and y_2 are solutions of (3.3), we have

$$y_1'' + p(x) y_1' + q(x) y_1 = 0, \tag{3.7}$$

$$y_2'' + p(x) y_2' + q(x) y_2 = 0. \tag{3.8}$$

Now multiply (3.7) by y_2 and multiply (3.8) by y_1. Subtract the resulting two equations to obtain

$$y_1 y_2'' - y_1'' y_2 + p(x)\left(y_1 y_2' - y_1' y_2\right) = 0. \tag{3.9}$$

Recall the definition (3.6) and observe that

$$\frac{dW}{dx} = y_1 y_2'' - y_1'' y_2.$$

Hence (3.9) is the equation

$$\frac{dW}{dx} + p(x)W(x) = 0, \tag{3.10}$$

whose solution is

$$W(y_1, y_2; x) = c \exp\left(-\int^x p(s)\, dx\right). \tag{3.11}$$

Since the exponential is never zero we see from (3.11) that either $W(y_1, y_2; x) \equiv 0$ or $W(y_1, y_2; x)$ is never zero.

To summarize, to determine if $\{y_1, y_2\}$ forms a basis for $\mathcal{V}$, one needs to check at *only one point* whether the Wronskian is zero or not.

Applications of Wronskians

1. Claim: Suppose $\{y_1, y_2\}$ form a basis of $\mathcal{V}$, then they cannot have a common point of inflection in $a < x < b$ unless $p(x)$ and $q(x)$ simultaneously vanish there. To prove this, suppose x_0 is a common point of inflection of y_1 and y_2. That is,

   $$y_1''(x_0) = 0 \;\text{ and }\; y_2''(x_0) = 0.$$

   Evaluating the differential equation (3.3) satisfied by both y_1 and y_2 at $x = x_0$ gives

   $$\begin{aligned} p(x_0)y_1'(x_0) + q(x_0)y_1(x_0) &= 0, \\ p(x_0)y_2'(x_0) + q(x_0)y_2(x_0) &= 0. \end{aligned}$$

   Assuming that $p(x_0)$ and $q(x_0)$ are not both zero at x_0, the above equations are a set of homogeneous equations for $p(x_0)$ and $q(x_0)$. The only way these equations can have a nontrivial solution is for the determinant

   $$\begin{vmatrix} y_1'(x_0) & y_1(x_0) \\ y_2'(x_0) & y_2(x_0) \end{vmatrix} = 0.$$

   That is, $W(y_1, y_2; x_0) = 0$. But this contradicts that $\{y_1, y_2\}$ forms a basis. Thus there can exist no such common inflection point.

2. Claim: Suppose $\{y_1, y_2\}$ form a basis of $\mathcal{V}$ and that y_1 has *consecutive* zeros at $x = x_1$ and $x = x_2$. Then y_2 has one and only one zero between x_1 and x_2. To prove this we first evaluate the Wronskian at $x = x_1$,

$$W(y_1, y_2; x_1) = y_1(x_1)y_2'(x_1) - y_1'(x_1)y_2(x_1) = -y_1'(x_1)y_2(x_1)$$

since $y_1(x_1) = 0$. Evaluating the Wronskian at $x = x_2$ gives

$$W(y_1, y_2; x_2) = -y_1'(x_2)y_2(x_2).$$

Now $W(y_1, y_2; x_1)$ is either positive or negative. (It can't be zero.) Let's assume it is positive. (The case when the Wronskian is negative is handled similarly. We leave this case to the reader.) Since the Wronskian is always of the same sign, $W(y_1, y_2; x_2)$ is also positive. Since x_1 and x_2 are consecutive zeros, the signs of $y_1'(x_1)$ and $y_1'(x_2)$ are opposite of each other. But this implies (from knowing that the two Wronskian expressions are both positive), that $y_2(x_1)$ and $y_2(x_2)$ have opposite signs. Thus there exists at least one zero of y_2 at $x = x_3$, $x_1 < x_3 < x_2$. If there exist two or more such zeros, then between any two of these zeros apply the above argument (with the roles of y_1 and y_2 reversed) to conclude that y_1 has a zero between x_1 and x_2. But x_1 and x_2 were assumed to be consecutive zeros. Thus y_2 has one and only one zero between x_1 and x_2.

In the case of the harmonic oscillator, $y_1(x) = \cos\omega_0 x$ and $y_2(x) = \sin\omega_0 x$, and the fact that the zeros of the sine function interlace those of the cosine function is well known.

3.2 Reduction of Order

Suppose y_1 is a solution of (3.3). Let

$$y(x) = v(x)y_1(x).$$

Then

$$y' = v'y_1 + vy_1' \text{ and } y'' = v''y_1 + 2v'y_1' + vy_1''.$$

Substitute these expressions for y and its first and second derivatives into (3.3) and make use of the fact that y_1 is a solution of (3.3). One obtains the following differential equation for v:

$$v'' + \left(p + 2\frac{y_1'}{y_1}\right) v' = 0,$$

or upon setting $u = v'$,

$$u' + \left(p + 2\frac{y_1'}{y_1}\right) u = 0.$$

This last equation is a first order linear equation. Its solution is

$$u(x) = c\exp\left(-\int\left(p + 2\frac{y_1'}{y_1}\right)\,dx\right) = \frac{c}{y_1^2(x)}\exp\left(-\int p(x)\,dx\right).$$

This implies

$$v(x) = \int u(x)\,dx,$$

so that

$$y(x) = cy_1(x)\int u(x)\,dx.$$

The point is, we have shown that if one solution to (3.3) is known, then a second solution can be found—expressed as an integral.

3.3 Constant Coefficients

We assume that $p(x)$ and $q(x)$ are constants independent of x. We write (3.3) in this case as[3]

$$ay'' + by' + cy = 0. \tag{3.12}$$

We "guess" a solution of the form

$$y(x) = e^{\lambda x}.$$

Substituting this into (3.12) gives

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0.$$

Since $e^{\lambda x}$ is never zero, the only way the above equation can be satisfied is if

$$a\lambda^2 + b\lambda + c = 0. \tag{3.13}$$

Let $\lambda_\pm$ denote the roots of this quadratic equation, i.e.

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We consider three cases.

1. Assume $b^2 - 4ac > 0$ so that the roots $\lambda_\pm$ are both real numbers. Then $\exp(\lambda_+ x)$ and $\exp(\lambda_- x)$ are two linearly independent solutions to (3.13). That

[3] This corresponds to $p(x) = b/a$ and $q(x) = c/a$. For applications it is convenient to introduce the constant a.

they are solutions follows from their construction. They are linearly independent since

$$W(e^{\lambda_+ x}, e^{\lambda_- x}; x) = (\lambda_- - \lambda_+)e^{\lambda_+ x}e^{\lambda_- x} \neq 0$$

Thus in this case, every solution of (3.12) is of the form

$$c_1 \exp(\lambda_+ x) + c_2 \exp(\lambda_- x)$$

for some constants c_1 and c_2.

2. Assume $b^2 - 4ac = 0$. In this case $\lambda_+ = \lambda_-$. Let λ denote their common value. Thus we have one solution $y_1(x) = e^{\lambda x}$. We could use the method of reduction of order to show that a second linearly independent solution is $y_2(x) = xe^{\lambda x}$. However, we choose to present a more intuitive way of seeing this is a second linearly independent solution. (One can always make it rigorous at the end by verifying that that it is indeed a solution.) Suppose we are in the distinct root case but that the two roots are very close in value: $\lambda_+ = \lambda + \varepsilon$ and $\lambda_- = \lambda$. Choosing $c_1 = -c_2 = 1/\varepsilon$, we know that

$$\begin{aligned} c_1 y_1 + c_2 y_2 &= \frac{1}{\varepsilon} e^{(\lambda+\varepsilon)x} - \frac{1}{\varepsilon} e^{\lambda x} \\ &= e^{\lambda x} \frac{e^{\varepsilon x} - 1}{\varepsilon} \end{aligned}$$

is also a solution. Letting $\varepsilon \to 0$ one easily checks that

$$\frac{e^{\varepsilon x} - 1}{\varepsilon} \to x,$$

so that the above solution tends to

$$xe^{\lambda x},$$

our second solution. That $\{e^{\lambda x}, xe^{\lambda x}\}$ is a basis is a simple Wronskian calculation.

3. We assume $b^2 - 4ac < 0$. In this case the roots $\lambda_\pm$ are complex. At this point we review the the exponential of a complex number.

Complex Exponentials

Let $z = x + iy$ (x, y real numbers, $i^2 = -1$) be a complex number. Recall that x is called the real part of z, $\Re z$, and y is called the imaginary part of z, $\Im z$. Just as we picture real numbers as points lying in a line, called the real line **R**; we picture complex numbers as points lying in the plane, called the complex plane **C**. The coordinates of z in the complex plane are (x, y). The absolute value of z, denoted $|z|$, is equal to $\sqrt{x^2+y^2}$. The complex conjugate of z, denoted $\overline{z}$, is equal to $x-iy$. Note the useful relation

$$z\,\overline{z} = |z|^2 .$$

In calculus, or certainly an advanced calculus class, one considers (simple) functions of a complex variable. For example the function

$$f(z) = z^2$$

takes a complex number, z, and returns it square, again a complex number. (Can you show that $\Re f = x^2 - y^2$ and $\Im f = 2xy$?). Using complex addition and multiplication, one can define *polynomials* of a complex variable

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 .$$

The next (big) step is to study power series

$$\sum_{n=0}^{\infty} a_n z^n .$$

With power series come issues of convergence. We defer these to your advanced calculus class.

With this as a background we are (almost) ready to define the exponential of a complex number z. First, we recall that the exponential of a real number x has the power series expansion

$$e^x = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (0! := 1).$$

In calculus classes, one normally defines the exponential in a different way[4] and then proves e^x has this Taylor expansion. However, one could *define* the exponential function by the above formula and then *prove* the various properties of e^x follow from this definition. This is the approach we take for defining the exponential of a complex number except now we use a power series in a complex variable:[5]

$$\boxed{e^z = \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbf{C}} \tag{3.14}$$

[4] A common definition is $e^x = \lim_{n\to\infty}(1+x/n)^n$.

[5] It can be proved that this infinite series converges for all complex values z.

We now derive some properties of $\exp(z)$ based upon this definition.

- Let $\theta \in \mathbf{R}$, then

$$\begin{aligned}\exp(i\theta) &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\ &= \cos\theta + i \sin\theta.\end{aligned}$$

  This last formula is called *Euler's Formula*. Two immediate consequences of Euler's formula (and the facts $\cos(-\theta) = \cos\theta$ and $\sin(\theta) = -\sin\theta$) are

$$\begin{aligned}\exp(-i\theta) &= \cos\theta - i\sin\theta \\ \overline{\exp(i\theta)} &= \exp(-i\theta)\end{aligned}$$

  Hence

$$|\exp(i\theta)|^2 = \exp(i\theta)\,\exp(-i\theta) = \cos^2\theta + \sin^2\theta = 1$$

  That is, the values of $\exp(i\theta)$ lie on the unit circle. The coordinates of the point $e^{i\theta}$ are $(\cos\theta, \sin\theta)$.

- We claim the addition formula for the exponential function, well-known for real values, also holds for complex values

$$\boxed{\exp(z+w) = \exp(z)\,\exp(w), \quad z, w \in \mathbf{C}.} \tag{3.15}$$

  We are to show

$$\begin{aligned}\exp(z+w) &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \quad \text{(binomial theorem)}\end{aligned}$$

  is equal to

$$\begin{aligned}\exp(z)\,\exp(w) &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{m=0}^{\infty} \frac{1}{m!} w^m \\ &= \sum_{k,m=0}^{\infty} \frac{1}{k!m!} z^k w^m\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} z^k w^{n-k} \quad n := k+m$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} .$$

Since

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} ,$$

we see the two expressions are equal as claimed.

- We can now use these two properties to understand better $\exp(z)$. Let $z = x + iy$, then

$$\exp(z) = \exp(x+iy) = \exp(x)\exp(iy) = e^x (\cos y + i \sin y) .$$

Observe the right hand side consists of functions from calculus. Thus with a calculator you could find the exponential of any complex number using this formula.[6]

A form of the complex exponential we frequently use is if $\lambda = \sigma + i\mu$ and $x \in \mathbf{R}$, then

$$\exp(\lambda x) = \exp((\sigma + i\mu)x)) = e^{\sigma x} (\cos(\mu x) + i \sin(\mu x)) .$$

Returning to (3.12) in case $b^2 - 4ac < 0$ and assuming a, b and c are all real, we see that the roots $\lambda_\pm$ are of the form[7]

$$\lambda_+ = \sigma + i\mu \;\text{ and }\; \lambda_- = \sigma - i\mu.$$

Thus $e^{\lambda_+ x}$ and $e^{\lambda_- x}$ are linear combinations of

$$e^{\sigma x}\cos(\mu x) \;\text{ and }\; e^{\sigma x}\sin(\mu x).$$

That they are linear independent follows from a Wronskian calculuation. To summarize, we have shown that every solution of (3.12) in the case $b^2 - 4ac < 0$ is of the form

$$c_1 e^{\sigma x}\cos(\mu x) + c_2 e^{\sigma x}\sin(\mu x)$$

for some constants c_1 and c_2.

[6] Of course, this assumes your calculator doesn't overflow or underflow in computing e^x.
[7] $\sigma = -b/2a$ and $\mu = \sqrt{4ac - b^2}/2a$.

3.4 Exercises

#1. Higher Order Equations

The third order homogeneous differential equation with constant coefficients is

$$a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0 \tag{3.16}$$

where a_i are constants. Assume a solution of the form

$$y(x) = e^{\lambda x}$$

and derive an equation that λ must satisfy in order that y is a solution. (You should get a cubic polynomial.) What is the form of the general solution to (3.16)?

Chapter 4

Difference Equations

4.1 Introduction

We have learned that the general inhomogeneous second order linear differential equation is of the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x).$$

The independent variable, x, takes values in $\mathbf{R}$. (We say x is a continuous variable.) Many applications lead to problems where the independent variable is *discrete*; that is, it takes values in the integers. Instead of $y(x)$ we now have y_n, n an integer. The discrete version of the above equation, called an *inhomogeneous second order linear difference equation*, is

$$a_n\, y_{n+2} + b_n\, y_{n+1} + c_n\, y_n = f_n \tag{4.1}$$

where we assume the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{f_n\}$ are known. For example,

$$(n^2+5)y_{n+2} + 2y_{n+1} + \frac{3}{n+1}y_n = e^n, \;\; n = 0, 1, 2, 3, \ldots$$

is such a difference equation. Usually we are given y_0 and y_1 (the initial values), and the problem is to solve the difference equation for y_n.

In this chapter we consider the special case of *constant coefficient* difference equations:

$$a\, y_{n+2} + b\, y_{n+1} + c\, y_n = f_n$$

where a, b, and c are constants independent of n. If $f_n = 0$ we say the difference equation is *homogeneous*. An example of a homogeneous second order constant coefficient difference equation is

$$6y_{n+2} + \frac{1}{3}y_{n+1} + 2y_n = 0.$$

4.2 Constant Coefficient Difference Equations

4.2.1 Solution of Constant Coefficient Difference Equations

In this section we give an algorithm to solve all second order homogeneous constant coefficient difference equations

$$a\,y_{n+2} + b\,y_{n+1} + c\,y_n = 0. \tag{4.2}$$

The method is the discrete version of the method we used to solve contant coefficient differential equations. We first guess a solution of the form

$$y_n = \lambda^n, \;\; \lambda \neq 0.$$

(For differential equations we guessed $y(x) = e^{\lambda x}$.) We now substitute this into (4.2) and require the result equal zero,

$$\begin{aligned} 0 &= a\lambda^{n+2} + b\lambda^{n+1} + c\lambda^n \\ &= \lambda^n \left(a\lambda^2 + b\lambda + c\right). \end{aligned}$$

This last equation is satisfied if and only if

$$a\lambda^2 + b\lambda + c = 0. \tag{4.3}$$

Let λ_1 and λ_2 denote the roots of this quadratic equation. (For the moment we consider only the case when the roots are distinct.) Then

$$\lambda_1^n \;\text{ and }\; \lambda_2^n$$

are both solutions to (4.2). Just as in our study of second order ODEs, the linear combination

$$c_1\lambda_1^n + c_2\lambda_2^n$$

is also a solution and every solution of (4.2) is of this form. The constants c_1 and c_2 are determined once we are given the initial values y_0 and y_1:

$$\begin{aligned} y_0 &= c_1 + c_2, \\ y_1 &= c_1\lambda_1 + c_2\lambda_2, \end{aligned}$$

are two equation that can be solved for c_1 and c_2.

4.2.2 Fibonnaci Numbers

Consider the sequence of numbers

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad \cdots$$

that is, each number is the sum of the preceding two numbers starting with

$$1 \quad 1$$

as initial values. These integers are called *Fibonnaci numbers* and are denoted F_n. From their definition, F_n satisfies the difference equation

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1$$

with

$$F_0 = 0, F_1 = 1.$$

The quadratic equation we must solve is

$$\lambda^2 = \lambda + 1,$$

whose roots are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

Setting

$$F_n = c_1\lambda_1^n + c_2\lambda_2^n,$$

we see that at $n = 0$ and 1 we require

$$\begin{aligned} 0 &= c_1 + c_2, \\ 1 &= c_1\lambda_1 + c_2\lambda_2. \end{aligned}$$

Solving these we find

$$c_1 = \frac{1}{\sqrt{5}}, \ c_2 = -\frac{1}{\sqrt{5}},$$

and hence

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right).$$

Since $\lambda_1 > 1$ and $|\lambda_2| < 1$, λ_1^n grows with increasing n whereas $\lambda_2^n \to 0$ as $n \to \infty$. Thus for large n

$$F_n \sim \frac{1}{\sqrt{5}} \lambda_1^n,$$

and

$$\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{\lambda_1} := \omega.$$

The number

$$\omega = \frac{\sqrt{5} - 1}{2} = 0.61803398\ldots\ldots$$

is called the golden mean.

4.3 Inhomogeneous Difference Equations

In a completely analogous way to the ODE case, one proves that every solution to the inhomogeneous linear difference equation (4.1) is of the form

$$(y_n)_{\text{homo}} + (y_n)_{\text{part}}$$

where $(y_n)_{\text{homo}}$ is a solution to the homogeneous equation (4.1) with $f_n = 0$ and $(y_n)_{\text{part}}$ is a particular solution to the inhomogeneous equation (4.1).

4.4 Exercises

#1. Degenerate Roots

Consider the constant coefficient difference equation (4.2) but now assume the two roots $\lambda_{1,2}$ are equal. Show that

$$n\lambda_1^n$$

is a second linearly independent solution to (4.2).

#2. Rational Approximations to $\sqrt{2}$

Solve the difference equation

$$x_{n+1} = 2x_n + x_{n-1}, \;\; n \geq 1$$

with initial conditions $x_0 = 0$ and $x_1 = 1$ that corresponds to the sequence 0, 1, 2, 5, 12, 29,.... Show that

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{x_n} = \sqrt{2}.$$

The rational numbers

$$\frac{x_{n+1} - x_n}{x_n}$$

provide us with very good approximations to $\sqrt{2}$.

Chapter 5

Matrix Differential Equations

5.1 The Matrix Exponential

Let A be a $n \times n$ matrix with constant entries. In this chapter we study the matrix differential equation

$$\boxed{\frac{dx}{dt} = Ax \quad \text{where} \quad x \in \mathbf{R}^n.} \tag{5.1}$$

We will present an algorithm that reduces solving (5.1) to problems in linear algebra.

The exponential of the matrix tA, $t \in \mathbf{R}$, is defined by the infinite series[1]

$$\boxed{e^{tA} = \exp(tA) := I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots .} \tag{5.2}$$

Remark: In an advanced course you will prove that this infinite series of matrices converges to a $n \times n$ matrix.

It is important to note that for matrices A and B, in general,

$$\exp(tA)\exp(tB) \neq \exp(tA + tB).$$

[1] We put the scalar factor t directly into the definition of the matrix exponential since it is in this form we will use the matrix exponential.

If A and B commute ($AB = BA$) then it is the case that

$$\exp(tA)\exp(tB) = \exp(tA + tB).$$

This last fact can be proved by examining the series expansion of both sides—on the left hand side one has to multiply two infinite series. You will find that by making use of $AB = BA$ the result follows precisely as in the case of complex exponentials.

Here are some examples:

1.

$$A = D = \text{diagonal matrix} = \text{diag}\left(\lambda_1, \lambda_2, \dots, \lambda_n\right).$$

Observe that

$$D^k = \text{diag}\left(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\right).$$

Thus

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k = \text{diag}\left(e^{t\lambda_1}, e^{t\lambda_2}, \dots, e^{t\lambda_n}\right).$$

2. Suppose that A is a diagonalizable matrix; that is, there exist matrices S and D with S invertible and D diagonal such that

$$A = SDS^{-1}.$$

Observe

$$A^2 = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1},$$

and more generally,

$$A^k = SD^kS^{-1}.$$

Thus

$$\begin{aligned}
\exp(tA) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} SD^kS^{-1} \\
&= S \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k S^{-1} \\
&= S \exp(tD) S^{-1}.
\end{aligned}$$

In the next to the last equality, we used the fact that S and S^{-1} do not depend upon the summation index k and can therefore be brought outside of the sum. The last equality makes use of the previous example where we computed the exponential of a diagonal matrix. This example shows that if one can find such S and D, then the computation of the $\exp(tA)$ is reduced to matrix multiplications.

3. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Matrix multiplication shows

$$A^2 = -I,$$

and thus

$$A^{2k} = \left(A^2\right)^k = (-I)^k = (-1)^k I,$$
$$A^{2k+1} = A^{2k} A = (-1)^k A.$$

Hence

$$\begin{aligned}
\exp(tA) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \qquad (5.3)\\
&= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (-1)^k I + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (-1)^k A \\
&= \cos t\, I + \sin t\, A \\
&= \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & -\sin t \\ \sin t & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \qquad (5.4)
\end{aligned}$$

Remark: You can also compute

$$\exp\left(t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

by the method of Example #2. Try it!

5.2 Application of e^{tA} to differential equations

5.2.1 Derivative of e^{tA} with respect to t

The following is the basic property of the exponential that we apply to differential equations. As before, A denotes a $n \times n$ matrix with constant coefficients.

$$\boxed{\frac{d}{dt} \exp(tA) = A \exp(tA) = \exp(tA) A.} \qquad (5.5)$$

Here is the proof: Differentiate

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

term-by-term[2] with the result

$$\begin{aligned}\frac{d}{dt}e^{tA} &= A + tA^2 + \frac{t^2}{2!}A^3 + \cdots \\ &= A\left(I + tA + \frac{t^2}{2!}A^2 + \cdots\right) \\ &= A\,e^{tA} \\ &= e^{tA}\,A.\end{aligned}$$

The last equality follows by factoring A out on the right instead of the left.

5.2.2 Solution to Matrix ODE with Constant Coefficients

We now use (5.5) to prove

Theorem: Let

$$\frac{dx}{dt} = Ax \tag{5.6}$$

where $x \in \mathbf{R}^n$ and A is a $n \times n$ matrix with constant coefficients. Then every solution of (5.6) is of the form

$$x(t) = \exp(tA)x_0 \tag{5.7}$$

for some constant vector $x_0 \in \mathbf{R}^n$.

Proof: (i) First we show that $x(t) = e^{tA}x_0$ is a solution:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\left(e^{tA}x_0\right) = \left(\frac{d}{dt}e^{tA}\right)x_0 \\ &= Ae^{tA}x_0 \\ &= Ax(t).\end{aligned}$$

(ii) We now show that *every* solution of (5.6) is of the form (5.7). Let $y(t)$ by any solution to (5.6). Let

$$\Delta(t) := e^{-tA}y(t).$$

[2] In a complex analysis course you will prove that convergent complex power series can be differentiated term-by-term and the resulting series has the same radius of convergence. Note there really is something to prove here since there is an interchange of two limits.

If we can show that $\Delta(t)$ is independent of t—that it is a constant vector which we call x_0, then we are done since multiplying both sides by e^{tA} shows

$$e^{tA}x_0 = e^{tA}\Delta(t) = e^{tA}e^{-tA}y(t) = y(t).$$

(We used the fact that tA and $-tA$ commute so that the addition formula for the matrix exponential is valid.) To show that $\Delta(t)$ is independent of t we show its derivative with respect to t is zero:

$$\begin{aligned} \frac{d\Delta}{dt} &= \frac{d}{dt}e^{-tA}y(t) \\ &= \left(\frac{d}{dt}e^{-tA}\right)y(t) + e^{-tA}\frac{dy}{dt} \quad \text{(chain rule)} \\ &= \left(-e^{-tA}A\right)y(t) + e^{-tA}\left(Ay(t)\right) \quad (y(t) \ \text{satisfies ODE}) \\ &= 0. \end{aligned}$$

The next theorem relates the solution $x(t)$ of (5.6) to the eigenvalues and eigenvectors of the matrix A (in the case A is diagonalizable).

Theorem: Let A be a diagonalizable matrix. Any solution to (5.6) can be written as

$$x(t) = c_1 e^{t\lambda_1}\psi_1 + c_2 e^{t\lambda_2}\psi_2 + \cdots + c_n e^{t\lambda_n}\psi_n \tag{5.8}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A with associated eigenvectors $\psi_1, \ldots, \psi_n$, and $c_1, \ldots, c_n$ are constants.

Proof: All solutions of (5.6) are of the form (5.7). Since A is diagonalizable, the eigenvectors of A can be used to form a basis: $\{\psi_1, \ldots, \psi_n\}$. Since this is a basis there exist constants $c_1, \ldots, c_n$ such that

$$x_0 = c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n.$$

(x_0 is the constant vector appearing in (5.7).)

For any eigenvector ψ of A with eigenvalue λ we have

$$e^{tA}\psi = e^{t\lambda}\psi.$$

(This can be proved by applying the infinite series (5.2) to the eigenvector ψ and noting $A^k\psi = \lambda^k\psi$ for all positive integers k.) Thus

$$\begin{aligned} e^{tA}x_0 &= c_1 e^{tA}\psi_1 + \cdots c_n e^{tA}\psi_n \\ &= c_1 e^{t\lambda_1}\psi_1 + \cdots + c_n e^{t\lambda_n}\psi_n. \end{aligned}$$

Here are two immediate corollaries of this theorem:

1. If A is diagonalizable and has only *real* eigenvalues, then any solution $x(t)$ of (5.1) will have no oscillations.

2. If A is diagonalizable and the real part of every eigenvalue is negative, then

$$x(t) \to 0 \quad \text{(zero vector), as} \quad t \to +\infty$$

To see this recall that if $\lambda = \sigma + i\mu$ (σ and μ both real), then

$$e^{\lambda t} = e^{\sigma t} e^{i\mu t}.$$

If $\sigma < 0$, $e^{\sigma t} \to 0$ as $t \to +\infty$. Now apply preceding theorem.

5.3 Relation to Earlier Methods of Solving Constant Coefficient DEs

Earlier we showed how to solve

$$ay'' + by' + cy = 0$$

where a, b and c are constants. Indeed, we proved that the general solution is of the form

$$y(t) = c_1 e^{t\lambda_1} + c_2 e^{t\lambda_2}$$

where λ_1 and λ_2 are the roots to

$$a\lambda^2 + b\lambda + c = 0.$$

(We consider here only the case of distinct roots.)

Let's analyze this familiar result using matrix methods. The $x \in \mathbf{R}^2$ is

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ dy/dt \end{pmatrix}$$

Therefore,

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} dy/dt \\ d^2y/dt^2 \end{pmatrix} \\ &= \begin{pmatrix} x_2 \\ -\frac{b}{a}x_2 - \frac{c}{a}x_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

This last equality defines the 2×2 matrix A. The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}.$$

Thus the eigenvalues of A are the same quantities λ_1 and λ_2 appearing above. Since

$$x(t) = e^{tA}x_0 = S \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} S^{-1}x_0,$$

$x_1(t)$ is a linear combination of $e^{t\lambda_1}$ and $e^{t\lambda_2}$.

5.4 Exercises

#1. Harmonic Oscillator via Matrix Exponentials

Write the oscillator equation

$$\ddot{x} + \omega_0^2 x = 0$$

as a first order system (5.1). (Explicitly find the matrix A.) Compute $\exp(tA)$ and show that $x(t) = \exp(tA)x_0$ gives the now familiar solution. Note that we computed $\exp(tA)$ in (5.4) for the case $\omega_0 = 1$.

#2. Exponential of Nilpotent Matrices

1. Using the series expansion for the matrix exponential, compute $\exp(tN)$ where

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Answer the same question for

$$N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

How do these answers differ from $\exp(tx)$ where x is any real number?

2. A $n \times n$ matrix N is called nilpotent[3] if there exists a positive integer k such that

$$N^k = 0$$

[3] In an advanced course in linear algebra, it will be proved that every matrix A can be written *uniquely* as $D + N$ where D is a diagonalizable matrix, N is a nilpotent matrix, *and* $DN = ND$. Furthermore, an algorithm will be given to find the matrices D and N from the matrix A. Once

where the 0 is the $n \times n$ zero matrix. If N is nilpotent let k be the smallest integer such that $N^k = 0$. Explain why $\exp(tN)$ is a matrix whose entries are polynomials in t of degree at most $k-1$.

---

this is done then one can compute $\exp(tA)$ as follows

$$\exp(tA) = \exp(tD + tN) = \exp(tD)\exp(tN).$$

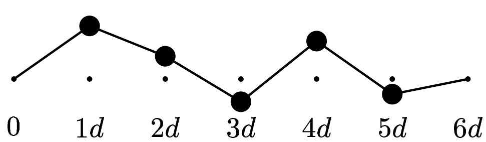
We showed above how to reduce the computation of $\exp(tD)$, D a diagonalizable matrix, to linear algebra. This problem shows that $\exp(tN)$ reduces to finitely many matrix multiplications. Thus the computation of both $\exp(tD)$ and $\exp(tN)$ are reduced to linear algebra and hence so is $\exp(tA)$. Observe that it is crucial that we know $DN = ND$.

Chapter 6

The Weighted String

6.1 Derivation of Differential Equations

The *weighted string* is a system in which the mass is concentrated in a set of equally spaced mass points, N in number with spacing d, imagined to be held together by massless springs of equal tension T. We further assume that the construction is such that the mass points move only in the vertical direction (y direction) and there is a constraining force to keep the mass points from moving in the horizontal direction (x direction). We call it a "string" since these mass points give a discrete string—the tension in the string is represented by the springs. The figure below illustrates the weighted string for $N = 5$.



The string is "tied down" at the endpoints 0 and $(N+1)d$. The horizontal coordinates of the mass points will be at $x = d, 2d, \ldots, Nd$. We let u_j denote the vertical displacement of the j^{th} mass point and F_j the transverse force on the j^{th} particle. To summarize the variables introduced so far:

$$
\begin{aligned}
m &= \text{mass of particle,} \\
N &= \text{total number of particles,} \\
T &= \text{tension of spring,} \\
d &= \text{horizontal distance between two particles,} \\
u_j &= \text{vertical displacement of } j^{th} \text{ particle,} \quad j = 1, 2, \ldots N, \\
F_j &= \text{transverse force on } j^{th} \text{ particle,} \quad j = 1, 2, \ldots N.
\end{aligned}
$$

To impose the boundary conditions that the ends of the string are rigidly fixed at $x = 0$ and $x = (N+1)d$, we take

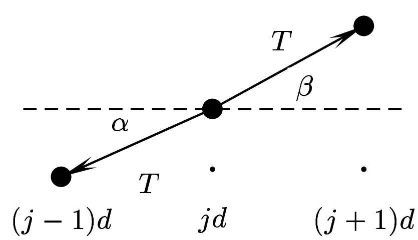
$$u_0 = 0 \quad \text{and} \quad u_{N+1} = 0.$$

Newton's equations for these mass points are

$$F_j = m\frac{d^2 u_j}{dt^2}, \quad j = 1, 2, \ldots, N.$$

This is a system of N second order differential equations. We now find an expression for the transverse force F_j in terms of the vertical displacements.

In the diagram below, the forces acting on the j^{th} particle are shown.



From the diagram,

$$F_j = T\sin\beta - T\sin\alpha.$$

We make the assumption that the angles α and β are small. (The string is not stretched too much!) In this small angle approximation we have

$$\sin\alpha \approx \tan\alpha \quad \text{and} \quad \sin\beta \approx \tan\beta.$$

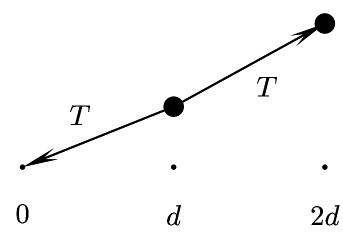
Therefore, in this small angle approximation

$$\begin{aligned} F_j &\approx T\tan\beta - T\tan\alpha \\ &= T\left(\frac{u_{j+1} - u_j}{d}\right) - T\left(\frac{u_j - u_{j-1}}{d}\right). \end{aligned}$$

Thus,

$$m\frac{d^2 u_j}{dt^2} = \frac{T}{d}\left(u_{j+1} - 2u_j + u_{j-1}\right), \quad j = 1, 2, \ldots, N. \tag{6.1}$$

Note that these equations are valid for $j = 1$ and $j = N$ when we interpret $u_0 = 0$ and $u_{N+1} = 0$. For example, for $j = 1$ the force F_1 is determined from the diagram:



$$
\begin{aligned}
F_1 &= T\frac{(u_2-u_1)}{d} - T\frac{u_1}{d} \\
&= \frac{T}{d}\left(u_2 - 2u_1 + u_0\right), \quad u_0 = 0.
\end{aligned}
$$

Equation (6.1) is a system of N second order linear differential equations. Thus the dimension of the vector space of solutions is $2N$; that is, it takes $2N$ real numbers to specify the initial conditions (N initial positions and N initial velocities). Define the $N \times N$ matrix

$$
V_N = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix} \tag{6.2}
$$

and the column vector $\mathbf{u}$

$$
\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_N \end{pmatrix}. \tag{6.3}
$$

Then (6.1) can be written in the compact matrix form

$$
\boxed{\frac{d^2\mathbf{u}}{dt^2} + \frac{T}{md} V_N \mathbf{u} = 0.} \tag{6.4}
$$

Note: We could also have written (6.1) as a *first* order matrix equation of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{6.5}$$

where A would be a $2N \times 2N$ matrix. However, for this application it is simpler to develop a special theory for (6.4) rather than to apply the general theory of (6.5) since the matrix manipulations with V_N will be a bit clearer than they would be with A.

6.2 Reduction to an Eigenvalue Problem

Equation (6.4) is the matrix version of the harmonic oscillator equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0, \quad \omega_0^2 = \frac{k}{m}\,. \tag{6.6}$$

Indeed, we will show that (6.4) is precisely N harmonic oscillators (6.6)—once one chooses the correct coordinates. We know that solutions to (6.6) are linear combinations of

$$\cos\omega_0 t \quad \text{and} \quad \sin\omega_0 t.$$

Thus we "guess" that solutions to (6.4) are linear combinations of the form

$$\cos\omega t\,\mathbf{f} \quad \text{and} \quad \sin\omega t\,\mathbf{f}$$

where ω is to be determined and $\mathbf{f}$ is a column vector of length N. (Such a "guess" can be theoretically deduced from the theory of the matrix exponential when (6.4) is rewritten in the form (6.5).)

Thus setting

$$\mathbf{u} = e^{i\omega t}\mathbf{f},$$

we see that (6.4) becomes the matrix equation

$$V_N\mathbf{f} = \frac{md}{T}\omega^2\mathbf{f}.$$

That is, we must find the eigenvalues and eigenvectors of the matrix V_N. Since V_N is a real symmetric matrix, it is diagonalizable with real eigenvalues. To each eigenvalue λ_n, i.e.

$$V_N\mathbf{f_n} = \lambda_n\mathbf{f_n}, \quad n = 1, 2, \ldots, N,$$

there will correspond a positive frequency

$$\omega_n^2 = \frac{T}{md}\lambda_n, \quad n = 1, 2, \ldots, N,$$

and a solution of (6.4) of the form

$$\mathbf{u_n} = (a_n \cos(\omega_n t) + b_n \sin(\omega_n t))\, \mathbf{f_n}$$

where a_n and b_n are constants. This can now be easily verified by substituting this above expression into the differential equation. To see we have enough constants of integration we observe that we have two constants, a_n and b_n, for each (vector) solution $\mathbf{u_n}$. And we have N vector solutions $\mathbf{u_n}$—thus $2N$ constants in all. We now turn to an explicit evaluation of the frequencies ω_n^2—such frequencies are called *normal modes*.

6.3 Computation of the Eigenvalues of V_N

We introduce the characteristic polynomial of the matrix V_N:

$$D_N(\lambda) = D_N = \det(V_N - \lambda I).$$

Expanding the determinant D_N in the last column, we see that it is a sum of two terms—each a determinant of matrices of size $(N-1)\times(N-1)$. One of these determinants equals $(2-\lambda)D_{N-1}$ and the other equals D_{N-2} as is seen after expanding again, this time by the last row. In this way one deduces

$$D_N = (2-\lambda)D_{N-1} - D_{N-2}, \quad N = 2, 3, 4, \ldots$$

with

$$D_0 = 1 \quad \text{and} \quad D_1 = 2 - \lambda.$$

We now proceed to solve this constant coefficient difference equation (in N). From earlier work we know that the general solution is of the form

$$c_1 \mu_1^N + c_2 \mu_2^N$$

where μ_1 and μ_2 are the roots of

$$\mu^2 - (2-\lambda)\mu + 1 = 0.$$

Solving this quadratic equation gives

$$\mu_{1,2} = 1 - \frac{\lambda}{2} \pm \frac{1}{2}\sqrt{(2-\lambda)^2 - 4}\,.$$

It will prove convenient to introduce an auxiliary variable θ through

$$2 - \lambda = 2\cos\theta,$$

A simple computation now shows

$$\mu_{1,2} = e^{\pm i\theta}.$$

Thus

$$D_N = c_1 e^{iN\theta} + c_2 e^{-iN\theta}.$$

To determine c_1 and c_2 we require that

$$D_0 = 1 \quad \text{and} \quad D_1 = 2 - \lambda.$$

That is,

$$\begin{aligned} c_1 + c_2 &= 1, \\ c_1 e^{i\theta} + c_2 e^{-i\theta} &= 2 - \lambda = 2\cos\theta. \end{aligned}$$

Solving for c_1 and c_2,

$$\begin{aligned} c_1 &= \frac{e^{i\theta}}{e^{i\theta} - e^{-i\theta}}, \\ c_2 &= -\frac{e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} D_N &= \frac{1}{e^{i\theta} - e^{-i\theta}} \left(e^{i(N+1)\theta} - e^{-i(N+1)\theta} \right) \\ &= \frac{\sin\left((N+1)\theta\right)}{\sin\theta}. \end{aligned}$$

The eigenvalues of V_N are solutions to

$$D_N(\lambda) = \det(V_N - \lambda I) = 0.$$

Thus we require

$$\sin\left((N+1)\theta\right) = 0,$$

which happens when

$$\theta = \theta_n := \frac{n\pi}{N+1}, \quad n = 1, 2, \ldots, N.$$

Thus the eigenvalues of V_N are

$$\lambda_n = 2 - 2\cos\theta_n = 4\sin^2(\theta_n/2), \;\; n = 1, 2, \ldots, N. \tag{6.7}$$

The eigenfrequencies are

$$\begin{aligned} \omega_n^2 &= \frac{T}{md}\lambda_n = \frac{2T}{md}(1 - \cos\theta_n) \\ &= \frac{2T}{md}\left(1 - \cos\frac{n\pi}{N+1}\right) = \frac{4T}{md}\sin^2\left(\frac{n\pi}{2(N+1)}\right). \end{aligned}$$

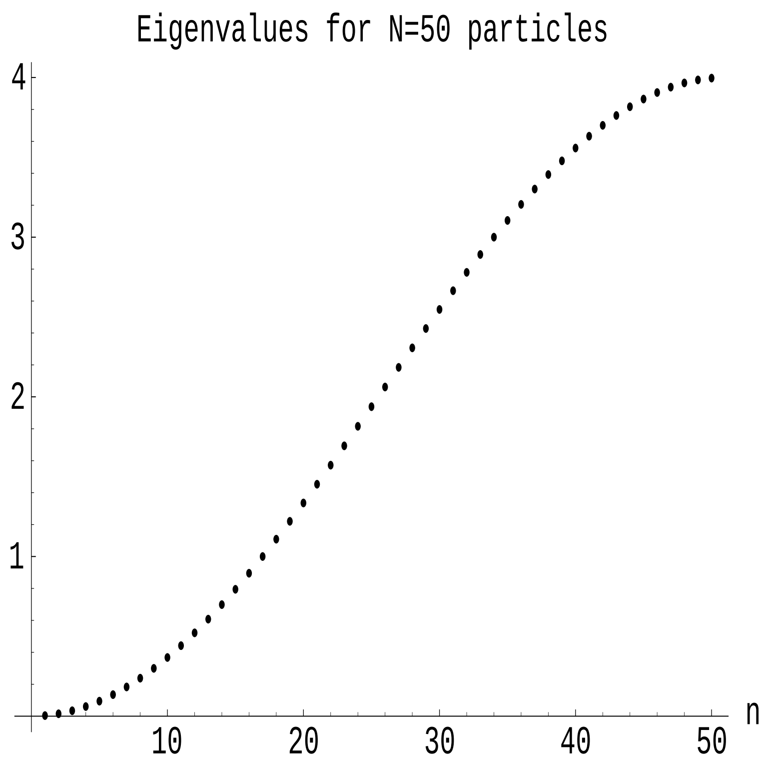


Figure 6.1: Eigenvalues λ_n, (6.7), for $N = 50$ particles.

Remark: We know there are at most N distinct eigenvalues of V_N. The index n does not start at zero because this would imply $\theta = 0$, but $\theta = 0$—due to the presence of $\sin\theta$ in the denominator of D_N—is *not* a zero of the determinant and hence does not correspond to an eigenvalue of V_N. We also conclude there are N distinct eigenvalues of V_N.

6.4 The Eigenvectors

6.4.1 Constructing the Eigenvectors $\mathbf{f_n}$

We now find the eigenvector $\mathbf{f}_n$ corresponding to eigenvalue λ_n. That is, we want a column vector $\mathbf{f_n}$ that satisfies

$$V_N \mathbf{f_n} = 2(1 - \cos\theta_n)\mathbf{f_n}, \quad n = 1, 2, \ldots, N.$$

Setting,

$$\mathbf{f_n} = \begin{pmatrix} f_{n1} \\ f_{n2} \\ \cdot \\ \cdot \\ f_{nN} \end{pmatrix},$$

the above equation in component form is

$$-f_{n,j-1} + 2f_{n,j} - f_{n,j+1} = 2(1 - \cos\theta_n) f_{n,j}$$

with

$$f_{n,0} = f_{n,N+1} = 0.$$

This is a constant coefficient difference equation *in the j index*. Assume, therefore, a solution of the form

$$f_{n,j} = e^{ij\varphi}.$$

The recursion relation becomes with this guess

$$-2\cos\varphi + 2 = 2(1 - \cos\theta_n),$$

i.e.

$$\varphi = \pm\theta_n.$$

The $f_{n,j}$ will be linear combinations of $e^{\pm ij\theta_n}$,

$$f_{n,j} = c_1 \sin(j\theta_n) + c_2 \cos(j\theta_n).$$

We require $f_{n,0} = f_{n,N+1} = 0$ which implies $c_2 = 0$.

To summarize,

$$\boxed{\begin{aligned} V_N \mathbf{f_n} &= \frac{md}{T}\omega_n^2 \mathbf{f_n}, \quad n = 1, 2, \ldots, N, \\ \omega_n^2 &= \frac{2T}{md}(1 - \cos\theta_n), \quad \theta_n = \frac{n\pi}{N+1}, \\ \mathbf{f_n} &= \begin{pmatrix} \sin(\theta_n) \\ \sin(2\theta_n) \\ \cdot \\ \cdot \\ \sin(N\theta_n) \end{pmatrix} \quad n = 1, 2, \ldots, N. \end{aligned}} \tag{6.8}$$

The general solution to (6.4) is

$$\mathbf{u}(t) = \sum_{n=1}^{N} \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \mathbf{f_n},$$

or in component form,

$$u_j(t) = \sum_{n=1}^{N} \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \sin(j\theta_n). \tag{6.9}$$

6.4.2 Orthogonality of Eigenvectors

The set of eigenvectors $\{\mathbf{f_n}\}_{n=1}^{N}$ forms a basis for $\mathbf{R}^N$ since the matrix V_N is symmetric. (Another reason they form a basis is that the eigenvalues of V_N are distinct.) We claim the eigenvectors have the additional (nice) property that they are orthogonal, i.e.

$$\mathbf{f_n} \cdot \mathbf{f_m} = 0, \quad n \neq m,$$

where $\cdot$ denotes the vector dot product. To prove this we use (6.8) to compute

$$\mathbf{f_n} \cdot \mathbf{f_m} = \sum_{j=1}^{N} \sin(j\theta_n) \sin(j\theta_m). \tag{6.10}$$

To see that this is zero for $n \neq m$, we leave as an exercise to prove the trigonometric identity

$$\sum_{j=1}^{N} \sin\left(\frac{nj\pi}{N+1}\right) \sin\left(\frac{mj\pi}{N+1}\right) = \frac{1}{2}(N+1)\delta_{n,m}$$

where $\delta_{n,m}$ is the Kronecker delta function. (One way to prove this identity is first to use the formula $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ to rewrite the above sum as a sum of exponentials. The resulting sums will be finite geometric series.) From this identity we also get that the length of each vector, $\|\mathbf{f}_n\|$, is

$$\|\mathbf{f}_n\| = \sqrt{\frac{N+1}{2}}.$$

6.5 Determination of constants a_n and b_n

Given the initial vectors $\mathbf{u}(0)$ and $\dot{\mathbf{u}}(0)$, we now show how to determine the constants a_n and b_n. At $t = 0$,

$$\mathbf{u}(0) = \sum_{n=1}^{N} a_n \mathbf{f_n}.$$

Dotting the vector $\mathbf{f_p}$ into both sides of this equation and using the orthogonality of the eigenvectors, we see that

$$a_p = \frac{2}{N+1} \sum_{j=1}^{N} \sin\left(\frac{pj\pi}{N+1}\right) u_j(0), \quad p = 1, 2, \ldots, N. \tag{6.11}$$

Differentiating $\mathbf{u}(t)$ with respect to t and then setting $t = 0$, we have

$$\dot{\mathbf{u}}(0) = \sum_{n=1} \omega_n b_n \mathbf{f_n}.$$

Likewise dotting $\mathbf{f_p}$ into both sides of this equation results in

$$b_p = \frac{2}{N+1} \frac{1}{\omega_p} \sum_{j=1}^{N} \sin\left(\frac{pj\pi}{N+1}\right) \dot{u}_j(0), \quad p = 1, 2, \ldots, N. \tag{6.12}$$

If we assume the weighted string starts in an initial state where all the initial velocities are zero,

$$\dot{u}_j(0) = 0,$$

then the solution $\mathbf{u}(t)$ has components

$$u_j(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t) \sin(j\theta_n)$$

where the constants a_n are given by (6.11) in terms of the initial displacements $u_j(0)$. The special solutions obtained by setting all the a_n except for one to zero, are called the *normal modes of oscillation* for the weighted string. They are most interesting to graph as a function both in space (the j index) and in time (the t variable). In figures we show a "snapshot" of various normal mode solutions at various times t.

6.6 Inhomogeneous Problem

The inhomogeneous version of (6.4) is

$$\frac{d^2\mathbf{u}}{dt^2} + \frac{T}{md} V_N \mathbf{u} = \mathbf{F}(t) \tag{6.13}$$

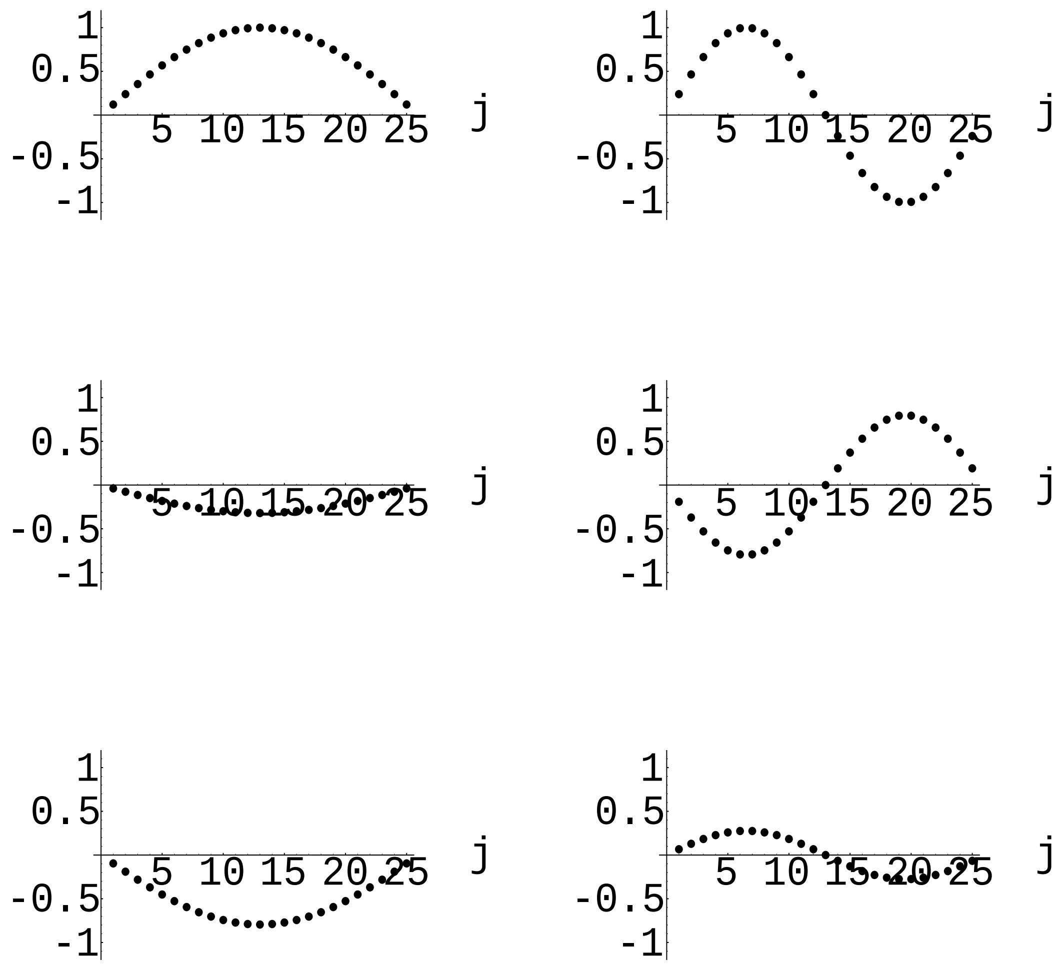


Figure 6.2: Vertical displacements u_j for the two lowest ($n = 1$ and $n = 2$) normal modes are plotted as function of the horizontal position index j. Each column gives the same normal mode but at different times t. System is for $N = 25$ particles.

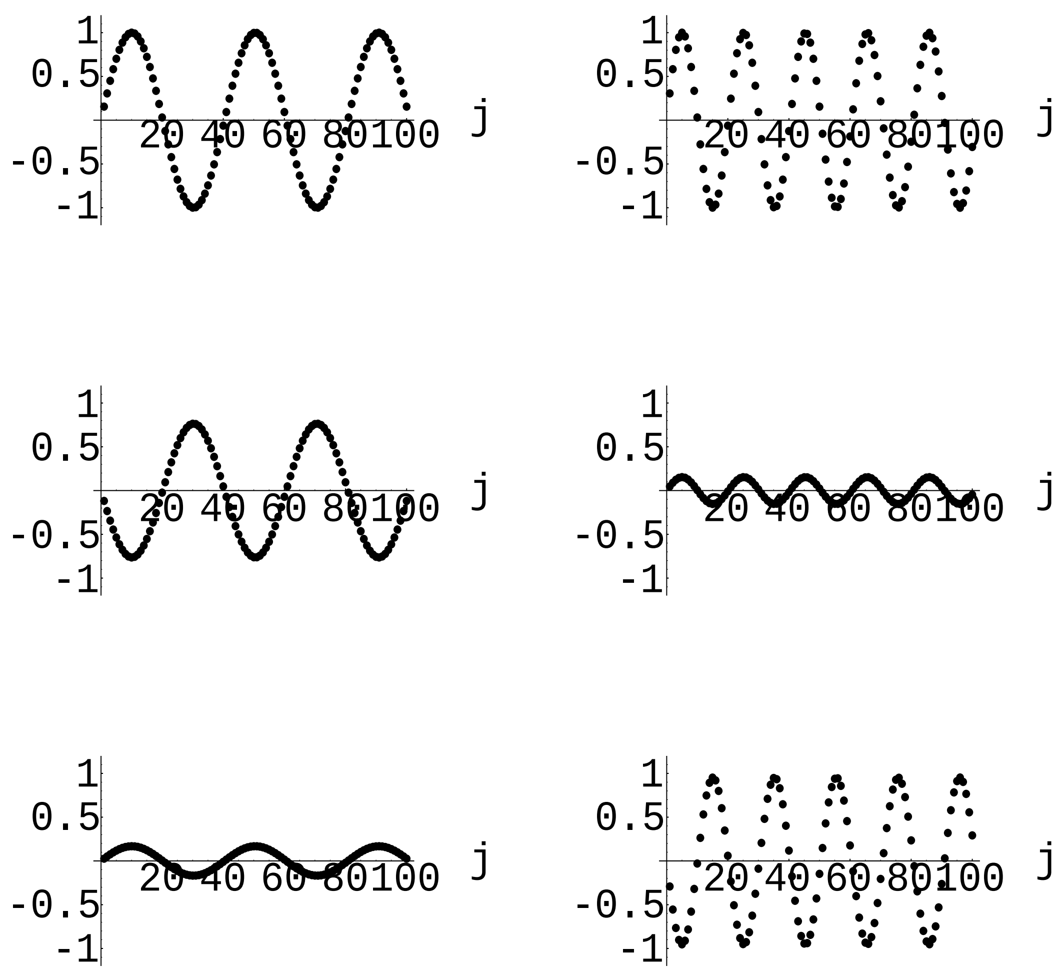


Figure 6.3: Vertical displacements u_j for the two normal modes $n = 5$ and $n = 10$ are plotted as function of the horizontal position index j. Each column gives the same normal mode but at different times t. System is for $N = 100$ particles.

where $\mathbf{F}(t)$ is a given driving term. The j^{th} component of $\mathbf{F}(t)$ is the external force acting on the particle at site j. An interesting case of (6.13) is

$$\mathbf{F}(t) = \cos\omega t\,\mathbf{f}$$

where $\mathbf{f}$ is a constant vector. The general solution to (6.13) is the sum of a particular solution and a solution to the homogeneous equation. For the particular solution we assume a solution of the form

$$\mathbf{u}_p(t) = \cos\omega t\,\mathbf{g}.$$

Substituting this into the differential equation we find that $\mathbf{g}$ satisfies

$$\left(V_N - \frac{md}{T}\omega^2 I\right)\mathbf{g} = \frac{md}{T}\mathbf{f}.$$

For $\omega^2 \neq \omega_n^2$, $n = 1, 2, \ldots, N$, the matrix

$$\left(V_N - \frac{md}{T}\omega^2 I\right)$$

is invertible and hence

$$\mathbf{g} = \frac{md}{T}\left(V_N - \frac{md}{T}\omega^2 I\right)^{-1}\mathbf{f}\,.$$

Writing (possible since the eigenvectors form a basis)

$$\mathbf{f} = \sum_{n=1}^{N}\alpha_n\mathbf{f_n},$$

we conclude that

$$\mathbf{g} = \sum_{n=1}^{N}\frac{\alpha_n}{\omega_n^2 - \omega^2}\mathbf{f_n}$$

for $\omega^2 \neq \omega_n^2$, $n = 1, 2, \ldots, N$. The solution with initial values

$$\mathbf{u}(0) = 0, \quad \dot{\mathbf{u}}(0) = 0 \tag{6.14}$$

is therefore of the form

$$\mathbf{u}(t) = \cos\omega t\sum_{n=1}^{N}\frac{\alpha_n}{\omega_n^2 - \omega^2}\mathbf{f_n} + \sum_{n=1}\left(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\right)\mathbf{f_n}\,.$$

Imposing the initial conditions (6.14) we obtain the two equations

$$\sum_{n=1}\left(\frac{\alpha_n}{\omega_n^2 - \omega^2} + a_n\right)\mathbf{f_n} = 0, \tag{6.15}$$

$$\sum_{n=1}\omega_n b_n\mathbf{f_n} = 0. \tag{6.16}$$

From the fact that $\{\mathbf{f_n}\}_{n=1}^N$ is a basis we conclude

$$a_n = -\frac{\alpha_n}{\omega_n^2 - \omega^2}, \quad b_n = 0 \quad \text{for} \quad n = 1, 2, \ldots, N.$$

Thus the solution is

$$\begin{aligned} \mathbf{u}(t) &= \sum_{n=1}^{N} \frac{\alpha_n}{\omega_n^2 - \omega^2} \left(\cos(\omega t) - \cos(\omega_n t)\right) \mathbf{f_n} & (6.17) \\ &= \sum_{n=1}^{N} \frac{2\alpha_n}{\omega_n^2 - \omega^2} \sin\left(\frac{1}{2}(\omega_n + \omega)t\right) \sin\left(\frac{1}{2}(\omega_n - \omega)t\right) \mathbf{f_n}\,. & (6.18) \end{aligned}$$

We observe that there is a *beat* whenever the driving frequency ω is close to a normal mode of oscillation ω_n. Compare this discussion with that on page 195 of Boyce-DiPrima.

6.7 Exercises

#1. Coupled Pendulums

Consider the system of two mathematical pendulums of lengths ℓ_1 and ℓ_2 and masses m_1 and m_2, respectively, in a gravitional field mg which move in two parallel vertical planes perpendicular to a common flexible support such as a string from which they are suspended. Denote by θ_1 (θ_2) the angle of deflection of pendulum #1 (#2). The kinetic energy of this system is

$$\mathrm{KE} = \frac{1}{2} m_1 \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2\,,$$

and the potential energy is

$$\mathrm{PE} = m_1 g\, \ell_1 (1 - \cos\theta_1) + m_2 g\, \ell_2 (1 - \cos\theta_2) + V_{int}$$

where V_{int} is the *interaction* potential energy.[1] If there is no twist of the support, then there is no interaction of the two pendulums. We also expect the amount of twist to depend upon the difference of the angles θ_1 and θ_2. It is reasonable to assume V_{int} to be an even function of $\theta_1 - \theta_2$. Thus

$$V_{int}(0) = 0, \quad V'_{int}(0) = 0.$$

For small deflection angles (the only case we consider) the simplest assumption is then to take

$$V_{int}(\theta_1 - \theta_2) = \frac{1}{2}\kappa(\theta_1 - \theta_2)^2$$

[1] These expressions should be compared with (2.24).

where κ is a positive constant. Since we are assuming the angles are small, the potential energy is then given, to a good approximation, by

$$\mathrm{PE} = \frac{1}{2} m_1 g \ell_1 \, \theta_1^2 + \frac{1}{2} m_2 g \ell_2 \, \theta_2^2 + \frac{1}{2} \kappa (\theta_1 - \theta_2)^2 .$$

Under these assumptions it can be shown that Newton's equations are

$$\begin{aligned} m_1 \ell_1^2 \ddot{\theta}_1 &= -(m_1 g \ell_1 + \kappa)\theta_1 + \kappa \theta_2 \,, \\ m_2 \ell_2^2 \ddot{\theta}_2 &= \kappa \theta_1 - (m_2 g \ell_2 + \kappa)\theta_2 \,. \end{aligned}$$

Observe that for $\kappa = 0$ the ODEs reduce to two uncoupled equations for the *linearized* mathematical pendulum. To simplify matters somewhat, we introduce

$$\omega_1^2 = \frac{g}{\ell_1}, \quad \omega_2 = \frac{g}{\ell_2} \quad k_1 = \frac{\kappa}{m_1 \ell_1^2}, \quad k_2 = \frac{\kappa}{m_2 \ell_2^2} \,.$$

Then it is not difficult to show (you need not do this) that the above differential equations become

$$\boxed{\begin{aligned} \ddot{\theta}_1 &= -(\omega_1^2 + k_1)\theta_1 + k_1 \theta_2 \\ \ddot{\theta}_2 &= k_2 \theta_1 - (\omega_2^2 + k_2)\theta_2 \,. \end{aligned}} \tag{6.19}$$

We could change this into a system of first order DEs (the matrix A would be 4×4). However, since equations of this form come up frequently in the theory of small oscillations, we proceed to develop a "mini theory" for these equations. Define

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} .$$

Show that the equations (6.19) can be written as

$$\ddot{\Theta} = A\Theta \tag{6.20}$$

where A is a 2×2 matrix. Find the matrix A. Assume a solution of (6.20) to be of the form

$$\Theta(t) = e^{i\omega t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} . \tag{6.21}$$

Using (6.21) in (6.20) show that (6.20) reduces to

$$A\Theta = -\omega^2 \Theta. \tag{6.22}$$

This is an eigenvalue problem. Show that ω^2 must equal

$$\begin{aligned}\omega_\pm^2 &= \frac{1}{2}(\omega_1^2+\omega_2^2+k_1+k_2)\\ &\quad\pm\frac{1}{2}\sqrt{(\omega_1^2-\omega_2^2)^2+2(\omega_1^2-\omega_2^2)(k_1-k_2)+(k_1+k_2)^2}\,. \end{aligned}\tag{6.23}$$

Show that an eigenvector for ω_+^2 is

$$f_1 = \begin{pmatrix} 1 \\ -k_2(\omega_+^2-\omega_2^2-k_2)^{-1} \end{pmatrix}, \tag{6.24}$$

and an eigenvector corresponding to ω_-^2 is

$$f_2 = \begin{pmatrix} -k_1(\omega_-^2-\omega_1^2-k_1)^{-1} \\ 1 \end{pmatrix}. \tag{6.25}$$

Now show that the general solution to (6.19) is

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = (c_1\cos(\omega_+ t)+c_2\sin(\omega_+ t))\,f_1 + (c_3\cos(\omega_- t)+c_4\sin(\omega_- t))\,f_2 \tag{6.26}$$

where c_i are real constants. One can determine these constants in terms of the initial data

$$\theta_1(0),\ \dot\theta_1(0),\ \theta_2(0),\ \dot\theta_2(0).$$

To get some feeling for these rather complicated expressions, we consider the special case

$$\theta_1(0)=\theta_0,\ \dot\theta_1(0)=0,\ \theta_2(0)=0,\ \dot\theta_2(0)=0 \tag{6.27}$$

with

$$m_1=m_2=m,\ \ell_1=\ell_2=\ell. \tag{6.28}$$

These last conditions imply

$$\omega_1=\omega_2:=\omega_0.$$

Explain in words what these initial conditions, (6.27), correspond to in the physical set up.

If we define

$$k=\frac{\kappa}{m\,\ell^2},$$

show that in the special case (6.27) and (6.28) that

$$\omega_+=\sqrt{\omega_0^2+2k}\ \text{ and }\ \omega_-=\omega_0. \tag{6.29}$$

In this same case solve for the coefficients c_1, c_2, c_3 and c_4 and show that

$$c_1 = \frac{1}{2}\theta_0,\ c_2 = 0,\ c_3 = \frac{1}{2}\theta_0,\ c_4 = 0,$$

and hence (6.26) becomes

$$\begin{aligned} \theta_1(t) &= \theta_0 \cos\left(\frac{1}{2}(\omega_+ + \omega_-)t\right) \cos\left(\frac{1}{2}(\omega_+ - \omega_-)t\right), \\ \theta_2(t) &= \theta_0 \sin\left(\frac{1}{2}(\omega_+ + \omega_-)t\right) \sin\left(\frac{1}{2}(\omega_+ - \omega_-)t\right). \end{aligned}$$

Suppose further that

$$\frac{k}{\omega_0^2} \ll 1. \tag{6.30}$$

What does this correspond to physically? Under assumption (6.30), show that *approximately*

$$\begin{aligned} \theta_1(t) &\approx \theta_0 \cos(\omega_0 t)\ \cos\left(\frac{k}{2\omega_0}t\right), \\ \theta_2(t) &\approx \theta_0 \sin(\omega_0 t)\ \sin\left(\frac{k}{2\omega_0}t\right). \end{aligned} \tag{6.31}$$

Discuss the implications of (6.31) in terms of the periods

$$T_0 = \frac{2\pi}{\omega_0} \quad \text{and} \quad T_1 = \frac{2\pi}{k/2\omega_0}.$$

Show that in this approximation

$$T_1 \gg T_0.$$

Draw plots of $\theta_1(t)$ and $\theta_2(t)$ using the approximate expressions (6.31).

Chapter 7

The Laplace Transform

7.1 Matrix version of the method of Laplace transforms for solving constant coefficient DE's

In §6.1 of Boyce and DiPrima we learned that the Laplace transform of a function $f(t)$ satisfying the hypotheses of Theorem 6.1.2. is

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-ts} f(t)\, dt \tag{7.1}$$

for s sufficiently large. We extend (7.1) to vector-valued functions $f(t)$,

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix} \tag{7.2}$$

(each of whose components satisfy the hypotheses of Theorem 6.1.2.) by

$$F(s) = \mathcal{L}(f)(s) = \begin{pmatrix} \int_0^\infty e^{-ts} f_1(t)\, dt \\ \int_0^\infty e^{-ts} f_2(t)\, dt \\ \vdots \\ \int_0^\infty e^{-ts} f_n(t)\, dt \end{pmatrix}. \tag{7.3}$$

Theorem 6.2.1 generalizes immediately to vector-valued functions; namely,

$$\mathcal{L}(\frac{df}{dt})(s) = s\mathcal{L}(f)(s) - f(0). \tag{7.4}$$

We now explain how *matrix Laplace transforms* are used to solve the matrix ODE

$$\frac{dx}{dt} = Ax + f(t) \tag{7.5}$$

where A is a constant coefficient $n \times n$ matrix, $f(t)$ is a vector-valued function of the independent variable t ("forcing term") with initial condition

$$x(0) = x_0. \tag{7.6}$$

First, we take the Laplace transform of both sides of (7.5). From (7.4) we see that the Laplace transform of the LHS of (7.5) is

$$\mathcal{L}(\frac{dx}{dt}) = s\mathcal{L}(x) - x_0.$$

The Laplace transform of the RHS of (7.5) is

$$\begin{aligned} \mathcal{L}(Ax + f) &= \mathcal{L}(Ax) + \mathcal{L}(f) \\ &= A\mathcal{L}(x) + F(s) \end{aligned}$$

where we set $F(s) = \mathcal{L}(f)(s)$ and we used the fact that A is independent of t to conclude[1]

$$\mathcal{L}(Ax) = A\mathcal{L}(x). \tag{7.7}$$

Thus the Laplace transform of (7.5) is

$$s\mathcal{L}(x) - x_0 = A\mathcal{L}(x) + F,$$

or

$$(sI_n - A)\mathcal{L}(x) = x_0 + F(s) \tag{7.8}$$

where I_n is the $n \times n$ identity matrix. Equation (7.8) is a linear system of algebraic equations for $\mathcal{L}(x)$. We now proceed to solve (7.8). This can be done once we know that $(sI_n - A)$ is invertible. Recall that a matrix is invertible if and only if the determinant of the matrix is nonzero. The determinant of the matrix in question is

$$p(s) := \det(sI_n - A), \tag{7.9}$$

which is the characteristic polynomial of the matrix A. We know that the zeros of $p(s)$ are the eigenvalues of A. If s is larger than the absolute value of the largest eigenvalue of A; in symbols,

$$s > \max|\lambda_i|, \tag{7.10}$$

then $p(s)$ cannot vanish and hence $(sI_n - A)^{-1}$ exists. We assume s satisfies this condition. Then multiplying both sides of (7.8) by $(sI_n - A)^{-1}$ results in

[1] You are asked to prove (7.7) in an exercise.

$$\boxed{\mathcal{L}(x)(s) = (sI_n - A)^{-1}x_0 + (sI_n - A)^{-1}F(s).} \tag{7.11}$$

Equation (7.11) is the basic result in the application of Laplace transforms to the solution of constant coefficient differential equations with an inhomogeneous forcing term. Equation (7.11) will be a quick way to solve initial value problems once we learn efficient methods to (i) compute $(sI_n - A)^{-1}$, (ii) compute the Laplace transform of various forcing terms $F(s) = \mathcal{L}(f)(s)$, and (iii) find the inverse Laplace transform. Step (iii) is made easier by the use of extensive Laplace transform tables. It should be noted that many of the DE techniques one learns in engineering courses can be described as efficient methods to do these three steps for examples that are of interest to engineers.

We now give two examples that apply (7.11).

7.1.1 Example 1

Consider the scalar ODE

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \tag{7.12}$$

where b and c are constants. We first rewrite this as a system

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix},$$

so that

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} x + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Then

$$sI_2 - A = \begin{pmatrix} s & -1 \\ c & s+b \end{pmatrix},$$

and

$$(sI_2 - A)^{-1} = \frac{1}{s^2 + bs + c}\begin{pmatrix} s+b & 1 \\ -c & s \end{pmatrix}.$$

Observe that the characteristic polynomial

$$p(s) = \det(sI_2 - A) = s^2 + bs + c$$

appears in the denominator of the matrix elements of $(sI_2 - A)^{-1}$. (This factor in Laplace transforms should be familiar from the scalar treatment—here we see it is the characteristic polynomial of A.) By (7.11)

$$\mathcal{L}(x)(s) = \frac{1}{s^2+bs+c}\begin{pmatrix} (s+b)y(0)+y'(0) \\ -cy(0)+sy'(0) \end{pmatrix} + \frac{F(s)}{s^2+bs+c}\begin{pmatrix} 1 \\ s \end{pmatrix}$$

where $F(s) = \mathcal{L}(f)(s)$. This implies that the Laplace transform of $y(t)$ is given by

$$\mathcal{L}(y)(s) = \frac{(s+b)y(0)+y'(0)}{s^2+bs+c} + \frac{F(s)}{s^2+bs+c}. \tag{7.13}$$

This derivation of (7.13) should be compared with the derivation of equation (16) on page 298 of Boyce and DiPrima (in our example $a = 1$).

7.1.2 Example 2

We consider the system (7.5) for the special case of $n = 3$ with $f(t) = 0$ and A given by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}. \tag{7.14}$$

The characteristic polynomial of (7.14) is

$$p(s) = s^3 - 2s^2 + s - 2 = (s^2+1)(s-2) \tag{7.15}$$

and so the matrix A has eigenvalues $\pm i$ and 2. A rather long linear algebra computation shows that

$$(sI_3 - A)^{-1} = \frac{1}{p(s)}\begin{pmatrix} s^2-s-1 & 1 & -s+2 \\ s+2 & s^2 & s-2 \\ s-3 & -s+1 & s^2-3s+2 \end{pmatrix}. \tag{7.16}$$

If one writes a partial fraction decomposition of each of the matrix elements appearing in (7.16) and collects together terms with like denominators, then (7.16) can be written as

$$\begin{aligned} (sI_3 - A)^{-1} &= \frac{1}{s-2}\begin{pmatrix} 1/5 & 1/5 & 0 \\ 4/5 & 4/5 & 0 \\ -1/5 & -1/5 & 0 \end{pmatrix} \\ &\quad + \frac{1}{s^2+1}\begin{pmatrix} (3+4s)/5 & -(2+s)/5 & -1 \\ -(3+4s)/5 & (2+s)/5 & 1 \\ (7+s)/5 & (-3+s)/5 & -1+s \end{pmatrix}. \end{aligned} \tag{7.17}$$

We now apply (7.17) to solve (7.5) with the above A and $f = 0$ for the case of initial conditions

$$x_0 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}. \tag{7.18}$$

We find

$$\mathcal{L}(x)(s) = (sI_3 - A)^{-1} x_0 = \frac{1}{s-2} \begin{pmatrix} -1/5 \\ -4/5 \\ 1/5 \end{pmatrix} + \frac{s}{s^2+1} \begin{pmatrix} 6/5 \\ -6/5 \\ 4/5 \end{pmatrix} + \frac{1}{s^2+1} \begin{pmatrix} 2/5 \\ -2/5 \\ 8/5 \end{pmatrix}. \tag{7.19}$$

To find $x(t)$ from (7.19) we use Table 6.2.1 on page 300 of Boyce and DiPrima; in particular, entries 2, 5, and 6. Thus

$$x(t) = e^{2t} \begin{pmatrix} -1/5 \\ -4/5 \\ 1/5 \end{pmatrix} + \cos t \begin{pmatrix} 6/5 \\ -6/5 \\ 4/5 \end{pmatrix} + \sin t \begin{pmatrix} 2/5 \\ -2/5 \\ 8/5 \end{pmatrix}.$$

We give now a second derivation of (7.19) using the eigenvectors of A. As noted above, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. If we denote by ϕ_j an eigenvector associated to eigenvalue λ_j $(j = 1, 2, 3)$, then a routine linear algebra computation gives the following possible choices for the ϕ_j:

$$\phi_1 = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} (1+i)/2 \\ -(1+i)/2 \\ 1 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} (1-i)/2 \\ (-1+i)/2 \\ 1 \end{pmatrix}.$$

Now for any eigenvector ϕ corresponding to eigenvalue λ of a matrix A we have

$$(sI_n - A)^{-1}\phi = (s-\lambda)^{-1}\phi.$$

To use this observation we first write

$$x_0 = c_1\phi_1 + c_2\phi_2 + c_3\phi_3.$$

A computation shows that

$$c_1 = 1/5, \quad c_2 = 2/5 - 4i/5, \quad \text{and} \quad c_3 = 2/5 + 4i/5.$$

Thus

$$(sI_3 - A)^{-1} x_0 = \frac{1}{5}(s-2)^{-1}\phi_1 + \frac{2-4i}{5}(s-i)^{-1}\phi_2 + \frac{2+4i}{5}(s+i)^{-1}\phi_3.$$

Combining the last two terms gives (7.19).

7.2 Structure of $(sI_n - A)^{-1}$ when A is diagonalizable

In this section we assume that the matrix A is diagonalizable; that is, we assume a set of linearly independent eigenvectors of A form a basis. Recall the following two theorems from linear algebra: (1) If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable; and (2) If the matrix A is symmetric (hermitian if the entries are complex), then A is diagonalizable.

Since A is assumed to be diagonalizable, there exists a nonsingular matrix P such that

$$A = PDP^{-1}$$

where D is

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

and each eigenvalue λ_i of A appears as many times as the (algebraic) multiplicity of λ_i. Thus

$$\begin{aligned} sI_n - A &= sI_n - PDP^{-1} \\ &= P(sI_n - D)P^{-1}, \end{aligned}$$

so that

$$\begin{aligned} (sI_n - A)^{-1} &= \left(P(sI_n - D)P^{-1}\right)^{-1} \\ &= P(sI_n - D)^{-1}P^{-1}. \end{aligned}$$

Since P and P^{-1} are independent of s, the s dependence of $(sI_n - A)^{-1}$ resides in the diagonal matrix $(sI_n - D)^{-1}$. This tells us that the partial fraction decomposition of the matrix $(sI_n - A)^{-1}$ is of the form

$$(sI_n - A)^{-1} = \sum_{j=1}^{n} \frac{1}{s - \lambda_j} P_j$$

where

$$P_j = PE_jP^{-1}$$

and E_j is the diagonal matrix with all zeros on the main diagonal except for 1 at the (j,j)th entry. This follows from the fact that

$$(sI_n - D)^{-1} = \sum_{j=1}^{n} \frac{1}{s - \lambda_j} E_j$$

Note that P_j have the property that

$$P_j^2 = P_j.$$

Such matrices are called *projection operators.*

In general, it follows from Cramer's method of computing the inverse of a matrix, that the general structure of $(sI_n - A)^{-1}$ will be $1/p(s)$ times a matrix whose entries are polynomials of at most degree $n-1$ in s. When an eigenvalue, say λ_1, is degenerate and of (algebraic) multiplicity m_1, then the characteristic polynomial will have a factor $(s-\lambda_1)^{m_1}$. We have seen that if the matrix is diagonalizable, upon a partial fraction decomposition only a single power of $(s-\lambda_1)$ will appear in the denominator of the partial fraction decompostion. Finally, we conclude by mentioning that when the matrix A is not diagonalizable, then this is reflected in the partial fraction decomposition of $(sI_n - A)^{-1}$ in that some powers of $(s-\lambda_j)$ occur to a higher degree than 1.

7.3 Exercises

#1.

Use the Laplace transform to find the solution of the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

#2.

Let A be a $n \times n$ matrix whose entries are real numbers and $x \in \mathbf{R}^n$. Prove that

$$\mathcal{L}(Ax) = A\mathcal{L}(x)$$

where $\mathcal{L}$ denotes the Laplace transform.

#3.

Let E_j denote the diagonal $n \times n$ matrix with all zeros on the main diagonal except for 1 at the (j,j) entry.

- Prove that $E_j^2 = E_j$.
- Show that if P is any invertible $n \times n$ matrix, then $P_j^2 = P_j$ where $P_j := PE_jP^{-1}$.

#4.

It is a fact that you will learn in an advanced linear algebra course, that *if* a 2×2 matrix A is *not* diagonalizable, then there exists a nonsingular matrix P such that

$$A = P\,B\,P^{-1}$$

where

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

for some constant λ.

- Show that λ must be an eigenvalue of A with algebraic multiplicity 2.
- Find an eigenvector of A (in terms of the matrix P), and show that A has no other eigenvectors (except, of course, scalar multiples of the vector you have already found).
- Show that

  $$(sI_2 - A)^{-1} = \frac{1}{s-\lambda} P E_1 P^{-1} + \frac{1}{s-\lambda} P E_2 P^{-1} + \frac{1}{(s-\lambda)^2} P N P^{-1}$$

  where

  $$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

- Relate what is said here to the footnote in Exercise 5.4.2.