19 The Fourier Transform

19.1 The Fourier transform of test functions

Proposition 19.1 *For any $f \in \mathcal{S}(\mathbb{R}^d)$, the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ defined by*

$$\hat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int e^{-ik\cdot x} f(x)\, dx$$

is an element of $\mathcal{S}(\mathbb{R}^d)$.

Proof: $\partial^\alpha \hat{f}$ is bounded for all $\alpha \in (\mathbb{Z}^+)^d$. This can be seen as follows:

$$\begin{aligned}
|\partial^\alpha \hat{f}(k)| &= \frac{1}{(2\pi)^{d/2}} \left| \int (-ix)^\alpha e^{-ik\cdot x} f(x)\, dx \right| \\
&\leq \frac{1}{(2\pi)^{d/2}} \int |x^\alpha f(x)|\, dx \\
&\leq \frac{1}{(2\pi)^{d/2}} \int (1+\|x\|)^{|\alpha|+d+1} |f(x)| \frac{1}{(1+\|x\|)^{d+1}}\, dx \\
&\leq \frac{1}{(2\pi)^{d/2}} \|f\|_{|\alpha|+d+1,0} \int_{\mathbb{R}^d} \frac{1}{(1+\|x\|)^{d+1}}\, dx
\end{aligned}$$

It then also follows that $\partial^\alpha \hat{f}$ is continuous for all α.

To prove that $\|\hat{f}\|_{n,\alpha} < \infty$ for all n, α, we use Exercise 18.6 and repeat the above argument combined with integration by parts: for any α, β

$$\begin{aligned}
|k^\beta \partial^\alpha \hat{f}(k)| &= \frac{1}{(2\pi)^{d/2}} \left| \int (-ix)^\alpha (\partial^\beta e^{-ik\cdot x}) f(x)\, dx \right| \\
&\leq \frac{1}{(2\pi)^{d/2}} \int |\partial^\beta (x^\alpha f(x))|\, dx
\end{aligned}$$

It is then clear that $|k^\beta \partial^\alpha \hat{f}(k)|$ can be bounded by a finite sum of the form

$$\sum_{0\leq m\leq|\alpha|,\, 0\leq\gamma\leq\beta} C(m,\gamma) \|f\|_{m,\gamma}$$

$\blacksquare$

From the proof of the previous proposition it also follows that the map $\mathcal{F} : \mathcal{S} \to \mathcal{S} : f \mapsto \hat{f}$ is a *continuous* linear transformation from $\mathcal{S}$ to itself with its standard topology.

Next, we prove that $\mathcal{F}$ is invertible with a continuous inverse.

Proposition 19.2 (Fourier Inversion Formula) *Define* $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ *by*

$$\mathcal{G}f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ik\cdot x} f(k)\, dk$$

Then $\mathcal{G}$ *is a continuous linear transformation from* $\mathcal{S}$ *to itself and* $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F} = \mathrm{id}$, *i.e.,* $\mathcal{G} = \mathcal{F}^{-1}$.

Proof: First, note that

$$\mathcal{G} = R \circ \mathcal{F} \tag{19.1}$$

where R is the reflection defined by $Rf(x) = f(-x)$. Clearly, R is linear and continuous. Therefore, (19.1) and the fact that implies that $\mathcal{F}$ is a continuous linear transformation implies that $\mathcal{G}$ is linear and continuous, too.

To prove that $\mathcal{G}$ is the inverse of $\mathcal{F}$ we will compute, for any $f \in \mathcal{S}$, $\mathcal{G}\mathcal{F}f = \mathcal{G}\hat{f}$. At first, suppose $d = 1$.

$$\begin{aligned}
\mathcal{G}\hat{f}(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik\cdot x} \left[\int_{-\infty}^{+\infty} e^{-ik\cdot y} f(y)\, dy \right] dk \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{+n} \left[\int_{\infty}^{+\infty} e^{ik\cdot(x-y)} f(y)\, dy \right] dk \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-n}^{+n} e^{ik\cdot(x-y)}\, dk \right] f(y)\, dy \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \int \left[\frac{2 \sin n(x-y)}{x-y}\, dx \right] f(y)\, dy
\end{aligned}$$

Note that by introducing the "ultraviolet cut-off" in the integral over k, we were able to interchange the order of integration. Now, we use the fact that $(\sin nx)/\pi x$ is a delta sequence (see the previous section). We conclude that $\mathcal{G}\hat{f}(x) = f(x)$, i.e., $\mathcal{G}\mathcal{F} = \mathrm{id}$. Because the relation (19.1) $\mathcal{F}\mathcal{G} = \mathrm{id}$ then easily follows.

For arbitrary dimensions d it suffices to observe that the multidimensional Fourier transform is nothing but the compositions of the one-dimensional Fourier transforms in each of the components x_i of $x \in \mathbb{R}^d$, $i = 1, \ldots, d$. ∎

It is convenient to denote the inverse Fourier transform $\mathcal{F}^{-1}f$ by $\tilde{f}$. So, we have

$$\tilde{\hat{f}} = \hat{\tilde{f}} = f, \quad \text{for all } f \in \mathcal{S} \tag{19.2}$$

Example: Fourier transform of a Gaussian: suppose $a > 0$, and let $f(x) = \exp -a(\frac{1}{2}x \cdot x)$. Then

$$\hat{f}(k) = \frac{1}{a^{d/2}} \exp(-\frac{1}{a}(\frac{1}{2}k \cdot k)) \tag{19.3}$$

To prove this formula it suffices to consider the case $a = 1$. The formula for all $a > 0$ then follows by a change of variables $x \mapsto \sqrt{a}x$. We can also reduce the case of arbitrary d to the the one-dimensional case because the Fourier transform of the multi-dimensional Gaussian factorizes into a product of one-dimensional integrals. So, we have to prove that

$$\frac{1}{\sqrt{2\pi}} \int e^{-ikx} e^{-\frac{1}{2}x^2}\, dx = e^{-\frac{1}{2}k^2}$$

The LHS can be written as

$$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x+ik)^2}\, dx \times e^{-\frac{1}{2}k^2}$$

So, the point is to prove that

$$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x+ik)^2}\, dx = 1$$

This integral in independent of k:

$$\begin{aligned}
\frac{d}{dk}\frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x+ik)^2}\, dx &= -i\frac{1}{\sqrt{2\pi}} \int (x+ik) e^{-\frac{1}{2}(x+ik)^2}\, dx \\
&= i\frac{1}{\sqrt{2\pi}} \int \frac{d}{dx} e^{-\frac{1}{2}(x+ik)^2}\, dx \\
&= i\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+ik)^2}\Big|_{-\infty}^{+\infty} = 0
\end{aligned}$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x+ik)^2}\, dx = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}x^2}\, dx = 1$$

Exercise 19.1 (Behavior under translations) *Recall the action of the translation over $a \in \mathbb{R}^d$ on $\mathcal{S}$: $T_a f(x) = f(x-a)$. Prove that*

$$\widehat{T_a f}(k) = e^{-ik\cdot a} \hat{f}(k) \tag{19.4}$$

Exercise 19.2 (Fourier transform of derivative) *Prove that for any multi-index α*

$$\widehat{\partial^\alpha f}(k) = (-ik)^\alpha \hat{f}(k) \tag{19.5}$$

Exercise 19.3 (Fourier transform of convolutions) *For all $f, g \in \mathcal{S}$ we have*

$$\frac{1}{(2\pi)^{d/2}} \widehat{f * g}(k) = \hat{f}(k)\hat{g}(k) \tag{19.6}$$

19.2 The Fourier transformation of tempered distributions

Suppose $\varphi \in \mathcal{S}^*$. We define the Fourier transform $\hat{\varphi}$ of φ by

$$\hat{\varphi}(f) = \varphi(\tilde{f}), \qquad \text{for all } f \in \mathcal{S} \tag{19.7}$$

The tilde in this formula is not a misprint. That $\hat{\varphi}$ is a tempered distribution follows directly from the fact that $\mathcal{F}^{-1}$ is a continuous linear transformation of $\mathcal{S}$. In fact, the map $\varphi \mapsto \hat{\varphi}$ is a continuous transformation of $\mathcal{S}^*$ to itself with its standard topology. It is also clear that the Fourier transformation on $\mathcal{S}^*$ is also invertible with a continuous inverse given by $\varphi \mapsto \tilde{\varphi}$, $\tilde{\varphi}(f) = \varphi(\hat{f})$.

Examples:

- For φ that are defined in terms of a function $g \in \mathcal{S}$ by

$$\varphi(f) = \int \overline{g(x)} f(x)\, dx$$

we have that

$$\begin{aligned}
\hat{\varphi}(f) &= \int \overline{g(x)} \tilde{f}(x)\, dx \\
&= \int \overline{g(x)} \frac{1}{\sqrt{2\pi}} \left[\int e^{ik\cdot x} f(k)\, dk \right] dx \\
&= \int \overline{\left[\frac{1}{\sqrt{2\pi}} \int g(x) e^{-ik\cdot x}\, dx \right]} f(k)\, dk \\
&= \int \overline{\hat{g}(x)} f(x)\, dx
\end{aligned}$$

So, by making the pairing sesquilinear (i.e., by introducing the complex conjugation), the Fourier transform of a distribution determined by a test function coincides with the distribution determined by the Fourier transform of the test function.

- The Fourier transform of the delta distribution at the origin:

$$\hat{\delta}_0(f) = \delta_0(\tilde{f}) = \tilde{f}(0) = \frac{1}{\sqrt{2\pi}} \int f(x)\, dx$$

which can be written with the pairing as

$$\int \hat{\delta}(x) f(x)\, dx = \int \frac{1}{\sqrt{2\pi}} f(x)\, dx$$

If you wish, you can think of $\hat{\delta}$ as the constant function $\frac{1}{\sqrt{2\pi}}$.

From the Fourier inversion formula (19.2) we immediately see that

$$\hat{\varphi}(\hat{f}) = \varphi(f) = \tilde{\varphi}(\tilde{f}) \tag{19.8}$$

This is a version of *Plancherel's formula.* If we write φ (formally or literally) as

$$\varphi(f) = \int \overline{\varphi(x)} f(x)\, dx$$

Plancherel's formula becomes

$$\int \overline{\varphi(x)} f(x)\, dx = \int \overline{\hat{\varphi}(x)} \hat{f}(x)\, dx = \int \overline{\tilde{\varphi}(x)} \tilde{f}(x) \tag{19.9}$$

Remark: The definition of the Fourier tranform of distributions in terms of the Fourier transform of test functions is a special case of the more general procedure of obtaining the *adjoint* of a (continuous) linear transformation $T : X \to X$, where X is a (topological) linear space, as a (continuous) linear transformation on the (topological) dual X^*:

$$T^* : X^* \to X^* : \xi \mapsto \xi \circ T$$

T^* is then called the *adjoint* of T. In our case the Fourier transform on $\mathcal{S}^*$ is defined as the adjoint of the *inverse* Fourier transform on $\mathcal{S}$.

19.3 The Fourier transform on L^2

There are several ways of defining the Fourier transform on $L^2(\mathbb{R}^d)$. For us the easiest method starts from the observation that $\mathcal{S}$ is a linear subspace of L^2, which is dense for the L^2-norm. The Plancherel formula (19.9) shows that $\mathcal{F}$ is isometric, and *a fortiori*, bounded. Therefore it has a unique extension to an isometric transformation of L^2 into itself. The same is true for $\mathcal{F}^{-1}$. Therefore, $\mathcal{F}$ is isometric and invertible, hence unitary.

The main drawback is that it does not give us a formula to compute the Fourier transform of arbitrary elements of L^2, represented by a function. In fact, the latter is not possible. Of course, we can always resort to an approximation argument. Any L^2 function can be approximated by a sequence of test functions. By continuity of the Fourier transform, the Fourier transform of the limit of any convergent sequence is then the limit of the sequence of the Fourier transforms of the elements in the sequence.

Examples:

1. Let $f = \chi_{-N,N}$ be the characteristic function of the interval $[-N, N]$. Then,

$$\hat{f}(k) = \frac{1}{\sqrt{\pi}} \frac{\sin Nk}{k}$$

2. For $a > 0$, define $f(x) = \exp(-a|x|)$. Then $f \in L^2(\mathbb{R})$ and

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + k^2}$$

We defined the Fourier transform as a unitary operator on $L^2(\mathbb{R}^d)$. What is its spectrum? What are its eigenvectors if any? It is sufficient to consider this question for $d = 1$ (Why?).

As before denote the Fourier transform by $\mathcal{F}$. As $R\mathcal{F}^{-1} = \mathcal{F}$, we have that $\mathcal{F}^2 = R$, and therefore $\mathcal{F}^4 = \mathbb{1}$. It follows that if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{F}$, then $\lambda \in \{1, i, -1, -i\}$. It turns out that the spectrum of $\mathcal{F}$ consist entirely of eigenvalues, more precisely, $\sigma(\mathcal{F}) = \{1, i, -1, -i\}$, each of which is an infinitely degenerated eigenvalue. The eigenvectors form an o.n. basis and can be taken to be the Hermite functions defined by

$$H_n(x) = \frac{(-1)^n}{n!} e^{\pi x^2} \frac{d^n}{dx^n} e^{-2\pi x^2}, \quad n = 0, 1, 2, \ldots \tag{19.10}$$

The H_n are not normalized. Their L^2-norm is given by

$$||H_n||_{L^2}^2 = \frac{(4\pi)^n}{\sqrt{2n!}} \tag{19.11}$$

One can then prove that

$$\mathcal{F} H_n = (-i)^n H_n$$

Suppose we have proved that the normalized Hermite functions are an o.n. basis of $L^2(\mathbb{R}^2)$. Then, we can define the Fourier transform on $L^2(\mathbb{R})$ in the following alternative way. For any $f \in L^2(\mathbb{R})$, f can be written as

$$f = \sum_{n=0}^{\infty} a_n H_n$$

with

$$a_n = \frac{1}{||H_n||_{L^2}^2} \int H_n(x) f(x)\, dx$$

Then

$$\mathcal{F} f = \sum_{n=0}^{\infty} (-i)^n a_n H_n$$

19.4 The Fourier transform on L^1

For any $f \in \mathcal{L}^1(\mathbb{R}^d)$, and $k \in \mathbb{R}^d$, the function $x \mapsto e^{-ik\cdot x} f(x)$ is also L^1. Therefore, we can define the Fourier transform of $f \in L^1$ simply by

$$\hat{f}(k) = \int e^{-ik\cdot x} f(x)\, dx$$

So, no appromation argument is needed for the definition of the Fourier transform on L^1. Approximations arguments are useful, however, to prove certain properties of the Fourier transform of L^1. E.g., in the proof of the following result we will use that $\mathcal{S}$ is dense in L^1 with the L^1-norm. Denote by $C_0(\mathbb{R}^d)$ the space of continuous functions tending to zero at infinity, i.e., $f \in C_0(\mathbb{R}^d)$ is continous and $\lim_{x\to\infty} f(x) = 0$. Recall that $C_0(\mathbb{R}^d)$ equipped with the supremum norm is a Banach space.

Theorem 19.3 (The Riemann-Lebesgue Lemma) *For all $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in C_0(\mathbb{R}^d)$.*

Proof: Let (f_n) be a sequence in $\mathcal{S}$ converging to f in the L^1-norm. We claim that $(\hat{f}_n)$ converges uniformly to $\hat{f}$. The theorem then follows as any uniform limit of continuous functions vanishing at ∞ is a continuous function vanishing at ∞. We now prove the claim.

First, we show that $\hat{f}_n$ is a Cauchy sequence for the supremum norm:

$$\begin{aligned}
|\hat{f}_n(k) - \hat{f}_m(k)| &= \left| \frac{1}{\sqrt{2\pi}} \int e^{-ik\cdot x}(f_n(x) - f_m(x))\, dx \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \int |f_n(x) - f_m(x)|\, dx \\
&= \|f_n - f_m\|_{L^1}
\end{aligned}$$

As we assumed that $f_n \to f$ in the L^1 norm, this estimate shows that $(\hat{f}_n)$ is Cauchy for the supremum norm.

As $C_0(\mathbb{R}^d)$ is complete for the supremum norm, there is $g \in C_0(\mathbb{R}^d)$, such that $\hat{f}_n \to g$ uniformly. It remains to show that $g = \hat{f}$. But this is quite obvious:

$$\begin{aligned}
&|g(k) - \hat{f}(k)| \\
&\quad \left| \lim_{n\to\infty} \hat{f}_n(k) - \lim_{n\to\infty} \left[\frac{1}{\sqrt{2\pi}} \int e^{-ik\cdot x} f_n(x)\, dx - \frac{1}{\sqrt{2\pi}} \int e^{-ik\cdot x}(f_n(x) - f(x))\, dx \right] \right| \\
&\quad \leq \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int |f_n(x) - f(x)|\, dx = \lim_{n\to\infty} \|f_n - f\|_{L^1} = 0
\end{aligned}$$

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Note that the Riemann-Lebesgue Lemma does not hold in general for functions in L^2 (give a counterexample).

The following obvious inequality shows that the Fourier transform is a bounded (hence continuous) linear map L^1 into C_0 equipped with their respective norm topologies. The following properties show that the Fourier transform is an isomorphism between the algebra L^1 considered with convolution (up to a factor $1/(2\pi)^{d/2}$) as the product and its image $\mathcal{F}(L^1)$ considered as a subalgebra of C_0 with the ordinary pointwise product of functions. It is known that $\mathcal{F}(L^1) \neq C_0$, but a precise description of it is lacking. In other words, $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ is ono-to-one but not onto.

Proposition 19.4 *The Fourier transform on $L^1(\mathbb{R}^d)$ has the following two properties:*
i) For any two functions $f, g \in L^1(\mathbb{R}^d)$, $f * g \in L^1(\mathbb{R}^d)$, and

$$\frac{1}{(2\pi)^{d/2}} \widehat{f * g} = \hat{f}\hat{g}$$

ii) $\hat{f} = 0$ iff $f = 0$.

Proof: The proof of i) just involves the interchange of order of integration, which is justified because the integral involved exist:

$$\begin{aligned}
\frac{1}{(2\pi)^{d/2}} \widehat{f * g}(k) &= \frac{1}{(2\pi)^d} \int e^{-ik\cdot x} \left[\int f(x-y) g(y)\, dy \right] dx \\
&= \frac{1}{(2\pi)^d} \int e^{-ik\cdot y} \left[\int e^{-ik\cdot(x-y)} f(x-y)\, dx \right] g(y)\, dx \\
&= \frac{1}{(2\pi)^{d/2}} \int e^{-ik\cdot z} f(z)\, dz \times \frac{1}{(2\pi)^{d/2}} \int e^{-ik\cdot y} g(y)\, dy
\end{aligned}$$

To prove ii) we consider $f \in L^1$ as a tempered distribution. Plancherel's formula then reads

$$\int f(x) g(x)\, dx = \int \hat{f}(k) \hat{g}(k)\, dk, \qquad \text{for all } g \in \mathcal{S}$$

from which we see that $\hat{f} = 0$ iff the tempered distribution determined by f vanishes. As the tempered distribution is uniquely determined by f as an L^1 function, this proves the statement. ∎

Exercise 19.4 *Let $f \in L^1(\mathbb{R}^3)$ be a rotation invariant function in the sense that there is a function $g : \mathbb{R}^+ \to \mathbb{C}$ such that*

$$f(x) = g(\|x\|)$$

Prove that the Fourier transform of f is a continuous function $\hat{f}$ that is also rotation invariant and more specifically

$$\hat{f}(k) = h(\|k\|)$$

with

$$h(k) = \frac{1}{k}\sqrt{\frac{2}{\pi}}\int_0^{+\infty} r\sin(kr)g(r)\,dr$$