Math 115B Homework 1

1) The following problem is open, meaning no one knows a solution: Let n $\in \mathbb{Z}$ with $n > 1$. If $\phi(n)|n-1$, then n is prime.

Prove the following weaker result: Let $n \in \mathbb{Z}$ with $n > 1$. If $\phi(n)|n-1$, then n is a product of distinct prime numbers.

2) Find $\sigma(n)$ and $\tau(n)$ for $n = 64, 105, 2592, 4851, 111111$, and $15!$.

3) a) Characterize the positive integers n for which $\tau(n)$ is odd.

b) Characterize the positive integers n for which $\sigma(n)$ is odd.

4) Let $k \in \mathbb{Z}$ with $k > 1$. Prove that the equation $\tau(n) = k$ has infinitely many solutions n.

5) Let $n \in \mathbb{Z}$ with $n > 0$.

a) Show that $\tau(n) \leq 2\sqrt{n}$.

b) Show that $\tau(n) \leq \tau(2^n - 1)$.

6) a) Let $n \in \mathbb{Z}$ with $n > 0$. Show that $\sum_{d|n,d>0} \frac{1}{d} = \frac{\sigma(n)}{n}$.

b) Let n be a perfect number. Show that $\sum_{d|n,d>0} \frac{1}{d} = 2$.

7) Let $n \in \mathbb{Z}$ with $n > 0$. The number n is said to be *superperfect* if $\sigma(\sigma(n)) = 2n$.

a) Prove that 16 is superperfect.

b) Prove that if $2^p - 1$ is a Mersenna prime, then 2^{p-1} is superperfect.

c) Prove that if 2^a is superperfect, then $2^{a+1} - 1$ is a Mersenne prime. (Note: it turns out that all even superperfect numbers are of the form 2^{p-1} where $2^p - 1$ is a Mersenne prime. No odd superperfect numbers are known)

8) How difficult was this homework? How long did it take?