Assignment 4

(due date: Monday, February 2)

Problem 1. For all $t > 0$, define $\phi_t : \mathbb{R} \to \mathbb{R}$ by

$$\phi_t(x) = \frac{1}{\sqrt{4\pi t}} \exp -\frac{x^2}{4t}$$

a) Show that for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$I_t f(x) = \int_{\mathbb{R}} \phi_t(x-y) f(y)\, dy$$

defines a bounded function in $C^\infty(\mathbb{R})$.

b) Prove that I_t is an approximate identity in the sense that for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{t \downarrow 0} I_t f(x) = f(x), \quad \text{for all } x \in \mathbb{R}$$

So, it makes sense to define $I_0 f = f$.

Problem 2. Show that for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, $u(x,t) = I_t f(x)$ solves the heat equation on $\mathbb{R}$ with initial condition f for all $t \geq 0$, i.e.,

$$\begin{aligned} u_t &= u_{xx} \quad (t > 0) \\ u(x,0) &= f(x) \end{aligned}$$

Problem 3. Prove the semigroup property of the solutions of the heat equation, i.e., for all $t, s \geq 0$, and any bounded continuous function f,

$$I_s(I_t f) = I_{s+t} f$$

Problem 4. Consider the heat equation on a ring with a time-independent heat source $f \in L^2(T)$, and initial condition $u(x,0) = 0$:

$$\begin{aligned} u_t &= u_{xx} + f(x) \quad (t > 0) \\ u(x,0) &= 0 \end{aligned}$$

Use Fourier series to solve for $u(\cdot, t) \in L^2(T)$ for all $t > 0$. Discuss the behavior of the solution for large t.