Assignment 5

(due date: Monday, February 9)

Problem 1. Let $f \in L^2([0,1])$ be such that

$$\int_0^t f(x)\,dx = 0 \qquad \text{for all } t \in [0,1]$$

Prove that $f = 0$. (Hint: E.g., expand f in the Haar basis defined in class.)

Problem 2. Let $\{u_n\}_{n=1}^{\infty}$ be an o.n. basis of $L^2([0,1])$. Show that

$$\sum_{m=1}^{\infty} \left| \int_0^t \overline{u_n(x)}\,dx \right|^2 = t \qquad \text{for all } t \in [0,1] \tag{1}$$

Problem 3. Suppose that $\{u_n\}_{n=1}^{\infty}$ is an o.n. set in $L^2([0,1])$, for which (1) of Problem 2 holds. Prove that $\{u_n\}_{n=1}^{\infty}$ is an o.n. basis, i.e., prove that it is complete. (Hint: Problem 1 can help.)

Problem 4. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two mutually orthogonal closed subspaces of a Hilbert space $\mathcal{H}$ and P and Q are the orthogonal projections with ran $P = \mathcal{H}_1$ and ran $Q = \mathcal{H}_2$.

a) Prove that $S = P + Q$ is an orthogonal projection.

b) What is the range of S?

Problem 5. Let P and Q be orthogonal projections on a Hilbert space $\mathcal{H}$ such that $S = P+Q$ is also an orthogonal projection. Prove that ran $P \perp$ ran Q.