Assignments 7 and 8

(due date: Monday, June 8)

Problem 1. As a corollary of Theorem 25.10 (approximation of elements in L^p by C^infty functions), prove that C_c^∞, the C^∞ functions with compact support, are also dense in L^p with its norm topology.

Problem 2. Write a careful proof of Theorem 25.11 of the notes (Approximation by simple functions.)

Problem 3. Prove that the unit ball in $L^p([a,b],dx)$, $1 \leq p \leq +\infty$, is not strongly compact (*strongly* refers to the norm topology on L^p.).

Problem 4. Prove that the statement of the Banach-Alaoglu theorem does not hold for $p = 1$. I.e., give an example of a bounded sequence in L^1 that does not have a weakly convergent subsequence.

Problem 5. Define the following two sequences of functions on the real line:
$(f_n)_{n=1}^\infty$ given by

$$f_n(x) = \begin{cases} n^{1/2} & \text{if } n \leq x \leq n + 1/n \\ 0 & \text{else} \end{cases}$$

and $(g_n)_{n=1}^\infty$ given by

$$g_n(x) = \begin{cases} n^{-2/3} & \text{if } n \leq x \leq 2n \\ 0 & \text{else} \end{cases}$$

Consider these sequences with each of the topologies given below and determine whether or not $f_n \to 0$, and/or $g_n \to 0$. Explain.

a) Pointwise on $\mathbb{R}$.

b) Uniformly on $\mathbb{R}$.

c) In the norm topology of $L^2(\mathbb{R})$.

d) Strongly in $L^{3/2}(\mathbb{R})$.

e) Weakly in $L^{3/2}(\mathbb{R})$.

Problem 6. Let $1 \leq p \leq \infty$. Prove that if $f \in C^1(0,1])$, its weak derivative coincides with the usual pointwise definition of derivative.

Problem 7. Let $1 \leq p \leq \infty$. Prove that if $f \in L^p([0,1], dx)$, and its weak derivative is identically zero, then there exists $c \in \mathbb{C}$, such that $f(x) = c$ æ.

Problem 8. Of the following functions in $L^2([-1,1], dx)$, which are in $H^1([-1,1], dx)$?
i) $x \mapsto |x|$;
ii) $x \mapsto$ sign x;
iii) $x \mapsto \sum_{n=1}^{\infty} (1/n)^{3/2} \sin(2\pi n x)$.