Assignment 6

(due date: Monday, Nov 17)

Problem 1. Is $(C^1([0,1]), \|\cdot\|_{\sup})$ a complete metric space? Prove or disprove.

Problem 2. Prove that if T is a contraction on a metric space X, then T is uniformly continuous.

Problem 3. Let A be an $n \times n$ matrix with real entries. Suppose that all the eigenvalues of A are real and of absolute value strictly less than 1. Prove that A, considered as a linear map $\mathbb{R}^n \to \mathbb{R}^n$, is a contraction.

Problem 4. Consider the map $T : (0,\infty) \to (0,\infty)$ defined by

$$Ty = \frac{1}{2}(y + \frac{2}{y})$$

We proved in class that $y_n := T^n(1) \to \sqrt{2}$, exponentially fast.

a) Show that for any $y > 0$, the convergence of $T^n y$ to $\sqrt{x}$ is in fact faster than exponential. Start from equation (5.2) in the notes to obtain a good estimate for $|Ty - \sqrt{x}|$.

b) Take $y = 1$ and $x = 2$. Find a good estimate of how large n needs to be for $T^n 1$ and $\sqrt{2}$ to have identical first d digits. E.g., how many iterations of the map T are sufficient to compute the first 63 digits of $\sqrt{2}$?

Problem 5. Consider the logistic map $T : [0,1] \to [0,1]$ defined by

$$Tx = 4\mu x(1 - x)$$

with $\mu \in [0,1]$.
a) Prove that there exists $\delta > 0$ such that for $\mu \in (.25, .25 + \delta)$, T has exactly two fixed points: 0 and x_0, with $0 < x_0 < 1$.
b) Prove that there exists $\delta_1 > 0$ such that for $\mu \in (.25, .25 + \delta_1)$, and any ϵ, $0 < \epsilon < .5$, $T^n x \to x_0$, for all $x \in [\epsilon, 1 - \epsilon]$.
c) What happens for larger values of μ? Investigate this question as far as you can. You are not expected to give a complete analysis.