Assignment 5

(due date: Monday, May 11)

Problem 1. Prove that

$$\mathcal{B}(\mathbb{R}) = \Sigma(\{(a,b] \mid a < b\}) = \Sigma(\{[a,b) \mid a < b\}) = \Sigma(\{[a,b] \mid a < b\}) = \Sigma(\{(a,+\infty) \mid a \in \mathbb{R}\})$$

Problem 2. Let Σ be a sigma-algebra of subsets of Ω, and suppose μ is a measure on Ω. Prove the following properties:

a) If $A, B \in \Sigma$, then $A \setminus B \in \Sigma$.

b) If $A, B \in \Sigma$, and $A \subset B$, then $\mu(A) \leq \mu(B)$.

c) If $A, B \in \Sigma$, then $\mu(A \cup B) \leq \mu(A) + \mu(B)$.

Problem 3. Is the counting measure on $[0,1]$ sigma-finite? Prove or disprove.

Problem 4. Give an example of a sequence of non-negative integrable functions $f_n : \mathbb{R} \to \mathbb{R}$, converging pointwise and monotonically to a function f, but such that

$$\lim_{n\to\infty} \int f_n(x)\, \mu(dx) \neq \int f(x)\, \mu(dx)$$