25 L^p spaces

For this chapter, *Analysis* by Lieb & Loss is recommended reading.

25.1 Definition

Let (Ω, Σ, μ) be a measure space. We are mainly interested in Ω that are a measurable subset of $\mathbb{R}^n$. Most of the results, however, are valid for an arbitrary measure space. We will introduce the spaces $L^p(\Omega, \mu)$, for $1 \leq p < +\infty$ and $p = +\infty$ separately, as they require a different approach.

Fix $1 \leq p < +\infty$. $L^p(\Omega)$, or $L^p(\Omega, \mu(dx))$ if it is desirably to express the measure μ explicitly, is the linear space of all measurable functions $f : \Omega \to \mathbb{C}$, such that $|f|^p$ is integrable, i.e.,

$$\int_\Omega |f(x)|^p \, \mu(dx) < +\infty \tag{25.1}$$

We will often write L^p instead of $L^p(\Omega)$ if this does not lead to confusion. Note that the Lebesgue measure λ, or $\lambda(dx)$, is often written as dx. So, $L^p([a, b], dx)$ means the L^p-space of complex-valued functions on the interval $[a, b]$ with the standard Lebesgue measure.

Exercise 25.1 *Prove that for all $1 \leq p < +\infty$, the space of measurable functions such that (25.1) holds is a linear space for the standard linear operations on functions: pointwise addition and multiplication by a scalar.*

The space $L^\infty(\Omega, \mu)$ consists of all functions $f : \Omega \to \mathbb{C}$ such that there exists a finite constant M such that

$$|f(x)| \leq M, \quad \mu - \text{a.e.}$$

For $1 \leq p < +\infty$, the L^p norm is defined by

$$\|f\|_p := \|f\|_{L^p} := \left[\int_\Omega |f(x)|^p \, \mu(dx)\right]^{1/p}, \quad \text{for all } f \in L^p(\Omega, \mu) \tag{25.2}$$

In fact, this is not a norm defined on the L^p-spaces as we have described them so far. This is because if $f = g$ μ-a.e., then $\|f - g\|_p = 0$. So, in order to make L^p into a normed space we need to consider all functions that are equal to f μ-a.e., as one element, i.e., L^p is really a space of equivalence classes of functions. We will prove that L^p, thus defined, and equipped with the L^p-norm, is a Banach space. Recall that in MAT203A we defined $L^p([0, 1])$ as the completion of the space of all continuous functions with the topology defined by the L^p-norm. The present definition has the advantage of giving us immediately a realization of L^p as a space of equivalence classes of functions, instead of a space of equivalence classes of Cauchy sequences of functions. These two definitions are equivalent however, i.e., they lead to the same Banach space.

Exercise 25.2 *Prove that $f \sim g$ iff $f = g$ μ-a.e., indeed defines an equivalence relation on the space of all measurable functions.*

Exercise 25.3 *Prove that $\|\cdot\|_p$ defined by (25.2) is a norm on $L^p(\Omega)$. Hint: use Minkowski's inequality (see the next subsection).*

Theorem 25.1 *For any measure space (Ω, Σ, μ), and $1 \leq p < +\infty$, $L^p(\Omega, \mu)$, with the L^p-norm, is a Banach space.*

Proof: We will abreviate $L^p(\Omega, \mu)$ by L^p and $\|\cdot\|_p$ by $\|\cdot\|$. Recall that for normed linear spaces completeness can be proved by showing that every absolutely convergent series converges. So, let $f_n \in L^p$, $n = 1, 2, \ldots$, be such that

$$\sum_{n=1}^{\infty} \|f_n\| = M < +\infty \tag{25.3}$$

We need to show that there exists $f \in L^p$ such that

$$\lim_{N\to\infty} \|f - \sum_{n=1}^{N} f_n\| = 0 \tag{25.4}$$

To this end, we first consider the sequence of functions g_N defind by

$$g_N(x) = \sum_{n=1}^{N} |f_n(x)| \quad .$$

Clearly, $g_N(x) \geq 0$, and pointwise convergence of (g_N) to a function $g : \Omega \to [0, +\infty]$, is guaranteed by the monotonicity of the sequence. We also have that

$$\|g_N\| \leq \sum_{n=1}^{N} \|f_n\| \leq M \quad .$$

Hence,

$$\int |g_N|^p d\mu = \|g_N\|^p \leq M$$

and, by the monotone convergence theorem (Theorem 24.4), this implies that

$$\|g\|^p = \lim_{N\to\infty} \|g_N\| \leq\leq M$$

In particular, $\|g\| < +\infty$, i.e., $g \in L^p$, and this implies that $g(x) < +\infty \mu -$ a.e., which, by the definition of g, means that $\sum_n f_n(x)$ is absolutely convergent $\mu -$ a.e., and, therefore, the series converges pointwise $\mu -$ a.e. and we can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Note that the partial sums of this series satisfy

$$\left| \sum_{n=1}^{N} f_n(x) \right|^p \leq g(x)^p$$

As, $g \in \mathcal{L}^p$, and $|f(x)| \leq g(x)$, we mist have that $f \in L^p$. Furthermore, this also means that $g^p \in L^1$, and we have

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^p \leq (2g(x))^p \quad .$$

Therefore, we can apply the dominated convergence theorem (Theorem 24.6) to show the convergence of the series in the L^p-norm:

$$\lim_{N\to\infty} \int \left| f - \sum_{n=1}^{N} f_n \right|^p \mu(dx) = \int 0\mu(dx) = 0.$$

■

Note that, in the proof above, we used the following trivial consequence of the dominated convergence theorem.

Exercise 25.4 *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of measurable functions converging to f pointwise μ-a.e.. Suppose there exists $g \in L^p(\Omega, \mu)$ such that $|f_n| \leq g$, μ-a.e.. Then $f_n \to f$ in the L^p-norm.*

There is a semi-converse to Exercise 25.4: it is also true that if $f_n \to f$ in L^p, then there exists a subsequence of (f_n) converging to f pointwise a.e.. We do not prove this here.

On $L^\infty(\Omega, \mu)$ the L^∞-norm is defined as follows:

$$\|f\|_\infty := \|f\|_{L^\infty} := \inf\{C \mid |f(x)| \leq C, \ \mu - \text{a.e.}\}$$

Clearly, if f is continuous, $\|f\|_\infty = \|f\|_{\text{sup}}$. In general, for $f \in L^\infty$, $\|f\|_\infty$ is the essential supremum of $|f|$, i.e.

$$\|f\|_\infty = \text{ess sup}\{|f(x)| \mid x \in \Omega\}$$

Note that $|f|$ may be greater than its essential supremum on a set of measure zero (but this is not essential!).

An important fact that we will not prove here is that $L^\infty(\Omega, \mu)$ with the L^∞-norm is also a Banach space, when considered as a space of equivalence classes of functions.

25.2 The basic inequalities

It has been said that Analysis is the art of estimating, i.e., proving inequalities. Inequalities are proved using other inequalities. Here are the Ur-inequalities of L^p theory.

25.2.1 Jensen's inequality

If (Ω, Σ, μ) is a measure space with $\mu(\Omega) < \infty$, we can define the *average* of any $f \in L^1(\Omega, \mu)$ by

$$\langle f \rangle_\mu := \frac{1}{\mu(\Omega)} \int_\Omega f(x)\, \mu(dx) \tag{25.5}$$

Jensen's inequality says that the average of a convex function of f is greater then or equal to the value of the convex function at the average of f.

Before we state and proof the theorem, we recall a few properties of convex functions. Let C be a *convex subset* of a linear space, i.e., for every pair $x, y \in C$, and any $t \in [0,1]$, we have $tx + (1-t)y \in C$. Geometrically this means that the line segment connecting x and y is entirely in C. By definition $J : C \to \mathbb{R}$ is *convex* iff for every $x, y \in C$, and $t \in [0,1]$,

$$J(tx + (1-t)y) \leq tJ(x) + (1-t)J(y) \tag{25.6}$$

which means that the subset of $C \times \mathbb{R}$ bounded from below by the graph (this is called the *epi-graph*) of J is convex.

Exercise 25.5 *Let $J : \mathbb{R} \to \mathbb{R}$ be a convex function. Prove the following properties:*
a) *For all $x \in \mathbb{R}$ the left- and right-derivatives, $J'_-(x)$ and $J'_+(x)$ exist and satisfy*

$$J'_-(x) \leq J'_+(x)$$

b) *J is continuous on $\mathbb{R}$.*
c) *For all $x \in \mathbb{R}$, there exists a constant $c \in \mathbb{R}$ such that*

$$J(y) \geq J(x) + c(y - x), \qquad \textit{for all } y \in \mathbb{R} \tag{25.7}$$

The graph of the function $y \mapsto c(y - x)$ satisfying (25.7) is called a *support line* of J at x.

Exercise 25.6 *Prove that a function $J : \mathbb{R} \to \mathbb{R}$ is convex iff J has at least one support line at each $x \in \mathbb{R}$.*

Theorem 25.2 (Jensen's inequality) *Let (Ω, Σ, μ) be a measure space with $\mu(\Omega) < \infty$. For any convex function $J : \mathbb{R} \to \mathbb{R}$ and any real-valued $f \in L^1(\Omega, \mu)$, one has the inequality*

$$\langle J \circ f \rangle_\mu \geq J(\langle f \rangle_\mu) \tag{25.8}$$

Proof: From Exercise 25.5 c) we know that there exists a constant $c \in \mathbb{R}$ such that, for all $y \in \mathbb{R}$,

$$J(y) \geq J(\langle f \rangle_\mu) - c(y - \langle f \rangle_\mu) \tag{25.9}$$

By putting $y = f(x)$ in this inequality and integrate both sides, we obtain

$$\begin{aligned} \int_\Omega J \circ f(x)\,\mu(dx) &\geq J(\langle f \rangle_\mu) \int_\Omega \mu(dx) + c\left(\int_\Omega f(x)\,\mu(dx) - \langle f \rangle_\mu \int_\Omega \mu(dx)\right) \\ &= J(\langle f \rangle_\mu) \int_\Omega \mu(dx) \end{aligned}$$

where we have used that $\int_\Omega \mu(dx) = \mu(\Omega) < \infty$ and (25.5). By dividing by $\mu(\Omega)$ one obtains Jensen's inequality. ∎

Note that the LHS of Jensen's inequality may be infinite. Although $J \circ f$ need not be in L^1, its negative part is due to (25.9). Therefore, its integral is well-defined.

One can also show that equality in Jensen's inequality can only occur in the following two circumstances: either $J(x)$ is not *strictly convex* at $x = \langle f \rangle_\mu$, meaning that J has a linear piece in its graph, or f is a constant function.

25.2.2 Hölder's inequality

The Hölder inequalities, and the special case of the Cauchy-Schwarz inequality, are probably the most important ones for proving estimates in L^p-spaces. Hölder's inequality is formulated for a pair of functions, one in L^p and one in L^q, where p and q satisfy: $1 \leq p, q \leq +\infty$, and

$$\frac{1}{p} + \frac{1}{q} = 1$$

with the convention that $1/(+\infty) = 0$. Such p and q are called *dual indices* or *dual exponents.*

Theorem 25.3 (Hölder's inequality) *Let $1 \leq p, q \leq +\infty$ and such that $1/p + 1/q = 1$. Then, for all $f \in L^p(\Omega, \mu)$, and $g \in L^q(\Omega, \mu)$, one has*

$$\left| \int f(x) g(x)\,\mu(dx) \right| \leq \|f\|_p \|g\|_q$$

Proof: For the dual pair of indices $\{p,q\}=\{1,+\infty\}$, Hölder's inequality reduces to the well-known inequality

$$\left|\int_\Omega f(x)g(x)\,\mu(dx)\right| \le \|g\|_\infty \int_\Omega |f(x)|\mu(dx)$$

So, we may as well assume that $1<p,q<\infty$.

Next, we observe that

$$\Omega_1 := \{x\in\Omega \mid g(x)\neq 0\}$$

is a measurable set and that

$$\left|\int_\Omega f(x)g(x)\,\mu(dx)\right| = \left|\int_{\Omega_1} f(x)g(x)\,\mu(dx)\right| \le \int_{\Omega_1} |f(x)|\,|g(x)|\,\mu(dx)$$

And for the RHS of Hölder's inequality we have

$$\|f\|_p^p = \int_\Omega |f(x)|^p\,\mu(dx) \ge \int_{\Omega_1} |f(x)|^p\,\mu(dx)$$

Therefore, without loss of generality, we may assume that $f(x)\ge 0$, and $g(x)>0$, for all $x\in\Omega$.

With these assumptions in mind we can define a new measure μ on Ω, by

$$\nu(dx) := g(x)^q\,\mu(dx)$$

Then, we apply Jensen's inequality (Theorem 25.2), with the convex function $J(x)=|x|^p$ $(1\le p<\infty\,!)$, and the function

$$h(x)=\frac{f(x)}{g(x)^{q/p}}$$

Note that $h\in L^1(\Omega,\nu)$. We obtain

$$\begin{aligned}
\frac{1}{\int_\Omega g(x)^q\,\mu(dx)}\int_\Omega f(x)^p\,\mu(dx) &= \frac{1}{\int_\Omega \nu(dx)}\int_\Omega \frac{f(x)^p}{g(x)^q}\,\nu(dx)\\
&= \langle J\circ h\rangle_\nu\\
&\ge J(\langle h\rangle_\nu)\\
&= \left|\frac{1}{\int_\Omega \nu(dx)}\int_\Omega \frac{f(x)}{g(x)^{q/p}}\,\nu(dx)\right|^p\\
&= \frac{1}{\left|\int_\Omega g(x)^q\,\mu(dx)\right|^p}\left|\int_\Omega f(x)g(x)^{q-q/p}\,\mu(dx)\right|^p\\
&= \left|\frac{\int_\Omega f(x)g(x)\,\mu(dx)}{\int_\Omega g(x)^q\,\mu(dx)}\right|^p
\end{aligned}$$

Note that we have used the fact that p and q are dual exponents for the last equality. By taking the pth root, rearranging and using again that p and q are dual exponents, we see that the inequality above is equivalent to Hölder's inequality. ∎

Note that the Hölder inequalities also imply that the LHS is finite if the RHS is.

We refer to Lieb & Loss for a discussion of the case of equality in the Hölder inequalities.

In the special case $p = q = 2$, Hölder's inequality is the Cauchy-Schwarz inequality for L^2-spaces.

Exercise 25.7 *Prove the following generalization of Hölder's inequality: for n functions, $f_i \in L^{p_i}(\Omega, \mu)$, with $1 \le p_i \ge +\infty$, such that $\sum_{i=1}^n 1/p_i = 1$, one has*

$$\left| \int f_1(x) \cdots f_n(x)\, \mu(dx) \right| \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}$$

As a simple application of Hölder's inequality we can show that, if $\mu(\Omega_1) < \infty$, $L^p(\Omega, \mu) \subset L^{p'}(\Omega, \mu)$, whenever $1 \le p' \le p \le +\infty$. Indeed, define $q \ge 1$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p'}$$

p/p' and q/p' are then a pair of dual indices and therefore we can apply Hölder's inequality (with $f \to |f|^{p'}$, and $g \to 1$):

$$\begin{aligned} \|f\|_{p'}^{p'} &= \int_\Omega |f(x)|^{p'}\, \mu(dx) \\ &\le \left[\int_\Omega (|f(x)|^{p'})^{p/p'}\, \mu(dx) \right]^{p'/p} \times \left[\int_\Omega \mu(dx) \right]^{p'/q} \\ &= (\mu(\Omega)^{p/(p-p')} \|f\|_p^{p'} \end{aligned}$$

This inequality between the norms implies that $\|f\|_p$ is finite for every $f \in L^p$, which is equivalent to the claimed inclusion.

Remark: It is crucial for these inclusions to hold that $\mu(\Omega) < \infty$. This is illustrated in the following exercise.

Exercise 25.8 *Prove that for any pair of distinct exponents $1 \le p, p' \le +\infty$, we have $L^p(\mathbb{R}, dx) \not\subset L^{p'}(\mathbb{R}, dx)$.*

25.2.3 Minkowski's inequality

Theorem 25.4 (Minkowski's inequality) *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two sigma-finite measure spaces, and suppose $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a non-negative, $\mu_1 \times \mu_2$ measurable function. Then*

$$\int_{\Omega_2} \left[\int_{\Omega_1} f(x,y)^p \, \mu_1(dx) \right]^{1/p} \mu_2(dy) \geq \left(\int_{\Omega_1} \left[\int_{\Omega_2} f(x,y) \, \mu_2(dy) \right]^p \mu_1(dx) \right)^{1/p}$$

We will omit the proof of this theorem, although it is a nice application of Hölder's inequality that also uses the Monotone Convergence Theorem and Fubini's Theorem. See Lieb & Loss if you are interested.

The following corollary is a special case of the previous theorem. It is also often referred to as Minkowski's inequality. It is the triangle inequality for the L^p-norm.

Corollary 25.5 *For $1 \leq p \leq +\infty$, and any $f, g \in L^p(\Omega, \mu)$ we have*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

25.2.4 Chebyshev's inequality

Chebyshev's inequality is an elementary but very useful inequality in probability theory.

Theorem 25.6 (Chebyshev's inequality) *For all $f \in L^p(\Omega, \mu)$, $0 < p < \infty$, and $\epsilon > 0$ one has*

$$\mu(\{x \in \Omega \mid |f(x)| > \alpha\}) \leq \frac{1}{\alpha^p} \|f\|_p^p$$

Proof: Define $S_\alpha \in \Sigma$ by

$$S_\alpha = \{x \in \Omega \mid |f(x)| > \alpha\}$$

Then

$$\begin{aligned} \|f\|_p^p &= \int_\Omega |f(x)|^p \, \mu(dx) \\ &\geq \int_{E_\alpha} |f(x)|^p \, \mu(dx) \\ &\geq \alpha^p \int_{E_\alpha} 1 \, \mu(dx) \\ &= \alpha^p \mu(E_\alpha) \end{aligned}$$

which is what we had to prove. ■

Note that Chebyshev's inequality holds for all $p \in (0, \infty)$.

25.3 Continuous linear functionals; weak convergence

Let $L^p(\Omega, \mu)^*$ denote the space of all continuous (i.e., bounded) linear functionals on $L^p(\Omega, \mu)$.

Example:

Suppose $g \in L^q(\Omega, \mu)$, with q the exponent dual to p, i.e., $1/p + 1/q = 1$. Define φ_g by

$$\varphi_g(f) = \int_\Omega g(x) f(x)\, \mu(dx) \tag{25.10}$$

Clearly, φ_g is a linear functional where it is well-defined. Hölder's inequality shows that φ_g is in fact well-defined on all of L^p, and bounded, with an explicit bound on its norm given by

$$||\varphi_g||_{(L^p)^*} \leq ||g||_q$$

In one of the following subsection we will see that, if $1 \leq p < +\infty$ (note the strict inequality!), the linear functionals of the form φ_g, with $g \in L^q$, are in fact all of $(L^p)^*$, but for the moment this is immaterial.

As we have done several times before, the continuous linear functionals can be used to define a new topology on L^p, the topology of *weak (L^p) convergence*. Let f_n be a sequence in L^p. If there exists $f \in L^p$ such that

$$\lim_n \varphi(f_n) = \varphi(f), \qquad \text{for all } \in (L^p)^*$$

we say that f_n converges to f, weakly in L^p. As expected, $f_n \to f$, weakly, does not imply $f_n \to f$, strongly, i.e., for the L^p-norm.

In order to understand how f_n may converge to f weakly but not strongly it is sufficient to consider the case $f = 0$. There are essentially three ways in which this can happen. These three are were fittingly baptized by Lieb & and Loss as

1. f_n *oscillates to death*: e.g. $f_n(x) = \sin nx$ in $L^p([0, \pi], dx)$, for any $1 \leq p < \infty$.
2. f_n *goes up the spout*: e.g. $f_n(x) = n^{1/p} g(nx)$, for a fixed $g \in L^p(\mathbb{R})$.
3. f_n *wanders off to infinity*: e.g. $f_n(x) = g(x - n)$, for a fixed $g \in L^p(\mathbb{R})$.

Note that, in the case of the Hilbert space $L^2(\Omega, \mu)$, all three of these possibilities lead to expansion coefficients of f_n in any o.n. basis that wander off to infinity as the bases index tends to infinity.

Exercise 25.9 *Prove that in each of the three examples above one has $f_n \to 0$ weakly, but not strongly.*

25.3.1 The Hahn-Banach Theorem

The Hahn-Banach theorem is a fundamental result in functional analysis, i.e., the theory of topological vector spaces (see. e.g., Rudin ***). We will prove it here for the special case of the L^p spaces. In words the Hahn-Banach Theorem says that *continuous linear functionals separate*, meaning that, if $f \neq g$, then there is a continuous linear functional φ, such that $\varphi(f) \neq \varphi(g)$. Due to linearity this can be expressed as follows.

Theorem 25.7 (Hahn-Banach Theorem) *Suppose $f \in L^p(\Omega, \mu)$ satisfies*

$$\varphi(f) = 0, \qquad \textit{for all } \varphi \in L^p(\Omega, \mu)^*$$

Then, $f = 0$.

Proof: The idea of the proof is to show that for every $f \in L^p$ there is an element $g \in L^q$, with q the exponent dual to p such that the functional φ_g defined in (25.10) satisfies

$$\varphi_g(f) = \|f\|_p^p$$

Clearly, the existence of such a functional for each f implies the statement of the theorem.

First, we treat the case $1 < p < \infty$. Simply put

$$g(x) = \begin{cases} |f(x)|^{p-2}\overline{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

Then $\varphi_g(f) = \|f\|_p^p$. We prove that φ_g is a continuous linear functional by showing that $g \in L^q$, with $q = (1 - 1/p)^{-1}$. But this is clear as

$$|g(x)|^q = |f(x)|^{(p-1)/(1-1/p)} = |f(x)|^p$$

This argument can be adapted to the special cases $p = 1$ and $p = \infty$ as follows. If $p = 1$, we define

$$g(x) = \begin{cases} \frac{\overline{f(x)}}{|f(x)|} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases} \tag{25.11}$$

Then, $|g(x)| \leq 1$, and hence $g \in L^\infty$. Again, $\varphi_g(f) = \|f\|_1$.

If $p = \infty$, we modify our strategy slightly as follows: for every $\epsilon > 0$ the level set

$$E_\epsilon = \{x \in \Omega \mid |f(x)| \geq \|f\|_\infty - \epsilon\}.$$

has non-vanishing measure. Then, from the sigma-finiteness and the sigma-additivity of μ it follows that there is $A \in \Sigma$ such that

$$\mu(A) < \infty, \quad |f(x)| \geq \|f\|_\infty - \epsilon$$

If $f \neq 0$, we have $\|f\|_\infty \neq 0$. So, we can take $0 < \epsilon < \|f\|_\infty$, and we can define

$$g(x) = \begin{cases} \frac{\overline{f(x)}}{\mu(A)|f(x)|} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then, $g \in L^1$ and

$$\varphi_g(f) = \frac{1}{\mu(A)} \int_A |f(x)|\, \mu(dx) \geq \|f\|_\infty - \epsilon > 0$$

and this concludes the proof. ■

Note that the sigma-finiteness of the measure was used only in the proof of the case $p = \infty$. Even in that case one can dispose of it, but then the proof is no longer constructive.

This theorem implies, among other things, that the weak topology is Hausdorff, e.g., a sequence in L^p can converge to at most one limit.

25.3.2 The Uniform Boundedness Principle

The following theorem is also a special case of a more general theorem (with the same name) in functional analysis.

Theorem 25.8 (Uniform Boundedness Principle for L^p-spaces) *Let $1 \leq p \leq +\infty$ and let f_n be a sequence in $L^p(\Omega, \mu)$, such that for all $\varphi \in (L^p)^*$, there exists $M_\varphi \geq 0$ such that*

$$|\varphi(f_n)| \leq M_\varphi, \qquad \textit{for all } n$$

Then f_n is a bounded sequence for the L^p-norm, i.e., there exists a constant $M \geq 0$ such that

$$\|f_n\|_p \leq M$$

Proof: Following Lieb & Loss, we work out the proof in detail for the case $1 < p < \infty$, and leave it to the reader to make the necessary modifications to treat the cases $p = 1$ and $p = \infty$, in the same spirit as in the proof of Theorem 25.7

Suppose that $\|f_n\|_p$ is *not* bounded. We will then construct a functional $\varphi \in (L^p)^*$ such that the sequence $\varphi(f_n)$ is unbounded as well.

First, we observe that for any sequence (f_n) in L^p such that $(\|f_n\|_p)$ is unbounded, we can find a sequence (g_n) in L^p, such that

$$\|g_n\| = 4^n, \qquad \text{and } |\varphi(g_n)| \leq |\varphi(f_{k_n})|$$

where (f_{k_n}) is a subsequence of (f_n). It suffices to choose (f_{k_n}) such that $\|f_{k_n}\|_p \geq 4^n$ and to define

$$g_n = \frac{4^n}{\|f_{k_n}\|_p} f_{k_n}$$

From this observation is follows that it is sufficient to show that for every sequence (g_n) in L^p such that $\|g_n\|_p = 4^n$, we can find a $\varphi \in (L^P)^*$ such that $\varphi(g_n)$ is unbounded. The proof of Theorem 25.7 suggests how to do this.

Define functions h_n by

$$h_n(x) = \frac{|g_n(x)|^{p-2}\overline{g_n(x)}}{\|g_n\|_p^{p-1}}$$

One has

$$\|h_n\|_q = 1 \tag{25.12}$$

where q is the exponent dual to p.

Next, we recursively choose complex numbers of modulus one, $e^{i\theta_n}$, $\theta_n \in \mathbb{R}$, such that

$$\begin{aligned}
\theta_1 &= 0 \\
\theta_2 &= \arg\left(\int_\Omega h_1(x)g_2(x)\,\mu(dx)\right) \\
&\vdots \\
\theta_n &= \arg\left(\sum_{k=1}^{n-1}\frac{e^{i\theta_k}}{3^k}\int_\Omega h_k(x)g_n(x)\,\mu(dx)\right) - \arg\left(\int_\Omega h_n(x)g_n(x)\,\mu(dx)\right) \\
&\vdots
\end{aligned}$$

As a result of this choice we have that, for all $n \geq 1$,

$$\int_\Omega h_n(x)g_n(x)\,\mu(dx)$$

and

$$\sum_{k=1}^{n-1}\frac{e^{i\theta_k}}{3^k}\int_\Omega h_k(x)g_n(x)\,\mu(dx)$$

have the same argument as complex numbers and, hence,

$$\begin{aligned}
\left|\sum_{k=1}^{n}\frac{e^{i\theta_k}}{3^k}\int_\Omega h_k(x)g_n(x)\,\mu(dx)\right| &\geq \frac{1}{3^n}\int_\Omega h_n(x)g_n(x)\,\mu(dx) \\
&= \frac{\|g_n\|_p}{3^n} \\
&= \left(\frac{4}{3}\right)^n
\end{aligned} \tag{25.13}$$

Now, we can define a functional φ by

$$\varphi(f) = \sum_{n=1}^{\infty} \frac{e^{i\theta_n}}{3^n} \int_\Omega h_n(x) f(x)\, \mu(dx)$$

By using Hölder's inequality and (25.12) we can bound φ as follows:

$$|\varphi(f)| \le \sum_{n=1}^{\infty} \frac{1}{3^n} \|f\|_p$$

and, therefore, $\varphi \in (L^p)^*$.

On the other hand, using (25.13), we find

$$\begin{aligned}
|\varphi(g_n)| &\ge \left| \sum_{k=1}^{n} \frac{e^{i\theta_k}}{3^k} \int_\Omega h_k(x) g_n(x)\, \mu(dx) \right| - 4^n \sum_{k=n+1}^{\infty} \frac{1}{3^n} \\
&\ge \left(\frac{4}{3}\right)^n - 4^n \frac{3^{-(n+1)}}{1-(1/3)} \\
&= \frac{1}{2}\left(\frac{4}{3}\right)^n
\end{aligned}$$

showing that $\varphi(g_n)$ is unbounded. ∎

25.3.3 L^p-L^q duality

We state without proof the Riesz Representation Theorem for L^p-spaces.

Theorem 25.9 (Riesz Representation Theorem) *If $1 \le p < \infty$, every $\varphi \in (L^p(\Omega, \mu))^*$ is of the form*

$$\varphi(f) = \int_\Omega g(x) f(x)\, \mu(dx$$

for some $g \in L^q(\Omega, \mu)$, with q such that $1/p + 1/q = 1$.

The topological dual of L^∞ contains L^1 as a strict subspace. Its full description is somewhat involved and rarely useful, so we do not give it here. If you are interested see e.g., K. Yosida, *Functional Analysis*, Springer Verlag, 5th edition (1980), section IV.9, Example 5.

For $1 < p < \infty$, we have $(L^p)^{**} = L^p$. One says that these spaces are *reflexive.*

It follows from this theorem that L^2 is its own dual, but, that $(L^p)^* \ne L^p$, if $p \ne q$ (except in the trivial case where these spaces are finite-dimensional). Therefore, L^p, with $p \ne 2$, is not a Hilbert space. It is clear that $\|\cdot\|_p$ does not satisfy the parallelogram identity if $p \neq 2$, and if the space is at least two-dimensional. The fact that the infinite-dimensional L^p spaces with $p \neq 2$ are not self-dual, , implies the stronger statement that the L^p topology cannot be derived from an equivalent norm that satisfies the parallelogram identity, i.e., one cannot find a norm such that it becomes a Hilbert space.

25.4 Approximations of L^p-functions

In this section we will suppose that Ω is a measurable subset of $\mathbb{R}^n$ (e.g. $\mathbb{R}^n$ itself). As we have seen many times, it is often useful to approximate an arbitrary element in some space as the limit of a sequence of elements with special properties. For lack of time, we will not give all proofs in this section.

25.4.1 Approximation by C^∞ functions

For $\alpha > 0$, let φ_α be the d-dimensional Gaussian given by

$$\varphi_\alpha(x) = \frac{1}{(2\pi\alpha)^{d/2}} e^{-\alpha \frac{1}{2}\|x\|^2}$$

For, $1 \leq p \leq \infty$, and $f \in L^p(\mathbb{R}^d)$, define f_α by

$$f_\alpha(x) = \varphi_\alpha * f(x) := \int_{\mathbb{R}^d} \varphi_\alpha(x-y) f(y)\, \mu(dy)$$

It is not hard to prove that, for all $\alpha > 0$, $f_\alpha \in C^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d, \mu)$ (See Exercise 25.10).

Exercise 25.10 *For, $\alpha > 0$, $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^d)$, and f_α defined above, prove that*
a) $f_\alpha \in C^\infty(\mathbb{R}^d)$. *(Hint: use that $\varphi_\alpha \in \mathcal{S}(\mathbb{R}^d)$)*
b) $f_\alpha \in L^p(\mathbb{R}^d, \mu)$. *(Hint: use Young's inequality, Proposition 21.4)*

Theorem 25.10 (Approximation by C^∞-functions) *If $1 \leq p < \infty$, $C^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d, \mu)$ is a dense subspace of $L^p(\mathbb{R}^d, \mu)$ with the L^p-norm. In particular, the functions f_α are C^∞ approximations of f in the sense of L^p:*

$$\lim_{\alpha \downarrow 0} \|f_\alpha - f\|_p = 0$$

Exercise 25.11 *As a corollary of Theorem 25.10, prove that C_c^∞, the C^∞ functions with compact support, are also dense in L^p.*

Exercise 25.12 *Are the C^∞ functions dense in L^∞? Or the continuous functions?*

25.4.2 Approximation by simple functions

Approximation by simple functions is another common tool in L^p theory. Simple functions are measurable functions $f : \Omega \to \mathbb{C}$ that take only finitely many values, i.e., the range of f is a finite set of complex numbers. Together with measurability this means that there are a finite number n of measurable sets $A_i \in \Sigma$, and complex numbers $c_i \in \mathbb{C}$, such that

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$$

where χ_A denotes the characteristic function of the set A.

Using the previous result on approximations by C_c^∞ functions, it is quite straigt-forward to prove that the simple functions are also dense in $L^p(\mathbb{R}^d)$, for all $1 \leq p < \infty$.

Theorem 25.11 (Approximation by simple functions) *Let $1 \leq p < \infty$, and Ω a measurable subset of $\mathbb{R}^d$. For all $f \in L^p(\Omega, dx)$, there exists a sequence (f_n) of simple functions such that*

$$\lim_{n\to\infty} \|f - f_n\|_p = 0$$

Exercise 25.13 *Prove Theorem 25.11.*

As an application of approximation by simple functions we can now prove that $L^p(\mathbb{R})$ is separable as a metric space. This extends a property that we already proved for $L^2(\mathbb{R}^d)$ in the context of Hilbert spaces.

Theorem 25.12 (Separability of L^p) *For $1 \leq p < \infty$, $L^p(\mathbb{R}^d)$ is a separable metric space.*

Proof: We have to show that $L^p(\mathbb{R}^d)$ contains a countable dense subset. The set of all simple functions that take values in the "rational complex numbers", i.e., the complex numbers with both real and imaginary part in $\mathbb{Q}$, is obviously dense in the space of all simple functions and, hence, due to Theorem 25.11, also dense in $L^p(\mathbb{R}^d, \mu)$. Of course, it is not countable, because there are far to many measurable sets. The proof of the theorem consists in showing that one can approximate any L^p function by a simple function of the form

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$$

with all A_i chosen from a suitable countable family of measurable sets. E.g., one can take cubes of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$,m where $a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{Q}$. To prove this, is straighforward but somewhat tedious. ∎

Note that this result extends immediately to $L^p(\Omega, \mu)$, with $1 \leq p < \infty$, and Ω an arbitrary measurable subset of $\mathbb{R}^d$.

25.5 Compactness in L^p-spaces

25.5.1 The Banach-Alaoglu Theorem

We will prove that every bounded sequence in L^p, with $1 < p < \infty$, has a *weakly* convergent subsequence. This usually proved as an application of the more general Banach-Alaoglu Theorem (see. e.g., W. Rudin, *Functional Analysis*, McGraw Hill, 2nd edition, 1991). Our proof will be based on the separability of the dual of L^p, i.e., L^q. The general Banach-Alaoglu Theorem does not use separability.

Theorem 25.13 (Banach-Alaoglu for L^p) *Let Ω be a measurable subset of $\mathbb{R}^d$, and (f_n) a norm-bounded sequence in $L^p(\Omega, \mu)$, with $1 < p < \infty$. Then, there exists a subsequence (f_{n_k}) weakly converging to $f \in L^p(\Omega)$.*

Proof: By the Riesz Representation Theorem 25.9 f_{n_k} converges weakly to f in L^p iff for all $g \in L^q$, with $1/p + 1/q = 1$,

$$\int_\Omega g(x) f_{n_k}(x)\, dx \to \int_\Omega g(x) f(x)\, dx \tag{25.14}$$

As L^q is separable (Theorem 25.12), there is a countable dense set of $g_i \in L^q$. This means that for any $g \in L^q$, there is a subsequence (g_{i_l}) of (g_i) such that

$$\lim_{l\to\infty} \|g_{i_l} - g\|_q = 0$$

It will be sufficient to prove (25.14) for g in that countable set, i.e., for $g = g_i$, $i = 1, 2, \ldots$ (see below). This can be done using Cantor's diagonal argument.

The sequence $(\int g_1(x) f_n(x)\, dx)$ is bounded. This follows from Hölders inequality and the boundedness of f_n in L^p. Therefore, it has a convergent subsequence, with limit L_1. This subsequence determines a corresponding subsequence of (f_n). Denote this subsequence by $(f_{1,n})$. Next, consider the sequence $(\int g_2(x) f_{1,n}(x)\, dx)$, which is again bounded and, therefore, has a convergent subsequence, with limit L_2, and denote the corresponding subsequence of $(f_{1,n})$ by $(f_{2,n})$. By repeating this argument we obtain a double sequence $(f_{m,n})$ with the following properties:

1. $(f_{1,n})_{n=1}^\infty$ is a subsequence of (f_n).
2. $(f_{m,n})_{n=1}^\infty$ is a subsequence of $(f_{m-1,n})_{n=1}^\infty$, for all $m \geq 2$.
3. $\lim_{n\to\infty} \int g_k(x) f_{m,n}(x)\, dx$ exists and equals L_k, for all $1 \leq k \leq m$.

The argument of Cantor's then shows that the subsequence $(f_{m,m})_{m=1}^\infty$ of $(f_n)_{n=1}^\infty$ satisfies

$$\lim_{m\to\infty} \int g_i(x) f_{m,m}(x)\, dx = L_i, \qquad \text{for all } i = 1, 2, \ldots$$

As the set $\{g_i\}$ is dense in L^q, and using the boundedness of (f_n), we see that $g_i \mapsto L_i$ defines a bounded linear functional on a dense subset of L^q. By Theorem 6.7 this functional extends uniquely to a bounded linear functional on all of L^q. Then, again by $L^p - L^q$ duality (Theorem 25.9), there is a element $f \in L^p$ such that

$$\lim_{m\to\infty} g(x) f_{m,m}(x)\, dx = \int g(x) f(x), dx$$

showing that the subsequence $(f_{m,m})$ converges weakly to f. ∎

Another way of expressing Theorem 25.13 is to say that bounded sets in L^p, with $1 < p < \infty$ are weakly compact.

Exercise 25.14 *Prove that the unit ball in $L^p([a,b], dx)$ is not strongly compact (strongly refers to the norm topology).*

Exercise 25.15 *Prove that Theorem 25.13 does not hold for $p = 1$. I.e., give an example of a bounded sequence in L^1 that does not have a weakly convergent subsequence.*

In the study of variantional problems the following theorem is sometimes useful. It shows that although a weakly convergent sequence is in general not strongly convergent, one can construct a strongly convergent sequence from it by considering convex combinations of the elements in the sequence. We say that g is a *convex combination* of $f_1, \ldots, f_n$, if there exists non-negative constants t_i, $i = 1, \ldots, n$, such that

$$\sum_{i=1}^n t_i = 1, \qquad \text{and} \qquad g = \sum_{i=1}^n t_i f_i \tag{25.15}$$

This is sometimes good enough, e.g., if one is looking for a minimizer of a convex functional. We skip the proof of this theorem.

Theorem 25.14 (Mazur's Theorem) *Let $1 < p < \infty$, and let (f_n) be a sequence in an arbitrary L^p space $L^p(\Omega, \mu)$ that converges weakly to $f \in L^p(\Omega, \mu)$. Then, there exists a sequence (g_n), where g_n is a convex combination of $f_1, \ldots, f_n$, such that (g_n) converges strongly to f.*

25.5.2 Strong compactness in L^p

Finally, we give a sufficient condition for a subset of $L^p(\mathbb{R}^d)$ to be compact in the norm topology. Note the similarity with the Arzelá-Ascoli Theorem, which characterizes compact sets of continuous functions with the sup-norm topology. We state the criterion for pre-compactness. Recall that a pre-compact set is a set with compact closure. So, a pre-compact set is compact iff it is closed.

Theorem 25.15 (Strong compactness in L^p spaces) *Let $1 \leq p < \infty$, and $\Omega \subset \mathbb{R}^d$. A set $A \subset L^p(\Omega, dx)$ is precompact iff the following three conditions are satisfied:*
i) A is bounded, i.e., there exists $M \geq 0$ such that

$$\|f\|_p \leq M, \quad \textit{for all } f \in A$$

ii) For all $\epsilon > 0$ there exists $\Delta > 0$ such that for all $a \in \mathbb{R}^d$,with $\|a\| < \delta$, one has

$$\|T_a f - f\|_p < \epsilon, \quad \textit{for all } f \in A$$

Here, T_a is the translation over a: $T_a f(x) = f(x-a)$, for all $x \in \Omega$.
iii) For all $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{\|x\| \geq R} |f(x)|^p \, dx < \epsilon, \quad \textit{for all } f \in A$$

The second condition is called *L^p-equicontinuity.* Note that the third condition is automatically satisfies if Ω is a bounded set, i.e., there exists $R > 0$ such that Ω is contained in a ball of radius R.