Assignment 3

(due date: Monday, Oct 20)

Below, $[a,b]$ denotes a finite closed interval in $\mathbb{R}$, and $C([a,b])$ is the space of all continuous real-valued functions on $[a,b]$, and $\|\cdot\|_{\sup}$ is the uniform norm on $C([a,b])$.

Problem 1. Let $(X,d_X),(Y,d_Y)$, and (Z,d_Z) be metric spaces and let $f: X \to Y$, and $g: Y \to Z$ be two continuous functions. Show that $h: X \to Z$, defined by $h(x) = g(f(x))$ is also continuous.

Problem 2. Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function f. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Problem 3. Consider the space $C^1([0,1])$ of continuously differentiable functions,

$$C^1([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \in C([0,1]) \text{ and } f' \in C([0,1])\}$$

with the C^1 norm defined by

$$\|f\| = \|f\|_{\sup} + \|f'\|_{\sup}$$

Prove that $(C^1([0,1]), \|\cdot\|)$ is a complete metric space.

Problem 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying

$$\lim_{\|x\|\to\infty} f(x) = +\infty$$

a) Prove that f is bounded below.

b) Prove that f attains its infimum.

Problem 5. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function of two real variables. Suppose $p(x,y) \geq 0$ for all $x, y \in \mathbb{R}$. Do all such functions p attain their infimum? Prove or disprove.