ORBITS ON K3 SURFACES OF MARKOFF TYPE

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Abstract. Let $\mathcal{W} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a surface given by the vanishing of a $(2, 2, 2)$-form. These surfaces admit three involutions coming from the three projections $\mathcal{W} \to \mathbb{P}^1 \times \mathbb{P}^1$, so we call them *tri-involutive K3* (TIK3) *surfaces*. By analogy with the classical Markoff equation, we say that $\mathcal{W}$ is of *Markoff type* (MK3) if it is symmetric in its three coordinates and invariant under double sign changes. An MK3 surface admits a group of automorphisms $\mathcal{G}$ generated by the three involutions, coordinate permutations, and sign changes. In this paper we study the $\mathcal{G}$-orbit structure of points on TIK3 and MK3 surfaces. Over finite fields, we study fibral connectivity and the existence of large orbits, analogous to work of Bourgain, Gamburd, Sarnak and others for the classical Markoff equation. For a particular 1-parameter family of MK3 surfaces $\mathcal{W}_k$, we compute the full $\mathcal{G}$-orbit structure of $\mathcal{W}_k(\mathbb{F}_p)$ for all primes $p \leq 113$, and we use this data as a guide to find many finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$, including a family of orbits of size 288 parameterized by a curve of genus 9.

Contents

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1. Introduction

The classical Markoff equation is the affine surface

$$\mathcal{M} : x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

It admits three involutions coming from the three projections $\mathcal{M} \to \mathbb{A}^2$, and these three involutions, together with double sign changes and coordinate permutations, generate the automorphism group $\mathcal{G}_{\mathcal{M}} := \operatorname{Aut}(\mathcal{M})$ of $\mathcal{M}$. A classical theorem of Markoff [27] says that the set of integer solutions in $(\mathbb{Z}^{\ge 0}, \mathbb{Z}^{\ge 0}, \mathbb{Z}^{\ge 0},)$, which we denote by $\mathcal{M}(\mathbb{Z})$, consists of two $\mathcal{G}_{\mathcal{M}}$-orbits, one "small" $\mathcal{G}_{\mathcal{M}}$-orbit containing the single point $(0,0,0)$, and one "large" $\mathcal{G}_{\mathcal{M}}$-orbit containing $(1,1,1)$.

The orbit structure structure of $\mathcal{M}(\mathbb{F}_p)$ under the action of $\mathcal{G}_{\mathcal{M}}$ has been studied by a number of authors. Baragar [1] conjectured that for every prime p, there is only one large orbit in $\mathcal{M}(\mathbb{F}_p)$, and this was proved for almost all p by Bourgain–Gambard–Sarnak [11] and subsequently for all sufficiently large p by Chen [16]. The proofs rely on an ingenious algorithm that jumps between differently oriented fibers, using the Hasse–Weil estimate to say that if a point on a "vertical" fiber has a large enough orbit, then one of the "horizontal" orbits consists of an entire "horizontal" fiber. The proof implicitly relies on the fact that each fiber of $\mathcal{M}$ is a torus and that the fibral automorphisms are toral translations (i.e., $\mathbb{G}_m$-translations), which in [11] are called rotations. See Section 2 for more details.

The first goal of this paper is to study similar questions on an analogous family of projective surfaces that admit three involutions. We define the family of *tri-involutive K3* (TIK3) *surfaces* to be the hypersurfaces

$$\mathcal{W} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \tag{2}$$

given by the vanishing of a $(2,2,2)$-form such that the three projection maps

$$\pi_{12}, \pi_{13}, \pi_{23} : \mathcal{W} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

are finite double covers. These three double covers induce three involutions

$$\sigma_1, \sigma_2, \sigma_3 : \mathcal{W} \longrightarrow \mathcal{W}$$

coming from switching the two sheets. The study of the geometry and arithmetic of these surfaces is of course not new; see Section 2 for a brief history.[1]

The first goal of this paper is to study the orbit structure of $\mathcal{W}(\mathbb{F}_p)$ under the action of $\mathrm{Aut}(\mathcal{W})$. To do this, we start by analyzing the connectivity of the fibers of $\mathcal{W}(\mathbb{F}_p)$ for the three projections

$$\pi_1, \pi_2, \pi_3 : \mathcal{W}(\mathbb{F}_p) \longrightarrow \mathbb{P}^1(\mathbb{F}_p).$$

We prove the following fibral linking result, which is a TIK3 analogue of [11, Proposition 6] for the Markoff equation. See Theorem 5.5 for further details and a proof.

Theorem 1.1. *Assume that $p > 100$, and let $\mathcal{W}/\mathbb{F}_p$ be a TIK3 surface. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fibers of $\mathcal{W}(\mathbb{F}_p)$ for any two (possibly identical) of the three projections $\pi_1, \pi_2, \pi_3 : \mathcal{W}(\mathbb{F}_p) \to \mathbb{P}^1(\mathbb{F}_p)$. Then there is a fiber $\mathcal{F}_3$ for one of the projections satisfying*

$$\mathcal{F}_1 \cap \mathcal{F}_3 \neq \emptyset \quad \textit{and} \quad \mathcal{F}_2 \cap \mathcal{F}_3 \neq \emptyset.$$

Our second goal is inspired by the classification of finite orbits on Markoff-type surfaces over $\mathbb{C}$. For example, the papers [7, 14, 21, 26] contain a detailed description of the $(a,b,c,d,e) \in \mathbb{C}^5$ for which the Markoff–Hurwitz surface

$$x^2 + y^2 + z^2 + ax + by + cz + dxyz + e = 0. \tag{3}$$

has one or more finite orbits. The existence of such orbits turns out to be related to algebraic solutions to Painlevé differential equations. It is likewise true [13] that a (non-degenerate) TIK3 surface $\mathcal{W}(\mathbb{C})$ has only finitely many finite orbits, but the methods used to classify the orbits for Markoff-type equations do not seem easily applicable to the TIK3 situation.

Generically, the automorphism group of $\mathcal{W}$ is generated by the three automorphisms. Since the Markoff equation (1) admits additional automorphisms, we consider an analogous family of TIK3 surfaces, which

[1]We remark that although the generic member of the family of surfaces given by the vanishing of a $(2,2,2)$-form is a K3 surface, there are special members that for not. For example, the classical Markoff equation defines a rational surface.

we call *Markoff-type K3* (MK3) *surfaces*. These are the TIK3 surfaces (2) that are invariant under coordinate permutations and double sign changes. See Proposition 6.5 for a description of the full 4-dimensional family of MK3 surfaces, and Proposition B.1 for a proof that a regular minimal model of a generic member of this family is a K3 surface..

A typical example, which we use as a prototype, is the following one-parameter family of MK3-surfaces $\mathcal{W}_k$. For non-zero k, we define $\mathcal{W}_k$ to be the projective closure in $(\mathbb{P}^1)^3$ of the affine surface

$$\mathcal{W}_k : x^2 + y^2 + z^2 + x^2y^2z^2 + kxyz = 0. \tag{4}$$

We note that for all $k \neq 0$, a regular minimal model of $\mathcal{W}_k$ is a K3 surface; see Proposition B.1. In order to understand the orbit structure in $\mathcal{W}_k(\mathbb{F}_p)$, we computed all orbits for $p \leq 113$ and all $k \in \mathbb{F}_p^*$; see Section 10 and Appendix C. We use these computations for two purposes.

First, by studying small orbit sizes that appear in $\mathcal{W}_k(\mathbb{F}_p)$ for many different p and k, we find patterns which we use to construct finite orbits in $\mathcal{W}_k(\mathbb{C})$. Proposition 1.2 illustrates most of our findings. Explicit equations for all of the orbits described in Proposition 1.2 may be found in Table 3, and Section 9 describes how we used the $\mathbb{F}_p$ data to find, or in some cases rule out, finite orbits over $\mathbb{C}$. We found especially interesting the examples of 1-parameter families having orbits of sizes 24, 192, and 288.

Proposition 1.2. *Let $\mathcal{W}_k$ be the projective closure in $(\mathbb{P}^1)^3$ of the affine surface* (4).

- $\mathcal{W}_4(\mathbb{Q})$ *contains an orbit of size* 4.
- $\mathcal{W}_k\big(\mathbb{Q}(i)\big)$ *contains an orbit of size* 48 *for every* $k \in \mathbb{Q}(i)$.
- *There is a field* $K/\mathbb{Q}$ *of degree* 3 *and an element* $k \in K$ *so that* $\mathcal{W}_k(K)$ *has an orbit of size* 64.
- *There is a* $k \in \mathbb{Q}(i, \sqrt{2})$ *so that* $\mathcal{W}_k\big(\mathbb{Q}(i, \sqrt{2})\big)$ *has an orbit of size* 96.
- *There is a field* $K/\mathbb{Q}$ *of degree* 8 *and an element* $k \in K$ *so that* $\mathcal{W}_k(K)$ *has an orbit of size* 144.
- *There is a field* $K/\mathbb{Q}$ *of degree* 8 *and an element* $k \in K$ *so that* $\mathcal{W}_k(K)$ *has an orbit of size* 160.
- *There is a* $k(t) \in \mathbb{Q}(t)$ *so that* $\mathcal{W}_{k(t)}\big(\mathbb{Q}(t)\big)$ *has an orbit of size* 24.
- *There is a* $k(t) \in \mathbb{Q}(i, t)$ *so that* $\mathcal{W}_{k(t)}\big(\mathbb{Q}(i, t)\big)$ *has an orbit of size* 192.

- *There is an irreducible curve $C/\mathbb{Q}$ of genus 9 and an element $k \in \mathbb{Q}(C)$ in the function field of C so that $\mathcal{W}_k\big(\mathbb{Q}(C)\big)$ has an orbit of size 288.*

In the spirit of the many uniform boundedness theorems and conjectures in arithmetic geometry and arithmetic dynamics, we pose the following question:

Question 1.3. Does there exist a constant N so that

$$\#\{P \in \mathcal{W}_k(\mathbb{C}) : \text{the orbit of } P \text{ is finite}\} \le N \quad \text{for all } k \in \mathbb{C}^*?$$

More generally, does there exist a constant N so that for every non-degenerate[2] TIK3 surface $\mathcal{W}$ we have

$$\#\{P \in \mathcal{W}(\mathbb{C}) : \text{the } \langle \sigma_1, \sigma_2, \sigma_3 \rangle\text{-orbit of } P \text{ is finite}\} \le N?$$

See Question 9.1 for a further discussion of uniform boundedness of finite orbits.

Second, we investigate large orbits in $\mathcal{W}_k(\mathbb{F}_p)$ to see if the methods employed in [11] for the Markoff equation are potentially applicable to the MK3 setting. The fiber-to-fiber jumping strategy employed by [11] uses the fact, which they prove for (3) with $(a, b, c, d, e) = (0, 0, 0, -3, 0)$, that if a vertical fibral orbit is sufficiently large, then at least one of the points in that vertical orbit has a horizontal orbit that consists of the entire horizontal fiber. (See Section 4 and Remark 4.5 for further details.) We are interested in the question of whether such a fiber-to-fiber jumping strategy will work on the MK3-surface $\mathcal{W}_k(\mathbb{F}_p)$. In Section 11 we show that the surface $\mathcal{W}_1(\mathbb{F}_{53})$ has an orbit of size 3456, but that the fiber-to-fiber jumping strategy cannot be used to prove that these 3456 points all lie in the same orbit. This suggests that additional ideas may be needed to prove the existence of a large orbit in $\mathcal{W}_k(\mathbb{F}_p)$.

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[2] See Definition 3.1, but briefly, non-degeneracy means that the three involutions are well-defined.

2. A brief survey of related work on the Markoff surface and on tri-involutive K3 surfaces

Definition 2.1. Let K be a field, for example a number field or a finite field. Let $a \in K^*$ and $k \in K$. The associated *Markoff equation* is

$$\mathcal{M}_{a,k} : x^2 + y^2 + z^2 = axyz + k, \tag{5}$$

and $\mathcal{G}_{\mathcal{M}}$ denotes the group of automorphisms of $\mathcal{M}_{a,k}$ generated by the involutions $\sigma_1, \sigma_2, \sigma_3$, double sign changes, and permutations of the coordinates.

Theorem 2.2. (a) (Markoff [27])

$$\mathcal{M}_{3,0}(\mathbb{Z}) = \bigl\{(0,0,0)\bigr\} \cup \mathcal{G}_{\mathcal{M}} \cdot (1,1,1).$$

(b) *More generally, for all $a, k \in \mathbb{Z}$ with $a \neq 0$, there is a finite set of points $P_1, \ldots, P_r \in \mathcal{M}_{a,k}(\mathbb{Z})$ such that*

$$\mathcal{M}_{a,k}(\mathbb{Z}) = \bigcup_{i=1}^{r} \mathcal{G}_{\mathcal{M}} \cdot P_i$$

except in the case of the so-called Cayley cubic $\mathcal{M}_{1,4}$.[3]

Conjecture 2.3. (Baragar [1, Section V.3], Bourgain–Gambard–Sarnak [10, 11]) *For all primes $p \geq 5$ we have*

$$\mathcal{M}_{3,0}(\mathbb{F}_p) = \bigl\{(0,0,0)\bigr\} \cup \bigl(\mathcal{G}_{\mathcal{M}} \cdot (1,1,1)\bigr).$$

As noted in Theorem 2.2(b), the set $\mathcal{M}_{a,k}(\mathbb{Z})$ generally consists of finitely many orbits. However, we may still ask to what extent the points in $\mathcal{M}_{a,k}(\mathbb{F}_p)$ lift to points in $\mathcal{M}_{a,k}(\mathbb{Z})$, or alternatively, to what extent $\mathcal{M}_{a,k}(\mathbb{F}_p)$ is essentially a single $\mathcal{G}_{\mathcal{M}}$-orbit. One difficulty that occurs comes from finite orbits in $\mathcal{M}_{a,k}(\overline{\mathbb{Q}})$, since their mod p reduction leads to (small) finite orbits in various $\mathcal{M}_{a,k}(\mathbb{F}_p)$. This leads to the following conjectures.

Conjecture 2.4. *Let $a, k \in \mathbb{Z}$.*

(a) *There is a constant $M_1(a,k)$ such that for all primes $p \nmid a$ we have*

$$\#\mathcal{M}_{a,k}(\mathbb{F}_p) \leq \#\Bigl(\text{largest } \mathcal{G}_{\mathcal{M}}\text{-orbit in } \mathcal{M}_{a,k}(\mathbb{F}_p)\Bigr) + M_1(a,k).$$

(b) *If $\#\mathcal{M}_{a,k}(\mathbb{Z}) = \infty$, then there is a constant $M_2(a,k)$ such that for all primes $p \nmid a$ we have*

$$\#\mathcal{M}_{a,k}(\mathbb{F}_p) \leq \#\bigl(\mathcal{M}_{a,k}(\mathbb{Z}) \bmod p\bigr) + M_2(a,k).$$

[3] For the Cayley cubic $\mathcal{M}_{1,4}$, the points $(2,t,t)$ for positive integers t generate distinct orbits, and their union is $\mathcal{M}_{1,4}(\mathbb{Z})$.

(*One might further ask whether* $M_1(a,k)$ *and* $M_2(a,k)$ *may be chosen independently of* a *and* k.)

Bourgain–Gambard–Sarnak and Chen have a number of results related to Conjectures 2.3 and 2.4, including the following:

Theorem 2.5. (a) [11, Theorem 1]

$$\#\mathcal{M}_{3,0}(\mathbb{F}_p) = \#\big(\mathcal{G}_{\mathcal{M}} \cdot (1,1,1)\big) + p^{o(1)}, \quad \textit{as } p \to \infty.$$

(b) [11, Theorem 2] *Conjecture* 2.3 *holds for all but possibly* $X^{o(1)}$ *primes* $p \le X$, *as* $X \to \infty$.

(c) [16] *Conjecture* 2.3 *holds for all but finitely many primes* p.

Remark 2.6. Chen's result (Theorem 2.5(c)) supersedes the results of Bourgain–Gambard–Sarnak (Theorem 2.5(a,b)), but Chen's proof depends strongly on the particular form of the equation $\mathcal{M}_{3,0}$. More precisely, Chen proves that the orbit of $(1,1,1)$ in $\mathcal{M}_{3,0}(\mathbb{F}_p)$ has cardinality divisible by p. This combined with the methods used to prove [11, Theorem 1] yield the desired result. However, we note that the methods used to prove the results in [11] should extend to give versions of Conjecture 2.4 analogous to Theorem 2.5(a,b) for all $\mathcal{M}_{a,k}$, while for now Chen's method seems to apply only to $\mathcal{M}_{3,0}$.

Remark 2.7. Other recent notable results include the following:

- Konyagin–Makarychev–Shparlinski–Vyugin [25] prove that

  $$\#\mathcal{M}_{3,0}(\mathbb{F}_p) \smallsetminus \big(\mathcal{G}_{\mathcal{M}} \cdot (1,1,1)\big) \le \exp\left((\log p)^{2/3+o(1)}\right).$$

  This improves Theorem 2.5, and the methods should extend to more general Markoff equations.
- Given a pseudo-Anosov element $g \in \mathrm{Out}(\mathbf{F}_2)$, g induces a permutation g_p on $\mathcal{M}_{1,k}(\mathbb{F}_p)$ for each prime p. Cerbu–Gunther–Magee–Peilen [15] prove that asymptotically, the action of g_p on $\mathcal{M}_{1,k}(\mathbb{F}_p)$ has an orbit of size at least $\frac{\log(p)}{\log|\lambda|} + O_g(1)$, where λ is the eigenvalue of largest modulus of g when viewed as an element of $\mathrm{GL}_2(\mathbb{Z})$.
- M. de Courcy-Ireland and S. Lee [19] verify strong approximation for the Markoff surface for all primes $p < 3000$. Additionally, they completely characterize the orbit structure of the degenerate Cayley cubic, $\mathcal{M}_{1,4}(\mathbb{F}_p)$, providing both the number of orbits as well as their sizes, given in terms of divisors of $p^2 - 1$.
- M. de Courcy-Ireland and M. Magee [20] demonstrate that the eigenvalues of the family of Markoff graphs modulo p converge to the Kesten-McKay measure, which is a heuristic indicator that Markoff graphs are suitably "random". This also provides a (very) weak bound on the spectral gap of such graphs.

- M. de Courcy-Ireland [18] shows that if $p > 7$, then the Markoff graph mod p is not planar.
- A. Gamburd , M. Magee and R. Ronan [22] prove an asymptotic formula for the function $N_{n,a,k}(R)$ that counts the number of integer solutions to $x_1^2 + \cdots + x_n^2 = ax_1 \cdots x_n + k$ with $\max |x_i| \le R$, excluding potential exceptional sets. They prove that $N_{n,a,k}(R)$ is asymptotic to $C(n,a,k)(\log R)^{\beta_n}$, where as indicated, the constant depends on n, a, k, while exponent β_n, which generally is not an integer, depends only on n. See also A. Baragar [2] for related work.

We conclude this section by briefly describing some earlier work on the geometry and arithmetic of tri-involutive K3 surfaces, which we recall are certain K3 surfaces admitting three non-commuting involutions. Wang [32] explicitly constructed canonical heights on TIK3 surfaces defined over number fields associated to the infinite order automorphisms $\sigma_i \circ \sigma_j$, similar to those constructed in [30] for K3 surfaces having two involutions. Baragar [3, 4, 5] further studied these height functions and asked, in particular, whether they fit together to form a vector canonical height. Kawaguchi [24] answered this in the negative for certain K3 surfaces, and Cantat and Dujardin [13] completely characterized the surfaces on which vector canonical heights exist.

We next state a recent result regarding finite orbits on TIK3 surfaces in characteristic 0.

Theorem 2.8 ([13, Cantat–Dujardin]). *Let $\mathcal{W}/\mathbb{C}$ be a TIK3 surface, and let $\langle\sigma_1, \sigma_2, \sigma_3\rangle \subseteq \operatorname{Aut}(\mathcal{W})$ be the subgroup of $\mathcal{W}$ generated by the three involutions $\sigma_1, \sigma_2, \sigma_3$. Then*

$$\bigl\{P \in \mathcal{W}(\mathbb{C}) : \textit{the } \langle\sigma_1, \sigma_2, \sigma_3\rangle\textit{-orbit of } P \textit{ is finite}\bigr\}$$

is a finite set.

Proof. This is a special case of the results in [13], since in the language of [13], the TIK3-surface $\mathcal{W}$ and its group of automorphisms $\langle\sigma_1, \sigma_2, \sigma_3\rangle$ do not form a Kummer group, and $\mathcal{W}$ contains no $\langle\sigma_1, \sigma_2, \sigma_3\rangle$-invariant curves. □

Finally, we mention Cantat's fundamental paper [12], although it is not specifically about TIK3 surfaces. Let $\varphi : \mathcal{X} \to \mathcal{X}$ be an automorphism of positive entropy of a K3 surface defined over $\mathbb{C}$, e.g., $\sigma_i \circ \sigma_j$ for a TIK3 surface. Then Cantat proves that there exists a unique invariant probability measure μ with maximal entropy, that (φ, μ) is measurably conjugate to a Bernoulli shift, and that μ gives the asymptotic distribution of periodic points.

3. Tri-Involutive K3 (TIK3) Surfaces

Definition 3.1. A *Tri-Involutive K3* (TIK3) *Surface* is a K3 surface[4]

$$\mathcal{W} = \{\overline{F} = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

defined by a $(2,2,2)$-form[5]

$$\overline{F}(X_1, X_2; Y_1, Y_2; Z_1, Z_2) \in K[X_1, X_2; Y_1, Y_2; Z_1, Z_2]. \tag{6}$$

For distinct $i, j \in \{1,2,3\}$, we denote the various projections of $\mathcal{W}$ onto one or two copies of $\mathbb{P}^1$ by

$$\pi_i : \mathcal{W} \longrightarrow \mathbb{P}^1 \quad \text{and} \quad \pi_{ij} : \mathcal{W} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

We say that the TIK3 is *non-degenerate* if it satisfies the following two conditions:

(i) The projection maps $\pi_{12}, \pi_{13}, \pi_{23}$ are finite.[6]

(ii) The generic fibers of the projection maps π_1, π_2, π_3 are (irreducible, geometrically connected) smooth curves, and thus the smooth fibers are curves of genus 1, since they are $(2,2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., curves given by the vanishing of a $(2,2)$-form.

Definition 3.2. To ease notation, we write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, and we let

$$F(x,y,z) = \overline{F}(x,1;y,1;z,1).$$

Then $\mathcal{W}$ is the closure in $(\mathbb{P}^1)^3$ of the affine surface, which by abuse of notation we also denote by $\mathcal{W}$,

$$\mathcal{W} : F(x,y,z) = 0.$$

Definition 3.3. Let $\mathcal{W}$ be a TIK3 surface with projections $\pi_1, \pi_2, \pi_3 : \mathcal{W} \to \mathbb{P}^1$ We define a *fiber of* $\mathcal{W}$ to be a set of the form

$$\pi_i^{-1}(t) \quad \text{for some } i \in \{1,2,3\} \text{ and some } t \in \mathbb{P}^1.$$

[4]We recall that an algebraic K3 surface is a smooth projective geometrically connected surface with trivial canonical bundle and irregularity zero. In this paper we work directly with equations of the form (6) satisfying the non-degeneracy condition, so it not important for our purposes that our surfaces are K3. However, for completeness, we show in Section B that minimal regular models of generic surfaces in our families are K3 surfaces.

[5]In general, an (a,b,c)-form is a global section to $\mathcal{O}_{(\mathbb{P}^1)^3}(a,b,c)$, or more prosaically, an (a,b,c)-form is a polynomial f in $K[X_1, X_2; Y_1, Y_2; Z_1, Z_2]$ satisfying $f(uX_1, uX_2; vY_1, vY_2; wZ_1, Zw_2) = u^a v^b w^c f(X_1, X_2; Y_1, Y_2; Z_1, Z_2)$.

[6]We note that $\pi_{12}, \pi_{13}, \pi_{23}$ are finite if and only if their fibers are 0-dimensional, in which case they are maps of degree 2.

Thus fibers may lie in any of three different directions, and we may view $\mathcal{W}$ as being triply cross-hatched by the various fibers. We denote the set of fibers by

$$\operatorname{Fiber}(\mathcal{W}) = \{\text{fibers of } \mathcal{W}\}.$$

If we need to refer to fibers over a particular point and corresponding to a particular projection, we use the following more precise notation. We denote the fibers of $\pi_1, \pi_2, \pi_3 : \mathcal{W} \to \mathbb{P}^1$ over points $x_0, y_0, z_0 \in \mathbb{P}^1$ by, respectively,

$$\mathcal{W}_{x_0}^{(1)} = \pi_1^{-1}(x_0), \qquad \mathcal{W}_{y_0}^{(2)} = \pi_2^{-1}(y_0), \qquad \mathcal{W}_{z_0}^{(3)} = \pi_3^{-1}(z_0).$$

For $P = (x_P, y_P, z_P) \in \mathcal{W}$, we let

$$\mathcal{W}_P^{(1)} = \mathcal{W}_{x_P}^{(1)}, \quad \mathcal{W}_P^{(2)} = \mathcal{W}_{y_P}^{(2)}, \quad \mathcal{W}_P^{(3)} = \mathcal{W}_{z_P}^{(3)}.$$

Definition 3.4. Let $\mathcal{W}$ be a non-degenerate TIK3 surface. For distinct $i, j, k \in \{1, 2, 3\}$, we write

$$\sigma_k : \mathcal{W} \longrightarrow \mathcal{W} \tag{7}$$

for the involution that swaps the sheets of π_{ij}, i.e., $\sigma_k \in \operatorname{Aut}(\mathcal{W})$ is the unique non-identity automorphism satisfying

$$\pi_{ij} \circ \sigma_k = \pi_{ij}.$$

The automorphism group of a TIK3 surface $\mathcal{W}$ contains the non-commuting involutions $\sigma_1, \sigma_2, \sigma_3$, and depending on the symmetries of $\mathcal{W}$'s defining polynomial F, the automorphism group may contain additional automorphisms. Typical examples include symmetry in x, y, z that allows permutation of the coordinates, and power symmetry that allows the signs of two of x, y, z to be reversed. For example, the Markoff equation (1) permits these extra automorphisms; and in Section 6 we consider analogous TIK3 surfaces. In any case, we will be interested in subgroups of the automorphism group that move points around individual fibers.

Definition 3.5. Let $\mathcal{W}$ be a non-degenerate TIK3 surface, let

$$\operatorname{Gen}(\mathcal{G}) \subset \operatorname{Aut}(\mathcal{W})$$

be a (finite) set of automorphisms of $\mathcal{W}$, and let

$$\mathcal{G} = \left\langle \varphi : \varphi \in \operatorname{Gen}(\mathcal{G}) \right\rangle$$

be the subgroup of $\mathrm{Aut}(\mathcal{W})$ generated by the elements of $\mathrm{Gen}(\mathcal{G})$. Let $\mathcal{F} \in \mathrm{Fiber}(\mathcal{W})$ be a fiber of $\mathcal{W}$. We denote the (restricted) stabilizer of $\mathcal{F}$ by[7]

$$\mathcal{G}_{\mathcal{F}} = \bigl\langle \varphi \in \mathrm{Gen}(\mathcal{G}) : \varphi(\mathcal{F}) = \mathcal{F} \bigr\rangle.$$

We further define (*restricted*) *fibral automorphism groups* in each of the three directions by[8]

$$\begin{aligned}
\mathcal{G}^{(1)} &= \bigl\langle \varphi \in \mathrm{Gen}(\mathcal{G}) : \varphi(\mathcal{W}_x^{(1)}) = \mathcal{W}_x^{(1)} \text{ for all } x \in \mathbb{P}^1 \bigr\rangle,\\
\mathcal{G}^{(2)} &= \bigl\langle \varphi \in \mathrm{Gen}(\mathcal{G}) : \varphi(\mathcal{W}_y^{(2)}) = \mathcal{W}_y^{(2)} \text{ for all } y \in \mathbb{P}^1 \bigr\rangle,\\
\mathcal{G}^{(3)} &= \bigl\langle \varphi \in \mathrm{Gen}(\mathcal{G}) : \varphi(\mathcal{W}_z^{(3)}) = \mathcal{W}_z^{(3)} \text{ for all } z \in \mathbb{P}^1 \bigr\rangle.
\end{aligned}$$

For example, if $\{i,j,k\} = \{1,2,3\}$ and $\mathcal{W}$ is generic, then typically we take $\mathcal{G}^{(k)} = \langle \sigma_i, \sigma_j \rangle$, since the k-direction fibers are invariant for σ_i and σ_j.

Definition 3.6. Let $\mathcal{W}$ be a non-degenerate TIK3 surface, let $\mathcal{G} \subseteq \mathrm{Aut}(\mathcal{W})$ be a group of automorphisms of $\mathcal{W}$, and let $P_0 = (x_0, y_0, z_0) \in \mathcal{W}(K)$. The *$\mathcal{G}$-orbit of P* is

$$\mathcal{G} \cdot P = \bigl\{ \varphi(P) : \varphi \in \mathcal{G} \bigr\}.$$

The *fibral $\mathcal{G}$-orbits of P* are

$$\mathcal{G}^{(k)} \cdot P = \bigl\{ \varphi(P) : \varphi \in \mathcal{G}^{(k)} \bigr\} \quad \text{for } k = 1, 2, 3.$$

4. A strategy for proving that $\mathcal{W}(\mathbb{F}_q)$ has a large $\mathcal{G}$-connected component

In this section we consider a non-degenerate TIK3-surface $\mathcal{W}$ defined over a finite field $\mathbb{F}_q$, and a group of automorphisms $\mathcal{G} \subseteq \mathrm{Aut}(\mathcal{W})$.

Definition 4.1. Let $t \in \mathbb{P}^1(\mathbb{F}_q)$, and let $i \in \{1,2,3\}$. We say that the fiber $\mathcal{W}_t^{(i)}(\mathbb{F}_q)$ is *$\mathcal{G}$-connected* if $\mathcal{G}^{(i)}$ acts transitively on $\mathcal{W}_t^{(i)}(\mathbb{F}_q)$. Following terminology from [10], we define the *$\mathcal{G}$-cage of* $\mathcal{W}(\mathbb{F}_q)$ to be the set

$$\mathsf{Cage}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) = \left\{ P \in \mathcal{W}(\mathbb{F}_q) : \begin{array}{l} \text{at least one of } \mathcal{W}_P^{(1)}(\mathbb{F}_q),\ \mathcal{W}_P^{(2)}(\mathbb{F}_q), \\ \text{and } \mathcal{W}_P^{(3)}(\mathbb{F}_q) \text{ is } \mathcal{G}\text{-connected} \end{array} \right\}.$$

[7] The reason that we do not use $\bigl\{\varphi \in \mathcal{G} : \varphi(\mathcal{F}) = \mathcal{F}\bigr\}$, which is the full subgroup that leaves $\mathcal{F}$ invariant, is because when using $\mathcal{G}$ to move around points in fibers of $\mathcal{W}$, we will want to apply one generator at a time.

[8] We do not include the set of generators $\mathrm{Gen}(\mathcal{G})$ in the notation for the fibral automorphism groups, since it will generally be clear from context. For example, for a generic TIK3 surface, we take $\mathrm{Gen}(\mathcal{G}) = \{\sigma_1, \sigma_2, \sigma_3\}$. If $\mathcal{W}$ has extra symmetries, for example if $\mathcal{W}$ is one of the MK3 surfaces described in Section 6, then $\mathrm{Gen}(\mathcal{G})$ will also include some coordinate permutations and sign shifts.

We denote the set of $\mathcal{G}$-connected fibers by

$$\mathsf{ConnFib}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) = \left\{ \mathcal{W}_t^{(i)}(\mathbb{F}_q) : \begin{array}{l} i \in \{1,2,3\},\ t \in \mathbb{P}^1(\mathbb{F}_q), \\ \mathcal{W}_t^{(i)}(\mathbb{F}_q) \text{ is } \mathcal{G}\text{-connected} \end{array} \right\}.$$

With this notation, an alternative description of the cage is as the union of the points in the fibers in $\mathsf{ConnFib}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$.

We further say that $\mathcal{W}(\mathbb{F}_q)$ is *cage-connected* if for every pair of points $P, Q \in \mathsf{Cage}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ there exists a sequence of $\mathcal{G}$-connected fibers $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n \in \mathsf{ConnFib}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ such that

$$P \in \mathcal{F}_1, \quad Q \in \mathcal{F}_n, \quad \mathcal{F}_i \cap \mathcal{F}_{i+1} \ne \emptyset \text{ for all } 1 \le i < n.$$

Remark 4.2. We can also describe cage-connectivity of $\mathcal{W}(\mathbb{F}_q)$ using a standard construction in graph theory. Let X be any set, and let $\mathcal{S} \subset 2^X$ be a collection of subsets of X. The *intersection graph of* $\mathcal{S}$ is the graph whose vertices are the elements of $\mathcal{S}$, and whose edges are all $[A, B]$ such that $A, B \in \mathcal{S}$ satisfy $A \cap B \ne \emptyset$. Then $\mathcal{W}(\mathbb{F}_q)$ is cage-connected if the intersection graph of its collection of $\mathcal{G}$-connected fibers $\mathsf{ConnFib}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ is a connected graph. Similarly, the content of Theorem 5.5 is that if $q > 100$, then the intersection graph of the collection of all $\mathcal{G}$-fibers of $\mathcal{W}(\mathbb{F}_q)$ is connected, and indeed its graph diameter is at most 2.

The starting point used in [10] to prove the connectivity of the Markoff graph $\mathcal{M}_{3,0}(\mathbb{F}_q) \smallsetminus \{(0,0,0)\}$ is to show that the associated cage is connected. This is done via a process that jumps from one connected fiber to another using a version of the following property:

Definition 4.3. We say that $\mathcal{W}(\mathbb{F}_q)$ has the *fiber-jumping property* if for all fibers $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{W}(\mathbb{F}_q)$ there exists a $\mathcal{G}$-connected fiber $\mathcal{F}_3 \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ satisfying

$$\mathcal{F}_1 \cap \mathcal{F}_3 \ne \emptyset \quad \text{and} \quad \mathcal{F}_2 \cap \mathcal{F}_3 \ne \emptyset.$$

As described in [10], the fiber-jumping property implies that $\mathcal{W}(\mathbb{F}_q)$ is cage-connected. For the convenience of the reader, we recall the short proof.

Proposition 4.4. *Suppose that $\mathcal{W}(\mathbb{F}_q)$ has the fiber-jumping property. Then $\mathcal{W}(\mathbb{F}_q)$ is cage-connected.*

Proof. Let $P, Q \in \mathsf{Cage}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$. By definition, this means that they lie on $\mathcal{G}$-connected fibers, say

$$P \in \mathcal{F}_1 \quad \text{and} \quad Q \in \mathcal{F}_2 \quad \text{with} \quad \mathcal{F}_1, \mathcal{F}_2 \in \mathsf{ConnFib}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr).$$

We apply the assumption that $\mathcal{W}(\mathbb{F}_q)$ has the fiber-jumping property to the fibers $\mathcal{F}_1$ and $\mathcal{F}_2$. This allows us to find a $\mathcal{G}$-connected fiber $\mathcal{F}_3 \in \mathsf{ConnFib}\big(\mathcal{W}(\mathbb{F}_q)\big)$ satisfying

$$\mathcal{F}_1 \cap \mathcal{F}_3 \neq \emptyset \quad \text{and} \quad \mathcal{F}_3 \cap \mathcal{F}_2 \neq \emptyset.$$

Then the sequence $\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_2$ takes us from P to Q, and since $P, Q \in \mathsf{Cage}_{\mathcal{G}}\big(\mathcal{W}(\mathbb{F}_q)\big)$ were arbitrary, this proves from the definition that $\mathcal{W}(\mathbb{F}_q)$ is cage-connected. □

The strategy that is employed in [10] to prove that most of the points in the Markoff set $\mathcal{M}_{3,0}(\mathbb{F}_q)$ form a connected set has several steps. We reformulate these steps for TIK3-surfaces, retaining (and expanding on) their chess terminology.

Setting the board (Cage connectivity):
$\mathcal{W}(\mathbb{F}_q)$ is cage-connected.

End game (Large fibral orbits):
Let $P \in \mathcal{W}_t^{(i)}(\mathbb{F}_q)$ be a point whose fibral orbit $\mathcal{G}^{(i)} \cdot P$ is moderately large. Then $\mathcal{G}^{(i)} \cdot P$ contains a point of the cage, i.e., it intersects a $\mathcal{G}$-connected fiber.

Middle game (Small fibral orbits):
Let $P \in \mathcal{W}_t^{(i)}(\mathbb{F}_q)$ be a point whose fibral orbit $\mathcal{G}^{(i)} \cdot P$ is of small, but non-negligible, size. Then $\mathcal{G}^{(i)} \cdot P$ contains a point lying in a fibral orbit of strictly larger size.

Opening (Tiny fibral orbits):
There are no non-trivial points $P \in \mathcal{W}_t^{(i)}(\mathbb{F}_q)$ whose fibral orbit $\mathcal{G}^{(i)} \cdot P$ is tiny.

Remark 4.5 (The Bourgain–Gamburd–Sarnak Connectivity Proof for the Markoff Equation)**.** We briefly sketch the connectivity proof for

$$\mathcal{M}^*(\mathbb{F}_p) = \mathcal{M}_{3,0}(\mathbb{F}_p) \setminus (0,0,0)$$

in [10]. They prove connectivity using the subgroup $\mathcal{G} \subset \operatorname{Aut}(\mathcal{M}_{3,0})$ generated by the compositions

$$\rho^{(i)} = \sigma_i \circ \tau_{jk}, \quad \text{where } \{i,j,k\} = \{1,2,3\},$$

and τ_{jk} denotes the transposition of the j and k coordinates. They call $\rho^{(i)}$ a *rotation*, since it acts on the fibers $(\mathcal{M}_{3,0})_t^{(i)}$ via a 2-by-2 (rotation) matrix acting on the jk-coordinates. Writing $\rho_t^{(i)}$ for the restriction of $\rho^{(i)}$ to this fiber, they note that the order of $\rho_t^{(i)}$ divides one of $p-1$, p, or $p+1$, with the exact order depending on the eigenvalues of the matrix $\rho_t^{(i)}$. It follows that

$$(\mathcal{M}_{3,0})_t^{(i)}(\mathbb{F}_p) \subset \mathsf{Cage}\big(\mathcal{M}_{3,0}(\mathbb{F}_p)\big) \quad \Longleftrightarrow \quad \rho_t^{(i)} \text{ has maximal order.}$$

The first step in proving that $\mathcal{M}^*(\mathbb{F}_p)$ is $\mathcal{G}$-connected is an argument that uses curve coverings, point counting, and inclusion/exclusion to show that $\mathcal{M}_{3,0}(\mathbb{F}_p)$ has the fiber jumping property for $\mathcal{G}$. It follows that $\mathsf{Cage}_{\mathcal{G}}\bigl(\mathcal{M}_{3,0}(\mathbb{F}_p)\bigr)$ is connected, cf. Proposition 4.4. They then use a similar argument for the endgame, where a fiber is deemed large if it has $p^{1/2+\epsilon}$ points. Next they consider the middle game, which consists of points whose (small) fibral orbit has at least p^ϵ points. This comes down to showing that certain equations have few solutions whose coordinates are elements of $\mathbb{F}_p^*$ of small order. They provide three proofs of the required statement, one via Stepanov's auxiliary polynomial proof of Weil's conjecture for curves over $\mathbb{F}_p$, one using directly a sharp estimate due to Corvaja and Zannier [17] for the gcd of polynomials over finite fields, and one using a projective Szemeredi-Trotter theorem due to Bourgain [9]. Indeed, they can handle the middle game for even smaller fibral components provided that p^2-1 does not have too many prime divisors. Finally, for the opening, they first observe that finite orbits in $\mathcal{M}_{a,k}(\overline{\mathbb{Q}})$ will cause tiny orbits in $\mathcal{M}_{a,k}(\mathbb{F}_p)$ for infinitely many p. However, in their case $\mathcal{M}_{3,0}(\overline{\mathbb{Q}})$ contains no finite orbits other than $\bigl\{(0,0,0)\bigr\}$, so this is not a problem. They next show that every point $P\in\mathcal{M}^*(\mathbb{F}_p)$ lies in a fibral component containing at least $(\log_{20}p)^{1/3}$ points. This and some further calculations suffice to prove that $\mathcal{M}^*(\mathbb{F}_p)$ is $\mathcal{G}$-connected unless p^2-1 is very smooth, i.e., is a product of a large number of small primes. (Conjecturally, there are only finitely many such primes.)

Remark 4.6 (Fiber Jumping and Cage Connectivity for TIK3-Surfaces)**.** As explained in Remark 4.5, Bourgain, Gamburd, and Sarnak [10] prove that the Markoff equation $\mathcal{M}_{3,0}(\mathbb{F}_p)\smallsetminus\bigl\{(0,0,0\bigr\}$ is $\mathcal{G}$-connected by first verifying the fiber-jumping property, which sets the board by implying that the cage is cage-connected. Later we will give an example showing that the analogous statement need not be true for TIK3 surfaces. More precisely, in Example 11.1 we describe a TIK3-surface $\mathcal{W}$ such that $\mathcal{W}(\mathbb{F}_{53})$ has one large $\mathcal{G}$-connected component, which we denote by $\mathcal{W}^*(\mathbb{F}_{53})$, that contains 3456 points. However, a direct calculation show that $\mathcal{W}^*(\mathbb{F}_{53})$ does not have the $\mathcal{G}$-fiber-jumping property. More precisely, the $\mathcal{G}$-connected fibers in $\mathcal{W}(\mathbb{F}_{53})$ form two connected components, so any proof that $\mathcal{W}^*(\mathbb{F}_{53})$ is $\mathcal{G}$-connected must find a way to connect points in $\mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_{53})\bigr)$ that does not travel purely along $\mathcal{G}$-connected fibers. Of course, the prime $p=53$ is not huge, so our example may simply be a small number phenomenon. However, other examples (see Table 5) suggest that the number of fibral components in a TIK3 cage tends to be smaller than the number of fibral components in a Markoff surface cage. So a proof that TIK3 surfaces over finite fields have large $\mathcal{G}$-connected components may need to find a way to expand the cage in order to fit it into a $\mathcal{G}$-connected set that can be used for the "setting the board" step.

In addition, the issue concerning smoothness of fibral group orders that arises in the method of BGS will be exacerbated for TIK3 surfaces. The analogous rotations (translations) on a TIK3 surface come from the actions of elliptic curves on homogeneous spaces. These actions are translations by a point whose order can range from $p+1-2\sqrt{p}$ to $p+1+2\sqrt{p}$. So now we are not concerned with smoothness of only $p \pm 1$, but instead with the smoothness of all numbers within this range. Ideally, we would like to restrict to values of p for which this range of numbers contains no smooth numbers, but there are unlikely to be infinitely many such p.

5. The incidence graph of the fibers of a TIK3 surface

Definition 5.1. A TIK3 surface has three fibral directions associated to the three projections onto $\mathbb{P}^1$. For expositional convenience, we will say that fibers corresponding to different projections are (*pairwise*) *orthogonal* to one another, while fibers corresponding to the same projection are *parallel*. So for example, the fibers $\mathcal{W}^{(1)}_{x_0}$ and $\mathcal{W}^{(2)}_{y_0}$ are orthogonal, while the fibers $\mathcal{W}^{(1)}_{x_0}$ and $\mathcal{W}^{(1)}_{x_1}$ are parallel.

Remark 5.2. Distinct parallel fibers clearly do not intersect, while orthogonal fibers in $\mathcal{W}(\mathbb{F}_q)$ may intersect in 0, 1, or 2 points. For example, if $x_0, y_0 \in \mathbb{P}^1(\mathbb{F}_q)$, then

$$\Bigl(\mathcal{W}^{(1)}_{x_0}(\mathbb{F}_q) \cap \mathcal{W}^{(2)}_{y_0}(\mathbb{F}_q)\Bigr) = \bigl\{(x_0, y_0, z) : F(x_0, y_0, z) = 0\bigr\}.$$

Thus the intersection is non-empty if and only if a certain quadratic form[9] has a zero in $\mathbb{P}^1(\mathbb{F}_q)$.

Our goal in this section is to give an easily verifiable condition which ensures that, given two orthogonal fibers $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathcal{W}(\mathbb{F}_q)$, there is a third fiber $\mathcal{F}_3 \subset \mathcal{W}(\mathbb{F}_q)$ satisfying

$$\mathcal{F}_1 \cap \mathcal{F}_3 \neq \emptyset \quad \text{and} \quad \mathcal{F}_2 \cap \mathcal{F}_3 \neq \emptyset.$$

In more evocative terms, although the union $\mathcal{F}_1 \cup \mathcal{F}_2$ of two orthogonal fibers may be "disconnected," there is a third fiber so that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is a "connected" set of orthogonal fibers. See Figure 1.

[9]We recall that although we write F using affine coordinates to ease notation, in our calculations it always represents a $(2,2,2)$ form. In particular, the polynomial $F(x_0, y_0, z)$ denotes a degree 2 homogeneous form in the variables Z_1 and Z_2; cf. Definition 3.2.

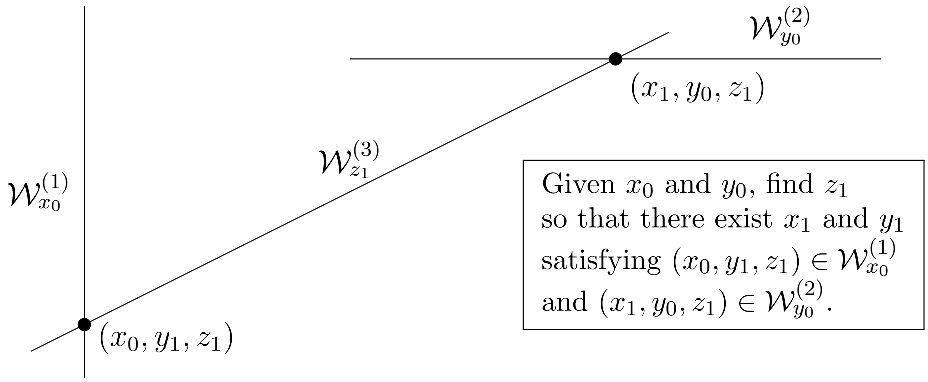

FIGURE 1. Finding a fiber $\mathcal{W}^{(3)}_{z_1}$ that intersects two given fibers $\mathcal{W}^{(1)}_{x_0}$ and $\mathcal{W}^{(2)}_{y_0}$

Definition 5.3. For $x_0, y_0, z_0 \in \mathbb{P}^1$, we define linking sets that describe how to link two given fibers via a third fiber.

$$\begin{aligned}
\mathcal{L}^{(1)}_{y_0,z_0} &= \bigl\{x \in \mathbb{P}^1 : \mathcal{W}^{(2)}_{y_0} \cap \mathcal{W}^{(1)}_{x} \ne \emptyset \text{ and } \mathcal{W}^{(3)}_{z_0} \cap \mathcal{W}^{(1)}_{x} \ne \emptyset\bigr\},\\
\mathcal{L}^{(2)}_{x_0,z_0} &= \bigl\{y \in \mathbb{P}^1 : \mathcal{W}^{(1)}_{x_0} \cap \mathcal{W}^{(2)}_{y} \ne \emptyset \text{ and } \mathcal{W}^{(3)}_{z_0} \cap \mathcal{W}^{(2)}_{y} \ne \emptyset\bigr\},\\
\mathcal{L}^{(3)}_{x_0,y_0} &= \bigl\{z \in \mathbb{P}^1 : \mathcal{W}^{(1)}_{x_0} \cap \mathcal{W}^{(3)}_{z} \ne \emptyset \text{ and } \mathcal{W}^{(2)}_{y_0} \cap \mathcal{W}^{(3)}_{z} \ne \emptyset\bigr\}.
\end{aligned}$$

Thus for example, the points in $\mathcal{L}^{(3)}_{x_0,y_0}$ tell us which z fibers can be used to link the $x = x_0$ fiber with the $y = y_0$ fiber.

Definition 5.4. For $x_0, y_0, z_0 \in \mathbb{P}^1$, we define the following algebraic sets[10] that are useful in creating fibral links:

$$\begin{aligned}
\mathcal{C}^{(1)}_{y_0,z_0} &= \bigl\{(x,y,z) \in (\mathbb{P}^1)^3 : F(x,y_0,z) = F(x,y,z_0) = 0\bigr\},\\
\mathcal{C}^{(2)}_{x_0,z_0} &= \bigl\{(x,y,z) \in (\mathbb{P}^1)^3 : F(x_0,y,z) = F(x,y,z_0) = 0\bigr\},\\
\mathcal{C}^{(3)}_{x_0,y_0} &= \bigl\{(x,y,z) \in (\mathbb{P}^1)^3 : F(x_0,y,z) = F(x,y_0,z) = 0\bigr\}.
\end{aligned}$$

We note that the $\mathcal{C}^{(1)}_{y_0,z_0}$ is the intersection in $(\mathbb{P}^1)^3$ of a hypersurface of type $(2,0,2)$ and a hypersurface of type $(2,2,0)$, and similarly for $\mathcal{C}^{(2)}_{x_0,z_0}$ and $\mathcal{C}^{(3)}_{x_0,y_0}$. See Lemma 5.6 for a proof that if $\mathcal{W}$ is a non-degenerate TIK3 surface, then these sets have dimension 1 and their irreducible components have geometric genus at most 5.

Theorem 5.5 (K3 Analogue of [11, Proposition 6])**.**
Let K be a field, and let $x_0, y_0, z_0 \in \mathbb{P}^1(K)$.

[10] Lemma 5.6(a) contains a proof that if $\mathcal{W}$ is a non-degenerate TIK3 surface, then these algebraic sets are 1-dimensional, although they need not be irreducible.

(a) *There are* surjective *maps*

$$\begin{aligned}\mathcal{C}^{(1)}_{y_0,z_0}(K) &\xrightarrow{(x,y,z)\mapsto x} \mathcal{L}^{(1)}_{y_0,z_0}(K),\\ \mathcal{C}^{(2)}_{x_0,z_0}(K) &\xrightarrow{(x,y,z)\mapsto y} \mathcal{L}^{(2)}_{x_0,z_0}(K),\\ \mathcal{C}^{(3)}_{x_0,y_0}(K) &\xrightarrow{(x,y,z)\mapsto z} \mathcal{L}^{(3)}_{x_0,y_0}(K).\end{aligned}$$

(b) *Assume that* $q \ge 100$. *Then*

$$\mathcal{L}^{(1)}_{y_0,z_0}(\mathbb{F}_q) \ne \emptyset, \qquad \mathcal{L}^{(2)}_{x_0,z_0}(\mathbb{F}_q) \ne \emptyset, \qquad \mathcal{L}^{(3)}_{x_0,y_0}(\mathbb{F}_q) \ne \emptyset.$$

Proof. (a) By symmetry, it suffices to consider the first map. We first show that the map is well-defined. Let $(x,y,z) \in \mathcal{C}^{(1)}_{y_0,z_0}(K)$. By definition of $\mathcal{C}^{(1)}_{y_0,z_0}$, this means that

$$F(x,y_0,z) = F(x,y,z_0) = 0, \quad \text{and thus} \quad (x,y_0,z),\ (x,y,z_0) \in \mathcal{W}(K).$$

Hence

$$(x,y_0,z) \in \mathcal{W}^{(2)}_{y_0}(K) \cap \mathcal{W}^{(1)}_{x}(K) \quad \text{and} \quad (x,y,z_0) \in \mathcal{W}^{(3)}_{z_0}(K) \cap \mathcal{W}^{(1)}_{x}(K),$$

which by definition of $\mathcal{L}^{(1)}_{y_0,z_0}$ shows that $x \in \mathcal{L}^{(1)}_{y_0,z_0}(K)$. This completes the proof that the projection map

$$\pi_1 : \mathcal{C}^{(1)}_{y_0,z_0}(K) \longrightarrow \mathcal{L}^{(1)}_{y_0,z_0}(K) \tag{8}$$

is well-defined.

To prove surjectivity, we start with some $x \in \mathcal{L}^{(1)}_{y_0,z_0}(K)$. By definition of $\mathcal{L}^{(1)}_{y_0,z_0}$, this means that we can find points

$$(x,y_0,z_1) \in \mathcal{W}^{(2)}_{y_0}(K) \cap \mathcal{W}^{(1)}_{x}(K) \quad \text{and} \quad (x,y_1,z_0) \in \mathcal{W}^{(3)}_{z_0}(K) \cap \mathcal{W}^{(1)}_{x}(K).$$

Then the definition of $\mathcal{C}^{(1)}_{y_0,z_0}$ tells us that

$$(x,y_1,z_1) \in \mathcal{C}^{(1)}_{y_0,z_0}(K).$$

We have thus constructed a point in $\mathcal{C}^{(1)}_{y_0,z_0}(K)$ whose image in $\mathcal{L}^{(1)}_{y_0,z_0}(K)$ is x, which completes the proof that the projection map (8) is surjective.
(b) We use (a) with $K = \mathbb{F}_q$. Again by symmetry, it suffices to prove the first assertion. And from the surjectivity of the map in (a), it suffices to prove that $\mathcal{C}^{(1)}_{y_0,z_0}(\mathbb{F}_q)$ is not empty.

Lemma 5.6(a) tells us that the algebraic set $\mathcal{C}^{(1)}_{y_0,z_0}$ has dimension 1. We let $\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}$ be a non-singular model of an irreducible component of $\mathcal{C}^{(1)}_{y_0,z_0}$. (Although generically $\mathcal{C}^{(1)}_{y_0,z_0}$ will be a smooth irreducible curve, there are cases in which it is singular and/or reducible; see Remark 5.7.) There is then a well-defined map

$$\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}(\mathbb{F}_q) \longrightarrow \mathcal{C}^{(1)}_{y_0,z_0}(\mathbb{F}_q).$$

Since our goal is simply to show that $\mathcal{C}^{(1)}_{y_0,z_0}(\mathbb{F}_q)$ is non-empty, it suffices to prove that $\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}(\mathbb{F}_q)$ is non-empty. Weil's estimate gives the inequality

$$\#\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}(\mathbb{F}_q) \ge q+1-2\cdot\bigl(\operatorname{genus}\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\bigr)\cdot\sqrt{q}. \tag{9}$$

In particular, we see that

$$q+1 > 2\cdot\bigl(\operatorname{genus}\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\bigr)\cdot\sqrt{q} \quad\Longrightarrow\quad \widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}(\mathbb{F}_q)\ne\emptyset. \tag{10}$$

Lemma 5.6(b) says that the genus of $\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}$ is at most 5. Hence (9) and (10) imply that $\mathcal{C}^{(1)}_{y_0,z_0}(\mathbb{F}_q)$ is non-empty provided $q+1>10\sqrt{q}$, which is true for all $q>100$. $\square$

We now prove the dimension and genus estimates used in the proof of Theorem 5.5.

Lemma 5.6. *Let $\mathcal{W}$ be a non-degenerate TIK3 surface defined over a field whose characteristic is not equal to 2 or 3, and let $\mathcal{C}$ be one of the algebraic sets $\mathcal{C}^{(1)}_{y_0,z_0}$, $\mathcal{C}^{(2)}_{x_0,z_0}$, or $\mathcal{C}^{(3)}_{x_0,y_0}$ described in Definition 5.4.*

- (a) *The algebraic sets $\mathcal{C}^{(1)}_{y_0,z_0}$, $\mathcal{C}^{(2)}_{x_0,z_0}$, $\mathcal{C}^{(3)}_{x_0,y_0}$ described in Definition 5.4 have dimension 1.*
- (b) *Each irreducible component of each of the algebraic sets $\mathcal{C}^{(1)}_{y_0,z_0}$, $\mathcal{C}^{(2)}_{x_0,z_0}$, $\mathcal{C}^{(3)}_{x_0,y_0}$ described in Definition 5.4 has geometric genus at most 5.*

Proof. Since this lemma is purely geometric, we assume that we are working over an algebraically closed field. By symmetry, it suffices to fix $y_0,z_0\in\mathbb{P}^1$ and to consider the algebraic set $\mathcal{C}^{(1)}_{y_0,z_0}$.
(a) We need to rule out the possibility that $\mathcal{C}^{(1)}_{y_0,z_0}$ is the empty set or has dimension 0 or 2 or 3. The algebraic set $\mathcal{C}^{(1)}_{y_0,z_0}$ is equal to the intersection of the following two algebraic sets:

$$V_1 := \bigl\{(x,y,z)\in(\mathbb{P}^1)^3 : F(x,y_0,z)=0\bigr\},$$
$$V_2 := \bigl\{(x,y,z)\in(\mathbb{P}^1)^3 : F(x,y,z_0)=0\bigr\}.$$

We first note that if $V_1=(\mathbb{P}^1)^3$, then $F(x,y_0,z)$ is identically 0, so for any value of x_0, the fiber $\pi_{12}^{-1}(x_0,y_0)$ is a copy of $\mathbb{P}^1$. This contradicts the assumed non-degeneracy of $\mathcal{W}$. Hence $V_1\ne(\mathbb{P}^1)^3$, and similarly $V_2\ne(\mathbb{P}^1)^3$. Thus

$$\dim(V_1)=2 \quad\text{and}\quad \dim(V_2)=2,$$

and hence

$$\dim(\mathcal{C}^{(1)}_{y_0,z_0}) = \dim(V_1\cap V_2)\le 2.$$

Suppose that $\mathcal{C}^{(1)}_{y_0,z_0}$ has dimension 2. This means that the algebraic sets V_1 and V_2 have a 2-dimensional component in common. Writing

$$(\mathbb{P}^1)^3 = \mathbb{P}^1_x \times \mathbb{P}^1_y \times \mathbb{P}^1_z$$

so that we can keep track of the three factors, we see from their definition that V_1 and V_2 are products,

$$V_1 = \{\text{curve in } \mathbb{P}^1_x \times \mathbb{P}^1_z\} \times \mathbb{P}^1_y \quad \text{and} \quad V_2 = \{\text{curve in } \mathbb{P}^1_x \times \mathbb{P}^1_y\} \times \mathbb{P}^1_z.$$

The assumption that V_1 and V_2 have a 2-dimensional component in common implies that they have one or more common components of the form $\{x_0\} \times \mathbb{P}^1_y \times \mathbb{P}^1_z$. There is thus a $(1,0,0)$-form $A(x)$ vanishing at x_0 and a $(1,0,2)$-form $B(x,z)$ and a $(1,2,0)$-form $C(x,y)$ so that the polynomial F defining $\mathcal{W}$ factors as both

$$F(x,y_0,z) = A(x)B(x,z) \quad \text{and} \quad F(x,y,z_0) = A(x)C(x,y).$$

But then $\mathcal{W}$ is degenerate, since it contains the lines $\{x_0\} \times \{y_0\} \times \mathbb{P}^1_z$ and $\{x_0\} \times \mathbb{P}^1_y \times \{z_0\}$, which means that the projection maps π_{12} and π_{13} have positive-dimensional fibers.

We now know that $\dim(\mathcal{C}^{(1)}_{y_0,z_0}) \le 1$. This implies in particular that V_1 and V_2 intersect properly (or not at all). We let

$$H_1 = \{\text{pt}\} \times \mathbb{P}^1 \times \mathbb{P}^1, \quad H_2 = \mathbb{P}^1 \times \{\text{pt}\} \times \mathbb{P}^1, \quad H_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \{\text{pt}\},$$

be generic hypersurfaces (divisors on $(\mathbb{P}^1)^3$) of the indicated form. Then usually V_1 and V_2 will be linearly equivalent to, respectively, $2H_1 + 2H_3$ and $2H_1 + 2H_2$, but there are potentially cases where $F(x,y_0,z)$ and/or $F(x,y,z_0)$ depends on only one of the variables x,y,z. In any case, we have

$$V_1 \sim aH_1 + bH_3 \quad \text{and} \quad V_2 \sim cH_1 + dH_2$$
$$\text{with } a,b,c,d \in \{0,2\},\ (a,b) \ne (0,0),\ (c,d) \ne (0,0).$$

Using $H_1 \cap H_2 \cap H_3 = 1$ and $H_i \cap H_j \cap H_k = 0$ if i,j,k are not distinct, we compute intersections

$$V_1 \cdot V_2 \cdot H_1 = bd, \quad V_1 \cdot V_2 \cdot H_2 = bc, \quad V_1 \cdot V_2 \cdot H_3 = ad. \tag{11}$$

Suppose now that the algebraic set

$$\mathcal{C}^{(1)}_{y_0,z_0} = V_1 \cap V_2$$

is a finite set of points or the empty set. This implies that the intersections in (11) all vanish, since they are intersections of points (or the empty set) with hypersurfaces. We consider three cases:

$$b \ne 0 \quad \Longrightarrow \quad c = d = 0 \quad \text{contradiction.}$$
$$d \ne 0 \quad \Longrightarrow \quad a = b = 0 \quad \text{contradiction.}$$

$$b = d = 0 \quad \Longrightarrow \quad a = c = 2 \quad \Longrightarrow \quad V_1 \sim V_2 \sim 2H_1.$$

This last case implies that $F(x, y_0, z)$ does not depend on z and that $F(x, y, z_0)$ does not depend on y. But then $\mathcal{W}$ is degenerate, since the fiber of π_{12} over any point of the form (x, y_0) has dimension 1, and similarly the fiber of π_{13} over any point of the form (x, z_0) has dimension 1. This concludes the proof that the algebraic set $\mathcal{C}^{(1)}_{y_0,z_0}$ has dimension 1.

(b) We let F be the $(2, 2, 2)$-form that defines the non-degenerate TIK3 surface $\mathcal{W}$. We define a projection map

$$\pi : \mathcal{C}^{(1)}_{y_0,z_0} \longrightarrow \mathbb{P}^1, \quad \pi(x, y, z) = x.$$

This map has degree 4. Keeping in mind that y_0 and z_0 are fixed, for $x_1 \in \mathbb{P}^1$ we have

$$\pi^{-1}(x_1) = \bigl\{(x_1, y, z) \in (\mathbb{P}^1)^3 : F(x_1, y_0, z) = F(x_1, y, z_0) = 0\bigr\}.$$

The equations for y and z are independent, so we find that

$$\#\pi^{-1}(x_1) = \#\bigl\{z \in \mathbb{P}^1 : F(x_1, y_0, z) = 0\bigr\} \cdot \#\bigl\{y \in \mathbb{P}^1 : F(x_1, y, z_0) = 0\bigr\}.$$

The non-degeneracy assumption tells us that $F(x_1, y_0, z)$ and $F(x_1, y, z_0)$ are not identically 0, so they are non-trivial quadratic forms in, respectively, z and y. As such, they have either 1 or 2 roots, and we can determine which is the case by computing an appropriate discriminant:

$$\#\bigl\{z \in \mathbb{P}^1 : F(x_1, y_0, z) = 0\bigr\} = \begin{cases} 1 & \text{if } \operatorname{Disc}_z F(x_1, y_0, z) = 0, \\ 2 & \text{if } \operatorname{Disc}_z F(x_1, y_0, z) \neq 0. \end{cases}$$

$$\#\bigl\{y \in \mathbb{P}^1 : F(x_1, y, z_0) = 0\bigr\} = \begin{cases} 1 & \text{if } \operatorname{Disc}_y F(x_1, y, z_0) = 0, \\ 2 & \text{if } \operatorname{Disc}_y F(x_1, y, z_0) \neq 0. \end{cases}$$

Combining these estimates yields the following formulas

$\#\pi^{-1}(x_1)$	$\operatorname{Disc}_y F(x_1, y, z_0)$	$\operatorname{Disc}_z F(x_1, y_0, z)$
4	$\neq 0$	$\neq 0$
2	$= 0$	$\neq 0$
2	$\neq 0$	$= 0$
1	$= 0$	$= 0$

We next observe that $\operatorname{Disc}_y F(x, y, z_0)$ is a degree 4 form in x, and thus has at most 4 roots in $\mathbb{P}^1$ when considered as a polynomial in x; and similarly for $\operatorname{Disc}_z F(x, y_0, z)$. So there are at most 8 points $x_1 \in \mathbb{P}^1$ with $\#\pi^{-1}(x_1) = 2$. Further, each time we get an x_1 with $\#\pi^{-1}(x_1) =$

1, we see that 2 of those 8 potential values of x_1 coalesce into 1 value. So if we let

$$\begin{aligned} A &= \#\bigl\{x_1 \in \mathbb{P}^1 : \pi^{-1}(x_1) = 2\bigr\}, \\ B &= \#\bigl\{x_1 \in \mathbb{P}^1 : \pi^{-1}(x_1) = 1\bigr\}, \end{aligned} \tag{12}$$

then we see that

B	0	1	2	3	4
A	≤ 8	≤ 6	≤ 4	≤ 2	$= 0$

(13)

We assume for the moment that $\mathcal{C}^{(1)}_{y_0,z_0}$ is irreducible,[11] and we let

$$\lambda : \widetilde{\mathcal{C}^{(1)}_{y_0,z_0}} \longrightarrow \mathcal{C}^{(1)}_{y_0,z_0}$$

be a desingularization of $\mathcal{C}^{(1)}_{y_0,z_0}$, so the geometric genus of $\mathcal{C}^{(1)}_{y_0,z_0}$ is simply the genus of $\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}$. We use the Riemann–Hurwitz genus formula

$$2\operatorname{genus}\bigl(\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\bigr) - 2 = -2\deg(\pi\circ\lambda) + \sum_{x_1\in\mathbb{P}^1}\Bigl(\deg(\pi\circ\lambda) - \#(\pi\circ\lambda)^{-1}(x_1)\Bigr).$$

(Our assumption that the characteristic is not 2 or 3 ensures that the degree 4 map $\pi\circ\lambda$ is tamely ramified.) Substituting

$$\deg(\pi\circ\lambda) = \deg(\pi)\cdot\deg(\lambda) = 4\cdot 1 = 4,$$

we get

$$\begin{aligned} \operatorname{genus}\bigl(\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\bigr) &= -3 + \frac{1}{2}\sum_{\substack{x_1\in\mathbb{P}^1 \\ \#(\pi\circ\lambda)^{-1}(x_1)<4}}\Bigl(4 - \#(\pi\circ\lambda)^{-1}(x_1)\Bigr) \\ &\le -3 + \frac{1}{2}\sum_{\substack{x_1\in\mathbb{P}^1 \\ \#\pi^{-1}(x_1)<4}}\Bigl(4 - \#\pi^{-1}(x_1)\Bigr) \\ &= -3 + \#\bigl\{x_1 \in \mathbb{P}^1 : \#\pi^{-1}(x_1) = 2\bigr\} \\ &\qquad + \frac{3}{2}\#\bigl\{x_1 \in \mathbb{P}^1 : \#\pi^{-1}(x_1) = 1\bigr\} \\ &= -3 + A + \frac{3}{2}B \quad \text{using the notation in (12),} \\ &\le 5 \quad \text{from (13), since the max is at } (A,B) = (8,0). \end{aligned}$$

Finally, we note that if $\mathcal{C}^{(1)}_{y_0,z_0}$ is reducible, then the above argument works mutatis mutandis if we replace $\mathcal{C}^{(1)}_{y_0,z_0}$ with any of its irreducible components and note that now the map π has degree 1 or 2. This completes the proof of Lemma 5.6. □

[11]See Remark 5.7 for examples where $\mathcal{C}^{(1)}_{y_0,z_0}$ is reducible.

Remark 5.7. Lemma 5.6(a) says that that the algebraic sets described in Definition 5.4 have dimension 1, but we note that they need not be irreducible. For example, let $\mathcal{W}$ be a TIK3 surface whose equation F is symmetric in y and z, i.e., $F(x,y,z) = F(x,z,y)$. Then for any $\xi \in K$ there is a factorization

$$F(x,\xi,z) - F(x,y,\xi) = F(x,z,\xi) - F(x,y,\xi) = (z-y)L(x,y,z),$$

where $L(x,y,z)$ has degree 1 in y and z. It follows that the algebraic set $\mathcal{C}_{\xi,\xi}^{(1)}$ is reducible, and indeed it is the union of two genus 1 curves, each of which is isomorphic to the fibral curve

$$\mathcal{W}_{\xi}^{(3)} \cong \bigl\{(x,y) \in \mathbb{A}^2 : F(x,y,\xi) = 0\bigr\}$$

6. Tri-Involutive Markoff-Type K3 (MK3) Surfaces

The Markoff equation (1) and many of its variants admit not only the involutions coming from the projections $\mathcal{M} \to \mathbb{A}^2$, they also admit sign-change involutions and coordinate permutations coming from the symmetry of the Markoff equation. We give a name to the TIK3 surfaces that have these extra automorphisms.

Definition 6.1. We let $\mathfrak{S}_3$, the symmetric group on 3 letters, act on $(\mathbb{P}^1)^3$ by permuting the coordinates, and we let the group

$$(\boldsymbol{\mu}_2^3)_1 := \bigl\{(\alpha,\beta,\gamma) : \alpha,\beta,\gamma \in \boldsymbol{\mu}_2 \text{ and } \alpha\beta\gamma = 1\bigr\} \tag{14}$$

act on $(\mathbb{P}^1)^3$ via sign changes,

$$\epsilon_{\alpha,\beta,\gamma}(x,y,z) = (\alpha x, \beta y, \gamma z). \tag{15}$$

In this way we obtain an embedding[12]

$$\mathcal{G}^\circ := (\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3 \hookrightarrow \operatorname{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1).$$

Definition 6.2. A *Markoff-type K3* (MK3) *surface* $\mathcal{W}$ is a TIK3 surface whose $(2,2,2)$-form (6) is invariant under the action of $\mathcal{G}^\circ$, i.e., the $(2,2,2)$-form F describing $\mathcal{W}$ satisfies

$$F(x,y,z) = F(-x,-y,z) = F(-x,y,-z) = F(x,-y,-z),$$
$$F(x,y,z) = F(z,x,y) = F(y,z,x) = F(x,z,y) = F(y,x,z) = F(z,y,x).$$

Definition 6.3. Let $\mathcal{W}$ be an MK3 surface. We define a set of generators

$$\operatorname{Gen}(\mathcal{G}) = \{\sigma_1,\sigma_2,\sigma_3\} \cup (\boldsymbol{\mu}_2^3)_1 \cup \mathfrak{S}_3, \tag{16}$$

[12]We remark that $(\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3$ is isomorphic to $\mathfrak{S}_4$, but for our applications the group $\mathcal{G}^\circ$ appears more naturally as the semi-direct product.

and we let

$$\mathcal{G}^\sigma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset \mathrm{Aut}(\mathcal{W}),$$
$$\mathcal{G} = \big\langle \varphi : \varphi \in \mathrm{Gen}(\mathcal{G}) \big\rangle = \big\langle \mathcal{G}^\sigma, \mathcal{G}^\circ \big\rangle \subseteq \mathrm{Aut}(\mathcal{W}).$$

We remind the reader that the choice of $\mathrm{Gen}(\mathcal{G})$ affects the description of fibral automorphism groups and of $\mathcal{G}$-connected fibers; see Definition 3.5.

We suspect that the full automorphism group of a generic MK3-surface is $\mathcal{G}$; but as we shall see in Remark 8.7, some MK3-surfaces admit additional automorphisms. We start by describing some elementary properties of the group $\mathcal{G}$.

Proposition 6.4. *Let $\mathcal{W}$ be an MK3-surface, and let $\mathcal{G}^\circ$, $\mathcal{G}^\sigma$, and $\mathcal{G}$ be the subgroups of* $\mathrm{Aut}(\mathcal{W})$ *described in Definitions 6.1 and 6.3.*

(a) *$\mathcal{G}^\sigma$ is a normal subgroup of $\mathcal{G}$.*
(b) *$\mathcal{G} = \mathcal{G}^\circ \mathcal{G}^\sigma$.*

Proof. (a) Since $\mathcal{G}$ is defined to be the group generated by $\mathcal{G}^\circ$ and $\mathcal{G}^\sigma$, it suffices to show that $\mathcal{G}^\circ$ is contained in the normalizer of $\mathcal{G}^\sigma$. We let $\{i,j,k\} = \{1,2,3\}$, and for the purposes of this proof, we define transpositions and sign changes

$$\tau_{ij} = \text{swap the } i \text{ and } j \text{ coordinates},$$
$$\epsilon_{ij} = \text{multiply the } i \text{ and } j \text{ coordinates by } -1.$$

Since $\mathfrak{S}_3$ is generated by transpositions and $(\boldsymbol{\mu}_2^3)_1$ is generated by the sign changes, it suffices to check that $\mathcal{G}^\sigma$ is normalized by the τ_{ij} and the ϵ_{ij}. This can be checked by an explicit computation, or alternatively we can use the defining property $\pi_{ij} \circ \sigma_k = \pi_{ij}$ of σ_k, where π_{ij} is the projection map; see Definition 3.4. Thus momentarily letting $\tau : (\mathbb{P}^1)^2 \to (\mathbb{P}^1)^2$ be the map that swaps the coordinates and $\epsilon_i : (\mathbb{P}^1)^2 \to (\mathbb{P}^1)^2$ be the map that changes the sign of the ith coordinate, we compute

$$\pi_{ij} \circ (\tau_{ij}^{-1} \circ \sigma_k \circ \tau_{ij}) = \tau \circ \pi_{ij} \circ \sigma_k \circ \tau_{ij} = \tau \circ \pi_{ij} \circ \tau_{ij} = \pi_{ij},$$
$$\pi_{jk} \circ (\tau_{ik}^{-1} \circ \sigma_k \circ \tau_{ik}) = \tau \circ \pi_{ij} \circ \sigma_k \circ \tau_{ik} = \tau \circ \pi_{ij} \circ \tau_{ik} = \pi_{jk}$$
$$\pi_{ij} \circ (\epsilon_{ij}^{-1} \circ \sigma_k \circ \epsilon_{ij}) = \epsilon_{ij} \circ \pi_{ij} \circ \sigma_k \circ \epsilon_{ij} = \epsilon_{ij} \circ \pi_{ij} \circ \epsilon_{ij} = \epsilon_{ij}^2 \circ \pi_{ij} = \pi_{ij},$$
$$\pi_{ij} \circ (\epsilon_{ik}^{-1} \circ \sigma_k \circ \epsilon_{ik}) = \epsilon_i \circ \pi_{ij} \circ \sigma_k \circ \epsilon_{ik} = \epsilon_i \circ \pi_{ij} \circ \epsilon_{ik} = \epsilon_i^2 \circ \pi_{ij} = \pi_{ij}.$$

It follows from the definitions of the σ_i that

$$\tau_{ij}^{-1} \circ \sigma_k \circ \tau_{ij} = \sigma_k, \qquad \epsilon_{ij}^{-1} \circ \sigma_k \circ \epsilon_{ij} = \sigma_k,$$
$$\tau_{ik}^{-1} \circ \sigma_k \circ \tau_{ik} = \sigma_i, \qquad \epsilon_{ik}^{-1} \circ \sigma_k \circ \epsilon_{ik} = \sigma_k.$$

Hence $\mathcal{G}^\circ$ normalizes $\mathcal{G}^\sigma$, and indeed, $(\boldsymbol{\mu}_2^3)_1$ is in the centralizer of $\mathcal{G}^\sigma$.
(b) By definition the group $\mathcal{G}$ is generated by $\mathcal{G}^\circ$ and $\mathcal{G}^\sigma$, and from (a), we know that $\mathcal{G}^\sigma$ is a normal subgroup of $\mathcal{G}$. It follows that every element of $\mathcal{G}$ can be written as $\gamma\sigma$ with $\gamma \in \mathcal{G}^\circ$ and $\sigma \in \mathcal{G}^\sigma$. Hence $\mathcal{G} = \mathcal{G}^\circ\mathcal{G}^\sigma$. $\square$

Proposition 6.5. *Let $\mathcal{W}/K$ be a (possibly degenerate) MK3-surface.*

(a) *There exist $a, b, c, d, e \in K$ so that the $(2,2,2)$-form F that defines $\mathcal{W}$ has the form*

$$\begin{aligned} F_{a,b,c,d,e}(x,y,z) &= ax^2y^2z^2 + b(x^2y^2 + x^2z^2 + y^2z^2) \\ &\qquad + cxyz + d(x^2+y^2+z^2) + e = 0. \end{aligned} \tag{17}$$

(b) *Let F be as in (a). Then $\mathcal{W}$ is non-degenerate, i.e., the projections $\pi_{ij} : \mathcal{W} \to (\mathbb{P}^1)^2$ are quasi-finite, if and only if*

$$c \ne 0 \quad \text{and} \quad be \ne d^2 \quad \text{and} \quad ad \ne b^2.$$

(c) *For generic values of $(a,b,c,d,e) \in \mathbb{A}^5$, a minimal regular model for the MK3-surface defined by (17) is a K3 surface.*

Remark 6.6. We can recover the classical (translated) Markoff equation for the surface $\mathcal{M}_{a,k}$ in Definition 1 as a special case of an $F_{a,b,c,d,e}$. Thus $\mathcal{M}_{a,k}$ is given by the affine equation

$$F_{0,0,-a,1,-k}(x,y,z) = x^2 + y^2 + z^2 - axyz - k = 0.$$

We note, however, that the Markoff equation is degenerate, despite the involutions being well-defined on the affine Markoff surface $\mathcal{M}_{a,k}$. This occurs because the involutions are not well-defined at some of the points at infinity in the closure of $\mathcal{M}_{a,k}$ in $(\mathbb{P}^1)^3$. Further, the Markoff surface is a rational surface, not a K3 surface.

Proof of 6.5. (a) The space of $\mathfrak{S}_3$-invariant quadratic polynomials in $\mathbb{Z}[x,y,z]$ is spanned by the following 10 polynomals:

$$\begin{array}{llll} (1) & x^2y^2z^2 & (2) & xyz^2 + xy^2z + x^2yz \\ (3) & xyz & (4) & x^2y^2z + x^2yz^2 + xy^2z^2 \\ (5) & x^2+y^2+z^2 & (6) & x^2y^2 + x^2z^2 + y^2z^2 \\ (7) & \multicolumn{3}{l}{x^2y + x^2z + xy^2 + xz^2 + yz^2 + y^2z} \\ (8) & xy + xz + yz & (9) & x + y + z \qquad (10) \quad 1 \end{array}$$

Of these, the polynomials that are also invariant for the double-sign changes in $(\boldsymbol{\mu}_2^3)_1$ are (1), (3), (5), (6), and (10). Hence all $\bigl((\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3\bigr)$-invariant $(2,2,2)$-polynomials have the form indicated in (a).

(b) By symmetry, it suffices to consider π_{12} and σ_3. The map π_{12} is quasi-finite if and only if the fibers of the map π_{12} are 0-dimensional. Let $\overline{F}$ be the homogenization of the polynomial in (a). Then π_{12} is quasi-finite over the point

$$\bigl([\alpha,\beta],[\gamma,\delta]\bigr) \in \mathbb{P}^1 \times \mathbb{P}^1$$

if and only if the polynomial $F(\alpha,\beta;\gamma,\delta;X_3,Y_3)$ is not identically 0.

Suppose first that $c \neq 0$. Since

$$\Bigl(\text{the } X_3Y_3 \text{ term of } F(\alpha,\beta;\gamma,\delta;X_3,Y_3)\Bigr) = c\alpha\beta\gamma\delta X_3Y_3,$$

we see that π_{12} is quasi-finite unless $\alpha\beta\gamma\delta = 0$. By the symmetry of F, it suffices to consider the cases that $\alpha = 0$ and $\beta = 0$.

If $\alpha = 0$, then

$$F(0,1;\gamma,\delta;X_3,Y_3) = (b\gamma^2 + d\delta^2)X_3^2 + (d\gamma^2 + e\delta^2)Y_3^2.$$

Hence π_{12} is quasi-finite at $\bigl([0,1],[\gamma,\delta],[\alpha_3,\gamma_3]\bigr)$ unless

$$b\gamma^2 + d\delta^2 = d\gamma^2 + e\delta^2 = 0.$$

Since $(\gamma,\delta) \neq (0,0)$, this is possible if and only if $be = d^2$.

Similarly, if $\beta = 0$, we look at

$$F(1,0;\gamma,\delta;X_3,Y_3) = (a\gamma^2 + b\delta^2)X_3^2 + (b\gamma^2 + d\delta^2)Y_3^2.$$

Thus σ_3 is well-defined at $\bigl([1,0],[\gamma,\delta],[\alpha_3,\gamma_3]\bigr)$ unless

$$a\gamma^2 + b\delta^2 = b\gamma^2 + d\delta^2 = 0.$$

Since $(\gamma,\delta) \neq (0,0)$, this is possible if and only if $ad = b^2$.

We next consider the case that $c = 0$. Then

$$\begin{aligned} F(\alpha,\beta;\gamma,\delta;X_3,Y_3) = {} & (a\alpha^2\gamma^2 + b\alpha^2\delta^2 + b\beta^2\gamma^2 + d\beta^2\delta^2)X_3^2 \\ & + (b\alpha^2\gamma^2 + d\alpha^2\delta^2 + d\beta^2\gamma^2 + e\beta^2\delta^2)Y_2^2. \end{aligned} \tag{18}$$

We claim that there is always a point $\bigl([\alpha,\beta],[\gamma,\delta]\bigr) \in (\mathbb{P}^1)^2$ such that (18) is identically 0. This follows from the fact that the $(1,1)$-forms

$$aU_1V_1 + bU_1V_2 + bV_1U_2 + dU_2V_2 = 0, \tag{19}$$

$$bU_1V_1 + dU_1V_2 + dV_1U_2 + eU_2V_2 = 0, \tag{20}$$

define a non-empty subvariety of $(\mathbb{P}^1)^2$, since taking $\bigl([u_1,v_1],[u_2,v_2]\bigr)$ to be a solution to (19) and (20), we see that $\bigl([u_1^{1/2},v_1^{1/2}],[u_2^{1/2},v_2^{1/2}]\bigr)$ is a point at which (18) is identically 0. (If one or both of (19) and (20) is identically 0, that makes it even easier to find a point on the subvariety that they define.)

This completes the proof that π_{12} is quasi-finite if and only if $c \neq 0$ and $be \neq d^2$ and $ad \neq b^2$.
(c) See Proposition B.3(a) for the proof of this assertion. $\square$

7. Connected Fibral Components and the Cage for MK3 Surfaces

For this section we let $\mathcal{W}$ be an MK3-surface, as described in Definition 6.2, defined over a finite field $\mathbb{F}_q$. We note that the $\mathfrak{S}_3$-symmetry of $\mathcal{W}$ implies that for any $t \in \mathbb{P}^1(\mathbb{F}_q)$, the three fibers $\mathcal{W}_t^{(1)}(\mathbb{F}_q)$, $\mathcal{W}_t^{(2)}(\mathbb{F}_q)$ and $\mathcal{W}_t^{(3)}(\mathbb{F}_q)$ have the same orbit structure, so in particular[13]

$$\begin{aligned}\mathcal{W}_t^{(i)}(\mathbb{F}_q) \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) &\text{ for some } i \in \{1,2,3\}\\ \iff \quad \mathcal{W}_t^{(i)}(\mathbb{F}_q) \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) &\text{ for all } i \in \{1,2,3\}.\end{aligned}$$

Thus the $\mathcal{G}$-connected fibers in $\mathcal{W}(\mathbb{F}_q)$ are determined by the projection to $\mathbb{P}^1(\mathbb{F}_q)$ of $\mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ onto any of its coordinates. We denote this set by

$$\pi\,\mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) = \Bigl\{t \in \mathbb{P}^1(\mathbb{F}_q) : \mathcal{W}_t^{(i)}(\mathbb{F}_q) \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)\Bigr\}.$$

Then we have the useful characterization (for MK3-surfaces):

$$P \in \mathsf{Cage}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) \iff \text{some coordinate of } P \text{ is in } \pi\,\mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr).$$

8. A One Parameter Family of MK3 Surfaces

In the next few sections we study an interesting 1-parameter family of MK3-surfaces. We assume throughout that K is a field with $\operatorname{char}(K) \neq 2$.

Definition 8.1. For $k \in K^*$ we define $\mathcal{W}_k$ to be the MK3-surface

$$\mathcal{W}_k : x^2 + y^2 + z^2 + x^2y^2z^2 + kxyz = 0.$$

Remark 8.2. We note that a minimal regular model for $\mathcal{W}_k$ is a K3 surface; see Proposition B.3(b). Further, in the notation of Proposition 6.5, the $(2,2,2)$-form defining $\mathcal{W}_k$ has $(a,b,c,d,e) = (1,0,k,1,0)$. Hence

$$c = k \neq 0 \quad \text{and} \quad be = 0 \neq 1^2 = d^2 \quad \text{and} \quad ad = 1 \neq 0^2 = b^2,$$

so Proposition 6.5(b) tells us that $\mathcal{W}_k$ is non-degenerate.

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[13]We note that for MK3-surfaces, we take $\operatorname{Gen}(\mathcal{G})$ as described in (16), so Γ-connectivity of fibers on MK3-surfaces may employ coordinate permutations and sign changes, as well as the usual σ_i automorphisms.

Remark 8.3. Let $\zeta \in K$ be an element satisfying $\zeta^4 = 1$. Then there is a K-isomorphism

$$\mathcal{W}_k \longrightarrow \mathcal{W}_{\zeta^3 k}, \quad (x, y, z) \longmapsto (\zeta x, \zeta y, \zeta z). \tag{21}$$

So we always have an identification $\mathcal{W}_k(K) \cong \mathcal{W}_{-k}(K)$, and if K contains $i = \sqrt{-1}$, then there are further identifications $\mathcal{W}_k(K) \cong \mathcal{W}_{\pm ik}(K)$.

Remark 8.4. The three involutions (7) on $\mathcal{W}_k$ are given explicitly by

$$\begin{aligned}
\sigma_1(x,y,z) &= \left(-\frac{kyz}{1+y^2z^2} - x, y, z\right),\\
\sigma_2(x,y,z) &= \left(x, -\frac{kxz}{1+x^2z^2} - y, z\right),\\
\sigma_3(x,y,z) &= \left(x, y, -\frac{kxy}{1+x^2y^2} - z\right).
\end{aligned}$$

We recall from Section 6 that $\mathcal{G}^\circ$ is the group $(\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3$ of order 24 sitting in $\mathrm{Aut}(\mathcal{W}_k)$ composed of sign changes and coordinate permutations, that $\mathcal{G}^\sigma$ is the normal subgroup of $\mathrm{Aut}(\mathcal{W}_k)$ generated by $\sigma_1, \sigma_2, \sigma_3$, and that $\mathcal{G} = \mathcal{G}^\circ\mathcal{G}^\sigma$ is the subgroup of $\mathrm{Aut}(\mathcal{W}_k)$ generated by $\mathcal{G}^\circ$ and $\mathcal{G}^\sigma$.

Remark 8.5. Let $\mathcal{W}_{k,\xi}^{(i)}$ be a fiber of $\mathcal{W}_k$. Then each of the involutions $\sigma_1, \sigma_2, \sigma_3$ and each of the automorphisms in $\mathcal{G}^\circ$ defines an isomorphism from $\mathcal{W}_{k,\xi}^{(i)}$ to some other (or possibly the same) fiber of $\mathcal{W}_k$. It follows that the singular points on a fiber are mapped to singular points on a fiber. Hence the set

$$\bigcup_{i=1}^{3} \bigcup_{\xi \in \mathbb{P}^1} \mathrm{Sing}(\mathcal{W}_{k,\xi}^{(i)})$$

of fibral singular points is a finite subset of $\mathcal{W}_k$ that is $\mathcal{G}$-invariant, so it breaks up into a finite number of finite $\mathcal{G}$-orbits. If $\xi \neq 0, \infty$ and $\xi^4 \neq 1$, then it will be a $\mathcal{G}$-orbit of size 24; cf. Table 3.

Proposition 8.6. *Let $k \in K^*$. The set of singular points of $\mathcal{W}_k$ always contains the 4 points*

$$\big\{(0,0,0),\ (0,\infty,\infty),\ (\infty,0,\infty),\ (\infty,\infty,0)\big\}. \tag{22}$$

The point $(0,0,0)$ is fixed by $\mathcal{G}$, and the other 3 singular points form a $\mathcal{G}$-orbit.[14] *If $k \notin \{\pm 4, \pm 4i\}$, then the set (22) is the full set of singular points of $\mathcal{W}_k$.*

[14] If we also allow the δ-inversion involutions described in Remark 8.7, then the 4 singular points form a single orbit.

For $k=4$ the set of singular points is

$$\begin{aligned}\operatorname{Sing}(\mathcal{W}_4)=\bigl\{&(0,0,0),\,(0,\infty,\infty),\,(\infty,0,\infty),\,(\infty,\infty,0)\\&(1,1,-1),\,(1,-1,1),\,(-1,1,1),\,(-1,-1,-1)\bigr\};\end{aligned}\tag{23}$$

and for the other $k\in\{\pm4,\pm4i\}$, the singular points can be found using the isomorphisms described in Remark 8.3. The points in (23) with non-zero coordinates form a single $\mathcal{G}$-orbit of size 4.

Remark 8.7 (MK3-Surfaces with Extra Involutions)**.** The family of MK3-surfaces $\mathcal{W}_k$ admit additional involutions in which two of x,y,z are replaced by their multiplicative inverses.[15] Thus analogously to (14) and (15), we can define another action of $(\boldsymbol{\mu}_2^3)_1$ on $(\mathbb{P}^1)^3$ via the formula

$$\delta_{\alpha,\beta,\gamma}(x,y,z)=(x^\alpha,y^\beta,z^\gamma),\quad\text{where }(\alpha,\beta,\gamma)\in(\boldsymbol{\mu}_2^3)_1.\tag{24}$$

We observe that the δ and ϵ actions commute (since $(-1)^{-1}=-1$), so we obtain an embedding

$$\hat{\mathcal{G}}^\circ:=\underbrace{\bigl((\boldsymbol{\mu}_2^3)_1\times(\boldsymbol{\mu}_2^3)_1\bigr)\rtimes\mathfrak{S}_3}_{\text{We view this as a subgroup of }\operatorname{Aut}((\mathbb{P}^1)^3).}\hookrightarrow\operatorname{Aut}(\mathcal{W}_k),$$

where $\hat{\mathcal{G}}^\circ$ has order 96. Since the classical Markoff equation (5) and general MK3-surfaces (17) do not admit these extra automorphisms, we will not include them when constructing orbits in $\mathcal{W}_k$. So for example, the finite orbits and $\mathcal{G}^\circ$-generators in $\mathcal{W}_k(\mathbb{C})$ that we list in Table 3 are $\mathcal{G}$-orbits, as are the finite field orbits in $\mathcal{W}_k(\mathbb{F}_p)$ in Appendix C. There would be some collapsing of generators and merging of orbits if we also used the δ-automorphisms. However, the existence of these extra automorphisms can aid in studying the geometry of $\mathcal{W}_k$, as will be illustrated in the proof of Propositions 8.6 and 8.8.

More generally, Proposition 6.5 says that MK3-surfaces $\mathcal{W}_{a,b,c,d,e}$ are described by $(2,2,2)$-forms $F_{a,b,c,d,e}(x,y,z)$ that depend on 5 homogeneous parameters $[a,b,c,d,e]$. Then the formula

$$\begin{aligned}F_{a,b,c,d,e}(x,y,z)&-F_{a,b,c,d,e}(x^{-1},y^{-1},z)x^2y^2\\&=\Bigl((a-d)z^2+(b-e)\Bigr)(x^2y^2-1),\end{aligned}$$

combined with the x,y,z symmetry of $F_{a,b,c,d,e}$, imply that

$$\delta_{\alpha,\beta,\gamma}\in\operatorname{Aut}(\mathcal{W}_{a,b,c,d,e})\quad\Longleftrightarrow\quad a=d\text{ and }b=e.$$

Thus $\mathcal{W}_k=\mathcal{W}_{1,0,k,1,0}$ corresponds to $a=d=1$ and $b=e=0$.

[15] Note that we're really working in $\mathbb{P}^1$, so we formally set $0^{-1}=\infty$ and $\infty^{-1}=0$.

Proof of Proposition 8.6:

We let

$$F(x, y, z) = x^2 + y^2 + z^2 + x^2y^2z^2 + kxyz \tag{25}$$

be the polynomial defining $\mathcal{W}_k$, and we use subscripts to denote partial derivatives. The singular points on this affine piece of $\mathcal{W}_k$ are the solutions to

$$F = F_x = F_y = F_z = 0. \tag{26}$$

The ideal of $\mathbb{Q}[x, y, z, k]$ generated by the four polynomials in (26) contains the following polynomials:[16]

$$\begin{array}{|c|c|c|c|} \hline x^2 - y^2 & x(x^4 - 1) & x(2^4x^2 - k^2) & x(k^4 - 2^8) \\ \hline x^2 - z^2 & y(y^4 - 1) & y(2^4y^2 - k^2) & y(k^4 - 2^8) \\ \hline y^2 - z^2 & z(z^4 - 1) & z(2^4z^2 - k^2) & z(k^4 - 2^8) \\ \hline \end{array} \tag{27}$$

The point $(0, 0, 0)$ is always singular. Since (27) says that singular points satisfy $x^2 = y^2 = z^2$, any other singular point (x, y, z) necessarily has $xyz \neq 0$, and then (27) forces

$$k^4 = 2^8, \quad 2^4x^2 = 2^4y^2 = 2^4z^2 = k^2, \quad \text{and} \quad x^4 = y^4 = z^4 = 1.$$

From $k^4 = 2^8$, we see that $k \in \{\pm 4, \pm 4i\}$; and from $x^4 = y^4 = z^4 = 1$, we see that $x, y, z \in \{\pm 1, \pm i\}$. For each of these 4 possible values of k, it can be directly checked that the points satisfying $F = F_x = F_y = F_z$ are those given in the table in the statement of the proposition.

It remains to check the points on the complement in $(\mathbb{P}^1)^3$ of the affine piece. To do that, we use the fact that $(0, 0, 0)$ is the only singular point of the affine piece of $\mathcal{W}_k$ that has a coordinate mapped to ∞ under the $\delta_{\alpha,\beta,\gamma}$ inversion maps described in Remark 8.7. By symmetry, it suffices to check points P of the following forms, where y and z are non-zero:

P	Singular?	Why?
(∞, y, z)	No	$\delta_{-1,-1,1}(P) = (0, y^{-1}, z)$
(∞, ∞, z)	No	$\delta_{-1,-1,1}(P) = (0, 0, z)$
$(\infty, y, 0)$	No	$\delta_{-1,-1,1}(P) = (0, y^{-1}, 0)$
$(\infty, \infty, 0)$	Yes	$\delta_{-1,-1,1}(P) = (0, 0, 0)$
$(\infty, 0, 0)$	–	$\notin \mathcal{W}_k$
(∞, ∞, ∞)	–	$\notin \mathcal{W}_k$

□

[16] Indeed, this is true in the ring $\mathbb{Z}[2^{-1}, x, y, z, k]$.

Proposition 8.8. *Let K be a field with $\operatorname{char}(K) \neq 2$, let $k \in K^*$, and let $\xi \in \mathbb{P}^1(K)$. Then the fiber $\mathcal{W}^{(1)}_{k,\xi}$ is singular if and only if*

$$\xi = 0 \quad \text{or} \quad \xi = \infty \quad \text{or} \quad k = \pm 2(\xi \pm \xi^{-1}).$$

The singular points on the singular fibers are as follows:

$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{k,0}\bigr) = \bigl\{(0,0,0),\, (0,\infty,\infty)\bigr),$$

$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{k,\infty}\bigr) = \bigl\{(\infty,\infty,0),\, (\infty,0,\infty)\bigr),$$

and for all $\xi \notin \{0,\infty\}$ and for all $u \in \{\pm 1\}$ and all $v \in \{\pm 1, \pm i\}$,

$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{u(\xi+v\xi^{-1}),\xi}\bigr) = \bigl\{(\xi, v, -uv^3),\, (\xi, -v, uv^3)\bigr\}.$$

By symmetry, analogous statements are true for $\mathcal{W}^{(2)}_{k,\xi}$ and $\mathcal{W}^{(3)}_{k,\xi}$.

Proof. As in the proof of Proposition 8.6, we let F be the polynomial (25) defining $\mathcal{W}_k$, and we use subscripts to denote partial derivatives. The fiber $\mathcal{W}^{(1)}_{k,\xi}$ is singular if and only if the simultaneous equations

$$F(\xi,y,z) = F_y(\xi,y,z) = F_z(\xi,y,z) = 0 \tag{28}$$

have a solution. We compute

$$\begin{aligned}\operatorname{Res}_y\Bigl(\operatorname{Res}_z(F,F_z), \operatorname{Res}_z(F_y,F_z)\Bigr) &= 2^{12}\cdot k^8 \cdot x^{26}\cdot (2x^2 - kx - 2)^2 \\ &\quad \cdot (2x^2 - kx + 2)^2 \cdot (2x^2 + kx - 2)^2 \cdot (2x^2 + kx + 2)^2.\end{aligned}$$

We first consider the case that $\xi = 0$. Then (28) forces $y = z = 0$, so the only affine singular point is $(0,0,0)$. Using the inversion automorphism fixing the x-coordinate that is described in Remark 8.7, there is an additional singular point $(0,\infty,\infty)$, so we find that

$$\operatorname{Sing}(\mathcal{W}^{(1)}_{k,0}) = \bigl\{(0,0,0),\, (0,\infty,\infty)\bigr\}.$$

And similarly, using the inversion automorphisms in Remark 8.7 that replace the x-coordinate with x^{-1}, we see that

$$\operatorname{Sing}(\mathcal{W}^{(1)}_{k,\infty}) = \bigl\{(\infty,\infty,0),\, (\infty,0,\infty)\bigr).$$

We now assume that $\xi \neq 0, \infty$. Then our assumptions that $\operatorname{char}(K) \neq 2$ and $\mathcal{W}^{(1)}_{k,x_0}$ is singular imply that ξ is a root of one of the polynomials $2x^2 \pm kx \pm 2$. We will consider the case that

$$2\xi^2 + k\xi + 2 = 0,$$

and leave the similar computation for the other three cases to the reader. Thus we assume that

$$k = -2(\xi + \xi^{-1}) \quad \text{and} \quad \mathcal{W}^{(1)}_{k,\xi} \text{ is singular.}$$

Substituting the expression for k into (28), we find that (y_0, z_0) is a singular point on the fiber $\mathcal{W}^{(1)}_{k,\xi}$ if and only if (y_0, z_0) satisfy

$$\begin{aligned}(y^2z^2 - 2yz + 1)\xi^2 - 2yz + y^2 + z^2 &= 0,\\ (yz^2 - z)\xi^2 - z + y &= 0,\\ (y^2z - y)\xi^2 - y + z &= 0.\end{aligned}$$

Eliminating x or y or z from these three equations, we find that (y_0, z_0) satisfy

$$y^2 - 1 = z^2 - 1 = (y - z)(yz - 1) = 0,$$

and these equations have two solutions,

$$(y_0, z_0) = (1, 1) \quad \text{and} \quad (y_0, z_0) = (-1, -1).$$

Finally, we substitute $k = -2(\xi + \xi^{-1})$ and $(x, y, z) = (\xi, \pm 1, \pm 1)$ into (28) and verify that F, F_y, and F_z vanish. This proves that

$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{-2(\xi+\xi^{-1}),\xi}\bigr) = \bigl\{(\xi, 1, 1),\ (\xi, -1, -1)\bigr\} \quad \text{for all } \xi \neq 0, \infty,$$

which completes the proof of Proposition 8.8. □

Remark 8.9. For a general TIK3-surface, the three projection maps $\mathcal{W} \to \mathbb{P}^1$ give $\mathcal{W}$ three different structures as a surface fibered by genus 1 curves, and the corresponding Jacobian variety has a section of infinite order whose translation action on $\mathcal{W}$ is the σ_i associated to the projection. For MK3-surfaces, the $\mathfrak{S}_3$-symmetry implies that the three structures are the same. Using the explicit description of the singular points on $\mathcal{W}_k$ in Proposition 8.6 and the singular fibers of $\mathcal{W}_k$ in Proposition 8.8, one could compute a Néron model for $\mathcal{W}_k \to \mathbb{P}^1$ and compute the canonical height of the point on its Jacobian, but we will not do this computation in the present article.

Proposition 8.10. *Let $\mathcal{W}_k$ be the MK3-surface given in Definition 8.1, let F be the associated polynomial, let $y_0, z_0 \in \mathbb{P}^1$, and let $\mathcal{C}^{(1)}_{y_0,z_0}$ be the curve associated to F as given in Definition 5.4. If $\mathcal{C}^{(1)}_{y_0,z_0}$ is singular, then one of the following is true:*

$$y_0 \text{ or } z_0 = 0 \text{ or } \infty, \quad y_0^2 = z_0^2, \quad y_0^2z_0^2 = 1, \quad y_0 \text{ or } z_0 = \frac{\pm k \pm \sqrt{k^2 \pm 16}}{4}.$$

By symmetry, analogous statements are true for $\mathcal{C}^{(2)}_{x_0,z_0}$ and $\mathcal{C}^{(3)}_{x_0,y_0}$.

Corollary 8.11. *Let $k \in \mathbb{F}_q^*$. Then*

$$\#\left\{(x_0, y_0, z_0) \in \mathcal{W}_k(\mathbb{F}_q) : \begin{array}{c}\textit{one or more of } \mathcal{C}^{(1)}_{y_0,z_0},\\ \mathcal{C}^{(2)}_{x_0,z_0},\ \mathcal{C}^{(3)}_{x_0,y_0} \textit{ is singular}\end{array}\right\} \leq 144q.$$

Proof of Proposition 8.10. To ease notation, we let $b = y_0$ and $c = z_0$. An affine piece of the curve $\mathcal{C}_{b,c}^{(1)}$ is given by the equations

$$F(x,b,z) = F(x,y,c) = 0.$$

Hence a point $(x,y,z) \in \mathcal{C}_{b,c}^{(1)}$ is a singular point if and only if

$$\operatorname{rank}\begin{bmatrix} F_x(x,b,z) & 0 & F_z(x,b,z) \\ F_x(x,y,c) & F_y(x,y,c) & 0 \end{bmatrix} \le 1.$$

The rank condition and a bit of algebra yields three cases, which we consider in turn.

Case 1: $\boldsymbol{F_z(x,b,z) = F_y(x,y,c) = 0}$. In this case we are looking for values of b, c, k such that the equations

$$F(x,b,z) = F(x,y,c) = F_z(x,b,z) = F_y(x,y,c) = 0$$

have a solution $(x,y,z) \in \mathbb{A}^3$. Eliminating x, y, z from these four equations gives the equation

$$(b^2 - c^2)(b^2c^2 - 1) = 0.$$

Hence if there is a singular point, then $c = \pm b^{\pm 1}$.

Case 2: $\boldsymbol{F_x(x,b,z) = F_z(x,b,z) = 0}$. In this case, which is a version of Proposition 8.8, we are looking for values of b, c, k such that the equations

$$F(x,b,z) = F(x,y,c) = F_x(x,b,z) = F_z(x,b,z) = 0$$

have a solution $(x,y,z) \in \mathbb{A}^3$. Eliminating x, y, z from these four equations gives the equation

$$b^2(2b^2 - bk - 2)(2b^2 - bk + 2)(2b^2 + bk - 2)(2b^2 + bk + 2) = 0.$$

Hence if there is a singular point, then

$$b = 0 \quad \text{or} \quad b = \frac{\pm k \pm \sqrt{k^2 \pm 16}}{4}.$$

Case 3: $\boldsymbol{F_x(x,y,c) = F_y(x,y,c) = 0}$. By symmetry, this is the same as Case 2 with $y \leftrightarrow z$ and $b \leftrightarrow c$. □

Proof of Corollary 8.11. It suffices to bound the number of $(y_0, z_0) \in \mathbb{P}^1(\mathbb{F}_q)$ such that $\mathcal{C}_{y_0,z_0}^{(1)}$ is singular, and then multiply by 3 for the xyz-symmetry and also multiply by 2 because each (y_0, z_0) may yield 2 points on $\mathcal{W}_k$. (This includes some duplicates, so some improvement is possible.)

According to Proposition 8.10, the singular cases are included in the following table, where again we do not worry that some points appear more than once:

(y_0, z_0)	# with $\mathcal{C}^{(1)}_{y_0,z_0}$ singular
y_0 or $z_0 = 0$ or ∞	$\le 4q$
$y_0^2 = z_0^2 \ne 0$ or ∞	$\le 2(q-1)$
$y_0^2 z_0^2 = 1$	$\le 2(q-1)$
y_0 or $z_0 = \frac{\pm k \pm \sqrt{k^2 \pm 16}}{4}$	$\le 16q$

Hence there are at most $24q$ pairs (y_0, z_0), and as noted earlier, this must be multiplied by 6 to account for the other cases. □

9. Finite Orbits in $\mathcal{W}_k(\mathbb{C})$

Table 3 describes finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$. We do not claim that this is the complete list of possibilities. However, we note that the varied nature of the finite orbits in the 1-parameter family $\mathcal{W}_k$ suggests that any description of finite orbits over $\mathbb{C}$ on general TIK3-surfaces, or even on MK3-surfaces, is likely to be quite complicated.

Most of the orbits in Table 3 were unearthed by examining small orbits in $\mathcal{W}_k(\mathbb{F}_p)$ that appear in Appendix C and looking at specific properties of the points in the orbits. We explain the process for a number of examples.

Question 9.1 (Uniform Boundedness Question)**.** For each $k \in \mathbb{C}$, we know from [13] that there are only finitely many finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$. Is there a bound that is independent of k for the largest such orbit? More generally, is there such a bound for finite orbits in $\mathcal{W}(\mathbb{C})$ as $\mathcal{W}$ runs over all MK3-surfaces? And even more generally, how about for all TIK3-surfaces, although in this case we look at orbits for the group generated by the three involutions $\sigma_1, \sigma_2, \sigma_3$?

Remark 9.2. We mention that if we consider $\langle\sigma_1, \sigma_2, \sigma_3\rangle$-orbits, then the orbit of size 144 in Remark 9.6 consist of 12 orbits of size 12, the orbit of size 160 in Remark 9.7 consist of 4 orbits of size 40, and the orbit of size 288 described in Remark 9.8 consist of 12 orbits of size 24. These provide lower bounds for the putative uniform bounds discussed in Questions 1.3 and 9.1.

Definition 9.3 (Trivial Orbits)**.** As noted in Proposition 8.6, the four singular points in $\mathcal{W}_k$ form two $\mathcal{G}$-orbits, namely the fixed point

$$\{(0,0,0)\}$$

and the orbit of size 3,

$$\bigl\{(0,\infty,\infty),\ (\infty,0,\infty),\ (\infty,\infty,0)\bigr\}.$$

We will call these orbits the *trivial orbits* in $\mathcal{W}_k$, and as such, we have not included them in the table in Appendix C.

Remark 9.4 (One-dimensional families of finite orbits in $\mathcal{W}_k(\mathbb{C})$)**.** Table 3 contains several examples of one-dimensional families of finite orbits in $\mathcal{W}_k(\mathbb{C})$, and indeed, these families are defined over $\mathbb{Q}$ or $\mathbb{Q}(i)$. Ignoring the trivial orbits described in Definition 9.3, we have the following examples:

Size 24: There is a $k \in \mathbb{Q}(t)$ such that $\mathcal{W}_k\bigl(\mathbb{Q}(t)\bigr)$ has a $\mathcal{G}$-orbit of size 24.
Size 48: The set $\mathcal{W}_k\bigl(\mathbb{Q}(i)\bigr)$ has a $\mathcal{G}$-orbit of size 48
Size 192: There is a $k \in \mathbb{Q}(t)$ such that $\mathcal{W}_k\bigl(\mathbb{Q}(t)\bigr)$ has a $\mathcal{G}$-orbit of size 192.
Size 288: There is a curve $C/\mathbb{Q}$ of genus 9 and an element $k \in \mathbb{Q}(C)$ in the function field of C so that $\mathcal{W}_k\bigl(\mathbb{Q}(C)\bigr)$ has a $\mathcal{G}$-orbit of size 288.

Remark 9.5 (Orbits of Size 64)**.** We describe the derivation of the orbit of size 64 in Table 3. Experimentally in Appendix C we see orbits of size 64 in $\mathcal{W}_k(\mathbb{F}_p)$ for various values of p and k, but the relation between p and k is not clear. Examining the actual orbits in several of these cases, we found that there was a single point in $\mathcal{W}_k(\mathbb{F}_p)$ of the form (β,β,β), and that the point $(\beta,\beta,1)$ also appeared in $\mathcal{W}_k(\mathbb{F}_p)$. We next computed

$$\begin{aligned}(\beta,\beta,\beta)\in\mathcal{W}_k \quad &\Longleftrightarrow \quad \beta^6+k\beta^3+3\beta^2=0,\\ (\beta,\beta,1)\in\mathcal{W}_k \quad &\Longleftrightarrow \quad \beta^4+(k+2)\beta^2+1=0.\end{aligned}$$

Eliminating k and the trivial solutions $\beta\in\{0,1\}$ gives the equation[17]

$$\beta^3+\beta^2+\beta-1=0.$$

This gives $k=-(\beta+\beta^{-1})^2$. It is then an exercise to compute the $\mathcal{G}$-orbit of (β,β,β). It turns out to be the union of the $\mathcal{G}^\circ$ orbits of the following five points:

Point	(β,β,β)	$(\beta,\frac{1}{\beta},\frac{1}{\beta})$	$(\beta,\beta,1)$	$(\frac{1}{\beta},\frac{1}{\beta},1)$	$(\beta,\frac{1}{\beta},1)$
Size of $\mathcal{G}^\circ$-orbit	4	12	12	12	24

[17]We note that $\beta=0$ gives the contradiction $1=0$, while $\beta=1$ yields $k=-4$ and an orbit with fewer than 64 elements.

Remark 9.6 (Orbits of Size 144)**.** The orbits of size 144 in Appendix C tend to feature points of the form $(\alpha,\beta,1)$ and $(\alpha,\beta,-\beta)$ that satisfy

$$\sigma_1(\alpha,\beta,-\beta)=(\alpha,\beta,-\beta)\quad\text{and}\quad\sigma_3(\alpha,\beta,-\beta)=(\alpha,\beta,1).$$

We assume that $\alpha,\beta\notin\{0,\infty\}$ and that $\beta\neq-1$, and then we obtain four conditions on k,α,β:

$$\begin{aligned}(\alpha,\beta,1)\in\mathcal{W}_k\quad&\Longleftrightarrow\quad k=-(\alpha+\alpha^{-1})(\beta+\beta^{-1}),\\(\alpha,\beta,-\beta)\in\mathcal{W}_k\quad&\Longleftrightarrow\quad \alpha\beta^2k=\alpha^2(\beta^4+1)+2\beta^2,\\\sigma_1(\alpha,\beta,-\beta)=(\alpha,\beta,-\beta)\quad&\Longleftrightarrow\quad \alpha^2\beta^2(\beta^4+1)=2\beta^2,\\\sigma_3(\alpha,\beta,-\beta)=(\alpha,\beta,1)\quad&\Longleftrightarrow\quad (\beta^2-\beta+1)\alpha^2+\beta=0.\end{aligned}$$

The ideal in $\mathbb{Z}[\alpha,\beta,k]$ generated by these four relations is also generated (according to Magma) by the three relations

$$\alpha^4+4\alpha^2-1=0,\quad k=4\alpha(\alpha^2+4),\quad \beta^2+(\alpha^2+3)\beta+1=0.$$

(We also note that since $\alpha\neq0$, we can replace the formula for k by $k=4\alpha^{-1}$.)

Remark 9.7 (Orbits of Size 160)**.** The orbits of size 160 in Appendix C tend to include a single point of the form (β,β,β) having the property that

$$\sigma_1\circ\sigma_3(\beta,\beta,\beta)=(1,\beta,*). \tag{29}$$

The assumption that $(\beta,\beta,\beta)\in\mathcal{W}_k$ gives $k=-(3+\beta^4)/\beta$, and then computing (29) explicitly gives

$$\sigma_1\circ\sigma_3(\beta,\beta,\beta)=\left(\frac{\beta^9+2\beta^5+5\beta}{\beta^8+6\beta^4+1},\beta,\frac{2\beta}{\beta^4+1}\right).$$

Setting the first coordinate to 1 and discarding the trivial solution $\beta=1$ yields the condition

$$\beta^8+2\beta^4-4\beta^3-4\beta^2-4\beta+1.$$

Setting $\gamma=2\beta/(\beta^4+1)$ for convenience, we find that the union of the $\mathcal{G}^\circ$-orbits of the following points is an orbit of size 160.

Point	Size of $\mathcal{G}^\circ$-orbit
(β,β,β)	4
$(\beta^{-1},\beta^{-1},\beta)$	12
(β,β,γ)	12
$(\beta^{-1},\beta^{-1},\gamma)$	12
$(\beta,\beta^{-1},\gamma^{-1})$	24

Point	Size of $\mathcal{G}^\circ$-orbit
$(1,\beta,\gamma)$	24
$(1,\beta^{-1},\gamma)$	24
$(1,\beta,\gamma^{-1})$	24
$(1,\beta^{-1},\gamma^{-1})$	24

Remark 9.8 (Orbits of Size 288)**.** There is an orbit of size 288 in $\mathcal{W}_{11}(\mathbb{F}_{47})$ whose points have coordinates in the following set of values:

	t	$-t$	t^{-1}	$-t^{-1}$
α	3	44	16	31
β	6	41	8	39
γ	11	36	30	17
δ	15	32	22	25

In particular, we find that

$$\sigma_3(3,6,11) = (3,6,15) \quad \text{in } \mathcal{W}_{11}(\mathbb{F}_{47}).$$

If we now treat α,β,γ as indeterminates and want to require that

$$(\alpha,\beta,\gamma) \in \mathcal{W}_k \quad \text{and that} \quad \sigma_3(\alpha,\beta,\gamma) = (\alpha,\beta,\delta),$$

then we find that k and δ are given by the formulas

$$k = -\frac{\alpha^2+\beta^2+\gamma^2+\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma}, \tag{30}$$

$$\delta = \frac{\alpha^2+\beta^2}{\gamma(\alpha^2\beta^2+1)}. \tag{31}$$

Let $P_1 = (3,6,11) \in \mathcal{W}_{11}(\mathbb{F}_{47})$. Then the $\mathcal{G}$-orbit of P_1 has size 288, while the sub-orbit for $\mathcal{G}^\sigma = \langle \sigma_1,\sigma_2,\sigma_3\rangle$ has size 24 and is described in detail in Table 1. We observe that the subgroup of $\mathcal{G}^\circ$ leaving the orbit $\mathcal{G}^\sigma \cdot P_1$ invariant is

$$\operatorname{Stab}_{\mathcal{G}^\circ}(\mathcal{G}^\sigma \cdot P_1) = \{e,\lambda\}, \quad \text{where} \quad \lambda : (x,y,z) \longmapsto (x,-z,-y).$$

Hence the full $\mathcal{G}$-orbit of $P_1 \in \mathcal{W}_{11}(\mathbb{F}_{47})$ has order

$$\#\mathcal{G}\cdot P_1 = \Bigl(\#\mathcal{G}^\sigma \cdot P_1\Bigr)\cdot\left(\frac{\#\mathcal{G}^\circ}{\#\operatorname{Stab}_{\mathcal{G}^\circ}(\mathcal{G}^\sigma\cdot P_1)}\right) = 24\cdot\frac{24}{2} = 288.$$

Looking at Table 1, we find many relations in $\mathcal{W}_{11}(\mathbb{F}_{47})$, including for example[18]

$$\delta = \sigma_1(\alpha,\beta,\gamma)[1]^{-1} = -\sigma_2(\alpha,\beta,\gamma)[2] = \sigma_3(\alpha,\beta,\gamma)[3], \tag{32}$$

and

$$\sigma_2\circ\sigma_3(\alpha,\beta,\gamma) = \sigma_1\circ\sigma_3(-\beta^{-1},-\gamma,\alpha^{-1}). \tag{33}$$

If we now view (32) and (33) as determining conditions on the indeterminate quanitites α,β,γ, we find that α,β,γ must satisfy certain equations, and restricting to those equations that are satisfied by $(3,6,11)$

[18]We use the convenient notation $\boldsymbol{v}[j]$ to denote the jth coordinate of the vector $\boldsymbol{v}$.

in $\mathbb{F}_{47}$, we find that α, β, γ must satisfy

$$\alpha^3\beta^2 - \alpha^2\beta + \alpha - \beta^3 = 0, \tag{34}$$

$$\beta^3\gamma^3 - \beta^2 + \beta\gamma - \gamma^2 = 0, \tag{35}$$

$$\alpha^3\gamma^2 + \alpha^2\gamma + \alpha + \gamma^3 = 0. \tag{36}$$

These three relations for α, β, γ define a reducible subset of $\mathbb{A}^3$, and a computation using Magma shows that this set consists of two pieces. There is a finite set of points defined by

$$3\alpha + \gamma^3 = \beta + \gamma = \gamma^4 + 3 = 0, \tag{37}$$

and there is a geometrically irreducible reduced affine curve in $\mathbb{A}^3$ given by the equations

$$C = \left\{ (\alpha, \beta, \gamma) : \begin{array}{r} \alpha^2\beta - \alpha^2\gamma + \alpha\beta^2\gamma^2 - \alpha + \beta^2\gamma - \beta\gamma^2 = 0 \\ \alpha^2\gamma^2 - \alpha\beta^2\gamma^3 + \alpha\beta + \beta\gamma^3 = 0 \\ \beta^3\gamma^3 - \beta^2 + \beta\gamma - \gamma^2 = 0 \end{array} \right\} \tag{38}$$

We discard the points (37), since the orbit collapses if $\beta = -\gamma$. A further computation shows that the affine curve C has a unique singular point at $(0, 0, 0)$ and that it has (geometric) genus 9.

We let I denote the ideal in $\mathbb{Q}[\alpha, \beta, \gamma]$ generated by the three polynomials (38) defining the curve C. Then for each of the points P_j in Table 1, treating α, β, γ as indeterminates and taking k and δ in $\mathbb{Q}(\alpha, \beta, \gamma)$ as specified by (30) and (31), we used Magma to check that $\sigma_i(P_j)$ is as specified in Table 1 if we work in the fraction field of the quotient ring $\mathbb{Q}[\alpha, \beta, \gamma]/I$. Hence the $\mathcal{G}^\sigma$-orbit of (α, β, γ) has size 24 when we work over this ring, and then as noted earlier, the full $\mathcal{G}$-orbit has size 288.

In summary, we have shown that there is an irreducible affine curve $C/\mathbb{Q}$ of geometric genus 9 and an element $k \in \mathbb{Q}(C)$ in the function field of C so that $\mathcal{W}_k\bigl(\mathbb{Q}(C)\bigr)$ contains twelve $\mathcal{G}^\sigma$-orbits of size 24 that combine to form one $\mathcal{G}$-orbit of size 288.

However, we note that there are points on the curve $C(\mathbb{C})$ for which the orbit collapses. Thus if we set δ to be equal to any of α^{-1}, $-\beta$, or γ, then the $\mathcal{G}^\circ$-orbits of the 12 points listed in Table 3 collapse pairwise, and we obtain a total $\mathcal{G}$-orbit of size 144, instead of 288. A short computation shows that if we don't allow α, β, γ to be in $\{0, \pm1, \pm i\}$, then

$$\delta = \alpha^{-1} \Longrightarrow 3\alpha^4 = -1, \quad \delta = -\beta \Longrightarrow \beta^4 = -3, \quad \delta = \gamma \Longrightarrow \gamma^4 = -3.$$

Remark 9.9 (Orbits of Size 288: A Cautionary Tale)**.** We have seen in Remark 9.8 that there is an entire 1-parameter family of orbits of size 288 in characteristic 0. However, there are also exceptional orbits

P	P	$\sigma_1(P)$	$\sigma_2(P)$	$\sigma_3(P)$
P_1	(α,β,γ)	P_2	P_5	P_7
P_2	$(\delta^{-1},\beta,\gamma)$	P_1	P_3	P_{11}
P_3	$(\delta^{-1},-\alpha^{-1},\gamma)$	P_4	P_2	λP_{11}
P_4	$(-\beta^{-1},-\alpha^{-1},\gamma)$	P_3	P_6	P_{10}
P_5	$(\alpha,-\delta,\gamma)$	P_6	P_1	λP_7
P_6	$(-\beta^{-1},-\delta,\gamma)$	P_5	P_4	λP_{10}
P_7	(α,β,δ)	P_8	λP_5	P_1
P_8	$(\gamma^{-1},\beta,\delta)$	P_7	P_9	P_{12}
P_9	$(\gamma^{-1},-\alpha^{-1},\delta)$	P_{10}	P_8	λP_{12}
P_{10}	$(-\beta^{-1},-\alpha^{-1},\delta)$	P_9	λP_6	P_4
P_{11}	$(\delta^{-1},\beta,\alpha^{-1})$	P_{12}	λP_3	P_2
P_{12}	$(\gamma^{-1},\beta,\alpha^{-1})$	P_{11}	λP_9	P_8

TABLE 1. The $\mathcal{G}^\sigma$-orbit of $(\alpha,\beta,\gamma) = (3,6,11) \in \mathcal{W}_{11}(\mathbb{F}_{47})$, which we want to lift to a $\mathcal{G}^\sigma$-orbit in characteristic 0. The map $\lambda \in \mathcal{G}^\circ$ is $\lambda(x,y,z) = (x,-z,-y)$.

of size 288 in finite characteristic that do not lift. For example, we consider the orbit of size 288 in $\mathcal{W}_{11}(\mathbb{F}_{53})$. This orbit contains many points of the form $(\alpha,-\alpha,1)$ and many points of the form $(0,\beta,i\beta)$. We note that an orbit containing points of this form does not fit into the family described in Remark 9.8, but this does not preclude it coming from some other characteristic 0 orbit, so we continue analyzing the present example. In particular, we see that $\mathcal{W}_{11}(\mathbb{F}_{53})$ contains the points

$$(-38,38,1) \xrightarrow{\sigma_3} (15,38,12) \xrightarrow{\sigma_2} (15,11,12) \xrightarrow{\sigma_1} (0,11,12).$$

This suggests that we should take a point $(\alpha,-\alpha,1) \in \mathcal{W}_k$ satisfying

$$\sigma_1\circ\sigma_2\circ\sigma_3(\alpha,-\alpha,1) = (0,\beta,i\beta). \tag{39}$$

The assumption that $(\alpha,-\alpha,1) \in \mathcal{W}_k$ forces $k = (\alpha+\alpha^{-1})^2$, and the assumption that the first coordinate in (39) is 0 forces

$$\alpha^{18}-3\alpha^{16}+12\alpha^{14}-16\alpha^{12}+62\alpha^{10}-38\alpha^8+44\alpha^6-8\alpha^4+9\alpha^2+1 = 0. \tag{40}$$

We next observe that in $\mathcal{W}_{11}(\mathbb{F}_{53})$, the orbit of $(38,-38,1)$ has a σ_3 fixed point, specifically

$$\sigma_2\circ\sigma_3(38,-38,1) = (15,11,12) \quad \text{is fixed by } \sigma_3. \tag{41}$$

So in general we might want to impose the further condition that

$$\sigma_3\circ\sigma_2\circ\sigma_3(\alpha,-\alpha,1) = \sigma_2\circ\sigma_3(\alpha,-\alpha,1) \tag{42}$$

to mirror the behavior in $\mathcal{W}_{11}(\mathbb{F}_{53})$. Assuming that $\alpha \neq \pm 1$, we find that (42) forces α to satisfy

$$\alpha^{12} + 2\alpha^{10} + 15\alpha^8 + 12\alpha^6 + 15\alpha^4 + 2\alpha^2 + 1 = 0. \tag{43}$$

However, the conditions (40) and (43) are incompatible in characteristic 0. Indeed, the resultant of the two polynomials in (40) and (43) is equal to $2^{80} \cdot 53^2$, so the fact that (41) is true in $\mathcal{W}_{11}(\mathbb{F}_{53})$ comes from our choice of the specific finite field $\mathbb{F}_{53}$.

Remark 9.10 (Orbits of size 256: Another Cautionary Tale)**.** There is an orbit of size 256 in $\mathcal{W}_8(\mathbb{F}_{53})$ whose points have coordinates in the following set of values:

$$\{\pm 1, \pm\alpha^{\pm 1}, \pm\beta^{\pm 1}, \pm\gamma^{\pm 1}\} \quad \text{with} \quad \alpha = 16,\ \beta = 21,\ \gamma = 39.$$

In particular, there are points

$$\begin{aligned}
P_1 &= (\alpha, \alpha, \alpha) = (16, 16, 16) \in \mathcal{W}_8(\mathbb{F}_{53}),\\
P_2 &= (\alpha, \alpha, \gamma^{-1}) = (16, 16, 34) \in \mathcal{W}_8(\mathbb{F}_{53}),\\
P_3 &= (1, \alpha, \beta) = (1, 16, 21) \in \mathcal{W}_8(\mathbb{F}_{53}),\\
P_4 &= (\alpha, \beta, \gamma) = (16, 21, 39) \in \mathcal{W}_8(\mathbb{F}_{53}).
\end{aligned}$$

We first note that

$$P_1 = (\alpha, \alpha, \alpha) \in \mathcal{W}_k \quad \Longrightarrow \quad k = -\frac{\alpha^4 + 3}{\alpha},$$

$$P_2 = (\alpha, \alpha, \gamma^{-1}) \in \mathcal{W}_k \quad \Longrightarrow \quad \alpha^4 + 1 - 2\alpha\gamma = 0 \quad (\text{assuming } P_2 \neq P_1), \tag{44}$$

$$P_3 = (1, \alpha, \beta) \in \mathcal{W}_k \quad \Longrightarrow \quad (\alpha^2 + 1)\beta^2 - (\alpha^4 + 3)\beta + \alpha^2 + 1 = 0, \tag{45}$$

$$P_4 = (\alpha, \beta, \gamma) \in \mathcal{W}_k \quad \Longrightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + \alpha^2\beta^2\gamma^2 - (\alpha^4 + 3)\beta\gamma = 0. \tag{46}$$

This gives three relations on α, β, γ. We can use the orbit structure of $\mathcal{W}_8(\mathbb{F}_{53})$ to generate additional relations such as

$$\begin{aligned}
\sigma_1(16, 16, 16) = (39^{-1}, 16, 16) &\in \mathcal{W}_8(\mathbb{F}_{53})\\
&\Longrightarrow \quad \sigma_1(\alpha, \alpha, \alpha) = (\gamma^{-1}, \alpha, \alpha) \in \mathcal{W}_k\\
&\Longrightarrow \quad \alpha^4 - 2\alpha\gamma + 1 = 0, \qquad (47)\\
\sigma_1(16, 21, 39) = (16, 21, 39) &\in \mathcal{W}_8(\mathbb{F}_{53})\\
&\Longrightarrow \quad \sigma_1(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma) \in \mathcal{W}_k\\
&\Longrightarrow \quad \alpha^2(\alpha^4 + 3)\beta^2 - (\alpha^4 - 1) = 0. \qquad (48)
\end{aligned}$$

The five relations (44)–(48) are incompatible in characteristic 0, although they do of course have the solution $(\alpha,\beta,\gamma) = (16, 21, 39)$ in $\mathbb{F}_{53}$. More precisely, the resultant of the five polynomials (44)–(48) is $9752 = 2^3 \cdot 23 \cdot 53$, and indeed in $\mathcal{W}_2(\mathbb{F}_{23})$ we find an orbit of size 256 corresponding to $(\alpha,\beta,\gamma) = (6, 11, 18)$. So the orbits of size 256 in $\mathcal{W}_2(\mathbb{F}_{23})$ and $\mathcal{W}_8(\mathbb{F}_{53})$ do not lift to characteristic 0.

Remark 9.11 (Orbits of Size 384: A Third Cautionary Tale)**.** There is a point $P_1 = (22, 22, -23) \in \mathcal{W}_{13}(\mathbb{F}_{71})$. A direct computation shows that $\#\mathcal{G}\cdot P_1 = 384$. We let $(\alpha,\beta,\gamma,\delta) = (22, 23, 9, 44)$, and we consider the six points $P_1 \ldots, P_6 \in \mathcal{W}_{13}(\mathbb{F}_{71})$ described in Table 2. We also let $\hat{\mathcal{G}}^\circ \subset \operatorname{Aut}(\mathcal{W}_k)$ be the subgroup containing 96 automorphisms that is described in Remark 8.7. Again by direct computation[19] we find that $\mathcal{G}\cdot P_1 \subset \mathcal{W}_{13}(\mathbb{F}_{71})$ is invariant for $\hat{\mathcal{G}}^\circ$, and that it splits up into six $\hat{\mathcal{G}}^\circ$-orbits with orbit representatives $P_1,\ldots,P_6$ and orbits of size 48 or 96 as indicated in Table 2.

We now try to lift to characteristic 0, so we view $\alpha,\beta,\gamma,\delta$ as indeterminates. However, it turns out that the six conditions

$$P_i \in \mathcal{W}_k \quad \text{for} \quad i = 1,\ldots,6$$

are inconsistent in $\mathbb{Q}[\alpha,\beta,\gamma,\delta,k]$.

$\#\hat{\mathcal{G}}^\circ P$	P	P	$\sigma_1(P)$	$\sigma_2(P)$	$\sigma_3(P)$
48	P_1	$(\alpha,\alpha,-\beta)$	$(\gamma^{-1},\alpha,-\beta)$	$(\alpha,\gamma^{-1},-\beta)$	$(\alpha,\alpha,-\gamma)$
48	P_2	$(\alpha,\alpha,-\gamma)$	$(\beta^{-1},\alpha,-\gamma)$	$(\alpha,\beta^{-1},-\gamma)$	$(\alpha,\alpha,-\beta)$
48	P_3	(β,β,γ)	$(-\alpha^{-1},\beta,\gamma)$	$(\beta,-\alpha^{-1},\gamma)$	(β,β,δ)
48	P_4	(β,β,δ)	$(-1,\beta,\delta)$	$(\beta,-1,\delta)$	(β,β,γ)
96	P_5	$(\alpha,-\beta,\gamma^{-1})$	$(-\beta^{-1},-\beta,\gamma^{-1})$	$(\alpha,-\alpha^{-1},\gamma^{-1})$	$(\alpha,-\beta,\alpha)$
96	P_6	$(\beta,-\delta,1)$	$(\beta^{-1},-\delta,1)$	$(\beta,-\delta^{-1},1)$	$(\beta,-\delta,-\beta)$

TABLE 2. The $\mathcal{G}$-orbit of $(\alpha,\alpha,-\beta) = (22, 22, -23) \in \mathcal{W}_{13}(\mathbb{F}_{71})$, with $\gamma = 9$ and $\delta = 44$. We want to lift it to a $\mathcal{G}$-orbit in characteristic 0. We note that every point in the last three columns is in the $\hat{\mathcal{G}}^\circ$-orbit of one of $P_1,\ldots,P_6$.

[19]Somewhat surprisingly, for this example we find that $\mathcal{G}^\sigma\cdot P_1 = \mathcal{G}\cdot P_1 = \hat{\mathcal{G}}^\circ\mathcal{G}^\sigma\cdot P_1$ in $\mathcal{W}_{13}(\mathbb{F}_{71})$.

orbit size	k	$\mathcal{G}^\circ$-generators
1	all k	$(0,0,0)$
3	all k	$(0,\infty,\infty)$
4	$k=4$	$(-1,-1,-1)$
24	$\xi^4 \neq 1$ $k=-2(\xi+\xi^{-1})$	$(\xi,1,1)$, $(\xi^{-1},1,1)$
48	all k	$(1,i,0)$, $(1,i,\infty)$
64	$\beta^3+\beta^2+\beta-1=0$ $k=-(\beta+\beta^{-1})^2$	(β,β,β), $(\beta,\beta,1)$ $(\beta^{-1},\beta^{-1},1)$ $(\beta,\beta^{-1},\beta^{-1})$ $(\beta,\beta^{-1},1)$
96	$\eta^4=-1$ $k=-2\eta^2$	$(\eta,\eta^3,0)$ (η,η^3,η^6) (η,η^2,η^5) (η,η^2,∞)
144	$\alpha^4+4\alpha^2-1=0$ $\beta^2+(\alpha^2+3)\beta+1=0$ $\beta^4+2\beta^3-2\beta^2+2\beta+1=0$ $k=4\alpha^{-1}$	$(\alpha,\beta,1)$, $(\alpha^{-1},\beta,1)$, $(\alpha,\beta^{-1},1)$, $(\alpha^{-1},\beta^{-1},1)$, $(\alpha,\beta,-\beta)$, $(\alpha^{-1},\beta^{-1},-\beta)$
160	$\beta^8+2\beta^4-4\beta^3$ $-4\beta^2-4\beta+1=0$ $\gamma=2\beta/(\beta^4+1)$ $k=-(3+\beta^4)/\beta$	(β,β,β) $(1,\beta,\gamma)$ $(\beta^{-1},\beta^{-1},\beta)$ $(1,\beta^{-1},\gamma)$ (β,β,γ) $(1,\beta,\gamma^{-1})$ $(\beta^{-1},\beta^{-1},\gamma)$ $(1,\beta^{-1},\gamma^{-1})$ $(\beta,\beta^{-1},\gamma^{-1})$
192	$\xi^8 \neq 1$ $k=i(\xi^2-\xi^{-2})$	$(\xi,i\xi,0)$, $(\xi,-i\xi,1)$, $(\xi,i\xi^{-1},1)$, $(\xi,i\xi^{-1},\infty)$, $(\xi^{-1},-i\xi,1)$, $(\xi^{-1},i\xi,\infty)$, $(\xi^{-1},i\xi^{-1},0)$, $(\xi^{-1},i\xi^{-1},1)$
288 or 144*	$\alpha^2\beta-\alpha^2\gamma+\alpha\beta^2\gamma^2$ $-\alpha+\beta^2\gamma-\beta\gamma^2=0$ $\alpha^2\gamma^2-\alpha\beta^2\gamma^3+\alpha\beta+\beta\gamma^3=0$ $\beta^3\gamma^3-\beta^2+\beta\gamma-\gamma^2=0$ $\delta=\dfrac{\alpha^2+\beta^2}{\gamma(\alpha^2\beta^2+1)}$ $k=-\dfrac{\alpha^2+\beta^2+\gamma^2+\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma}$	(α,β,γ) $(\delta^{-1},\beta,\gamma)$ $(\delta^{-1},-\alpha^{-1},\gamma)$ $(-\beta^{-1},-\alpha^{-1},\gamma)$ (α,β,δ) $(\gamma^{-1},\beta,\delta)$ $(\gamma^{-1},-\alpha^{-1},\delta)$ $(-\beta^{-1},-\alpha^{-1},\delta)$ $(\alpha,-\gamma,\delta)$ $(-\beta^{-1},-\gamma,\delta)$ $(\delta^{-1},\beta,\alpha^{-1})$ $(\gamma^{-1},\beta,\alpha^{-1})$ *Orbit size 144 if $3\alpha^4=-1$ or $\beta^4=-3$ or $\gamma^4=-3$

TABLE 3. Examples of finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$, where in each case we list only one of $\mathcal{W}_{\pm k}$ and $\mathcal{W}_{\pm ik}$; cf. Remark 8.3.

10. $\mathcal{G}$-Orbits in $\mathcal{W}_k(\mathbb{F}_p)$

In this section we consider $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{F}_p)$, where we recall that $\mathcal{G}$ is the set of automorphisms of $\mathcal{W}_k$ generated by the three involutions $\sigma_1, \sigma_2, \sigma_3$, permutations of the three coordinates, and double sign changes. Orbits in $\mathcal{W}_k(\mathbb{F}_p)$ are necessarily finite, since $\mathcal{W}_k(\mathbb{F}_p)$ is itself a finite set. In Appendix C we list the $\mathcal{G}$-orbit structure for each $3 \le p \le 113$. We first did these computations with a custom program that we wrote in PARI-GP [31]. This program used a straightforward algorithm to compute the points in $\mathcal{W}_k(\mathbb{F}_p)$, and then a hash table to optimize finding and checking off points in orbits. This program allowed us to compute the components of $\mathcal{W}_k(\mathbb{F}_p)$ for $p \le 79$. We then reprogrammed the problem in Magma [8]. This allowed us to double-check the PARI-GP program, and ultimately to extend the computations to larger primes. Our first Magma implementation used the permutation group package in Magma and was a bit slower than PARI-GP. When we replaced the Magma permutation group package with the Magma graph theory package, the computations were roughly 10 times faster. This implementation used a Magma function that computes points on projective subvarieties of $(\mathbb{P}^1)^3(\mathbb{F}_p)$. When we replaced this with a Magma function that computes points on affine subvarieties of $\mathbb{A}^3(\mathbb{F}_p)$ and filled in the few extra points on $\mathcal{W}_k(\mathbb{F}_p)$ lying at infinity, we picked up roughly another order of magnitude in speed. To give an idea of the resources used, we note that the program computed the orbits in $\mathcal{W}_k(\mathbb{F}_{113})$ for 29 values of k in roughly 31 minutes on a MacBook Pro (2021) using an Apple M1 Pro chip.

In view of the isomorphisms provided by Remark 8.3, for $p \equiv 3 \pmod 4$ we compute the orbit structure of $\mathcal{W}_k(\mathbb{F}_p)$ for only one of $\pm k \in \mathbb{F}_p^*$; and for $p \equiv 1 \pmod 4$, we compute the orbit structure of $\mathcal{W}_k(\mathbb{F}_p)$ for only one of $\pm k, \pm ik \in \mathbb{F}_p^*$, where $i = \sqrt{-1} \in \mathbb{F}_p$. In the tables in Appendix C, we have also omitted the trivial orbits of size 1 and 3 described in Definition 9.3.

Reducing the characteristic 0 orbits in Table 3 modulo p yields some of the small characteristic p orbits in Appendix C. In particular, Table 4 lists the characteristic p orbits of sizes 144, 160 and 288 for $p \le 79$ that come from characteristic 0.

11. Fibral Orbits in $\mathcal{W}_k(\mathbb{F}_p)$

As usual, we let

$$\mathcal{G} = \langle \sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{13}, \tau_{23}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23} \rangle \subset \mathrm{Aut}(\mathcal{W}_k).$$

p	k	α	β	Orbit size
11	1	4	5	144
19	8	11	4	144
29	1	4	18	144
29	11	3	2	144
31	2	2	3	144
59	9	7	21	144
71	34	21	59	144
79	6	27	63	144

Orbits of size 144: Remark 9.6

p	k	β	γ	Orbit size
19	2	6	10	160
23	5	20	19	160
31	6	22	8	160
41	1	25	35	160
41	4	31	34	160
59	8	36	38	160
67	27	11	49	160
73	18	9	16	160

Orbits of size 160: Remark 9.7

p	k	α	β	γ	Orbit size	
19	9	7	2	3	144	$\beta^4 = -3$
23	4	10	8	9	288	
43	2	28	13	14	144	$3\alpha^4 = -1$
47	11	3	6	11	288	
59	23	13	33	8	288	
61	15	4	7	18	288	
67	31	5	30	12	144	$3\alpha^4 = -1$
71	13	10	44	16	288	
79	35	36	8	59	288	
79	36	12	19	51	288	

Orbits of sizes 144 and 288: Remark 9.8

TABLE 4. $\mathcal{W}(\mathbb{F}_p)$ orbits of sizes 144, 160 and 288 in Tables 8–11 coming from $\mathcal{W}(\overline{\mathbb{Q}})$ orbits in Table 3.

For $x_0, y_0, z_0 \in K$, we denote the fibers of $\mathcal{W}_k(K)$ as usual by

$$\mathcal{W}^{(1)}_{k,x_0} = \bigl\{(x_0, y, z) \in \mathcal{W}_k(K)\bigr\},$$
$$\mathcal{W}^{(2)}_{k,y_0} = \bigl\{(x, y_0, z) \in \mathcal{W}_k(K)\bigr\},$$
$$\mathcal{W}^{(3)}_{k,z_0} = \bigl\{(x, y, z_0) \in \mathcal{W}_k(K)\bigr\}.$$

The $\mathcal{G}$-fibral automorphism group of a fiber is the subgroup of $\mathcal{G}$ that maps the fiber to itself, and we use the action of $\mathcal{G}$-fibral automorphism group to define the fibral orbit(s) of the points on the fiber. See Definitions (3.5) and (3.6) for further details.

The $\mathcal{G}$-fibral automorphism group of the fiber $\mathcal{W}^{(1)}_{k,x_0}$ is generated by the two involutions σ_2 and σ_3 that fix x_0, the transposition τ_{23} that swaps the y and z coordinates, and the map ϵ_{23} that changes the sign of y and z; and similarly for the other fibers. Thus[20]

$$\begin{aligned}\mathcal{G}^{(1)}_{x_0} &= \langle \sigma_2, \sigma_3, \tau_{23}, \epsilon_{23} \rangle \subset \operatorname{Aut}\bigl(\mathcal{W}^{(1)}_{x_0}\bigr),\\ \mathcal{G}^{(2)}_{y_0} &= \langle \sigma_1, \sigma_3, \tau_{13}, \epsilon_{13} \rangle \subset \operatorname{Aut}\bigl(\mathcal{W}^{(2)}_{y_0}\bigr),\\ \mathcal{G}^{(3)}_{z_0} &= \langle \sigma_1, \sigma_2, \tau_{12}, \epsilon_{12} \rangle \subset \operatorname{Aut}\bigl(\mathcal{W}^{(3)}_{z_0}\bigr).\end{aligned}$$

We recall that since $\mathcal{W}_k$ is an MK3-surface, there is a set of points

$$\pi\,\mathsf{ConnFib}\bigl(\mathcal{W}_k(\mathbb{F}_q)\bigr) \subset \mathbb{P}^1(\mathbb{F}_q)$$

such that

$$\begin{aligned}&t \in \pi\,\mathsf{ConnFib}\bigl(\mathcal{W}_k(\mathbb{F}_q)\bigr) \quad \Longleftrightarrow\\ &\qquad \mathcal{W}^{(i)}_t(\mathbb{F}_q) \subseteq \mathsf{Cage}\bigl(\mathcal{W}_k(\mathbb{F}_q)\bigr) \text{ for one (equivalently all) } i \in \{1,2,3\}.\end{aligned}$$

Example 11.1. We consider the surface $\mathcal{W}_1$ over the finite field $\mathbb{F}_{53}$. The set $\mathcal{W}_1(\mathbb{F}_{53})$ has six $\mathcal{G}$-orbits of sizes, respectively, 1, 3, 24, 24, 48 and 3456. We are going to show that the cage in the big $\mathcal{G}$-component of $\mathcal{W}_1(\mathbb{F}_{53})$ is not cage-connected, and hence from Proposition 4.4, the big $\mathcal{G}$-component of $\mathcal{W}_1(\mathbb{F}_{53})$ does not have the fiber jumping property.

We compute the number of components on the various fibers, and when we do so, we find that

$$\pi\,\mathsf{ConnFib}\bigl(\mathcal{W}_1(\mathbb{F}_{53})\bigr) = \{\pm2, \pm4, \pm6, \pm8, \pm9, \pm11, \pm13, \pm20, \pm24, \pm26\}. \tag{49}$$

Next, for each t in $\pi\,\mathsf{ConnFib}\bigl(\mathcal{W}_1(\mathbb{F}_{53})\bigr)$, we would like to know which of the coordinates in $\pi\,\mathsf{ConnFib}\bigl(\mathcal{W}_1(\mathbb{F}_{53})\bigr)$ appear as the coordinate of some point in the (connected) fiber $\mathcal{W}^{(i)}_t(\mathbb{F}_{53})$. In general, if S is any set of points in $(\mathbb{P}^1)^3$, we define

$$\mathsf{Flatten}(S) = \text{the set of all coordinates of all points in } S.$$

[20]We have listed more generators than needed. For example, $\sigma_3 = \tau_{23} \circ \sigma_2 \circ \tau_{23}$, so $\operatorname{Aut}\bigl(\mathcal{W}^{(1)}_{x_0}\bigr) = \langle \sigma_2, \tau_{23}, \epsilon_{23} \rangle$, and similarly for the other fibers.

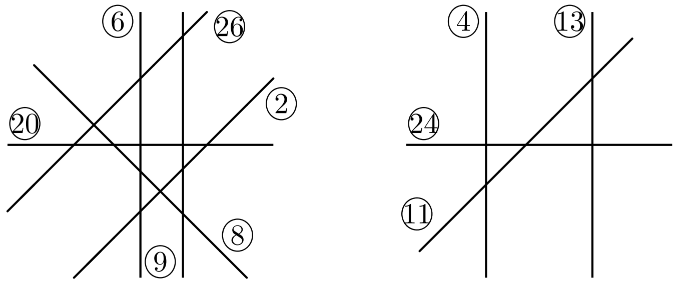

FIGURE 2. The two cage-connected components of the cage of the big $\mathcal{G}$-connected component of $\mathcal{W}_1(\mathbb{F}_{53})$, where the segment labeled ⓣ denotes the union of the six connected fibers $\cup_{i=1,2,3} \cup_{\epsilon=\pm 1} \mathcal{W}^{(i)}_{1,\epsilon t}(\mathbb{F}_{53})$

Then we may compute the connectivity of the cage of $\mathcal{W}_1(\mathbb{F}_{53})$ using the data in the following table.

t	$\mathsf{Flatten}\big(\mathcal{W}^{(1)}_{1,t}(\mathbb{F}_{53})\big) \cap \pi\,\mathsf{ConnFib}\big(\mathcal{W}_1(\mathbb{F}_{53})\big) \smallsetminus \{t\}$
± 2	$\{\pm 6, \pm 8, \pm 9, \pm 20\}$
± 4	$\{\pm 11, \pm 24\}$
± 6	$\{\pm 2, \pm 8, \pm 20, \pm 26\}$
± 8	$\{\pm 2, \pm 6, \pm 9, \pm 20, \pm 26\}$
± 9	$\{\pm 2, \pm 8, \pm 20, \pm 26\}$
± 11	$\{\pm 4, \pm 13, \pm 24\}$
± 13	$\{\pm 11, \pm 24\}$
± 20	$\{\pm 2, \pm 6, \pm 8, \pm 9, \pm 26\}$
± 24	$\{\pm 4, \pm 11, \pm 13\}$
± 26	$\{\pm 6, \pm 8, \pm 9, \pm 20\}$

Thus the cage in the big component of $\mathcal{W}_1(\mathbb{F}_{53})$ is not cage-connected. It consists of the following two pieces, which are illustrated in Figure 2:

$$\bigcup_{t\in\{\pm 2,\pm 6,\pm 8,\pm 9,\pm 20,\pm 26\}} \bigcup_{i\in\{1,2,3\}} \mathcal{W}^{(i)}_{1,t} \quad \text{and} \quad \bigcup_{t\in\{\pm 4,\pm 11,\pm 13,\pm 24\}} \bigcup_{i\in\{1,2,3\}} \mathcal{W}^{(i)}_{1,t}$$

$t_0 \backslash p$	5	7	11	13	17	19	23	29	31	37	41
∞	2	1	1	4	6	1	1	8	1	10	12
0	3	2	2	5	6	2	2	9	2	11	12
1	2	1	1	2	2	2	2	3	3	4	3
2	1	1	1	2	3	1	1	1	1	2	3
3	1	1	1	2	2	0	1	2	1	3	1
4	2	1	1	2	4	1	1	2	1	6	2
5		1	1	2	3	1	1	1	1	4	2
6		1	1	1	2	0	1	2	1	3	2
7			1	1	2	1	1	3	1	1	1
8			1	2	2	1	1	2	1	2	1
9			1	2	2	1	1	2	1	4	4
10			1	2	2	1	1	1	1	3	2
11				2	2	1	2	2	2	2	1
12				2	3	1	2	2	1	3	1
13					4	0	1	2	1	3	4
14					2	1	1	1	1	3	1
15					3	1	1	1	1	2	2
16					2	0	1	2	1	1	1
17						1	1	2	1	3	1
18						2	1	2	1	1	1
19							1	1	1	1	6
20							1	2	2	3	2
21							1	2	1	1	2
22							2	3	1	2	6
23								2	1	3	1
24								1	1	3	1
25								2	1	3	1
26								2	1	2	2
27								1	1	3	1
28								3	1	4	4
29									1	2	1
30									3	1	1
31										3	2
32										4	4
33										6	1
34										3	1
35										2	2
36										4	2
37											2
38											1
39											3
40											3

TABLE 5. # of fibral $\mathrm{Aut}(\mathcal{W}_{1,t_0}^{(i)})$-orbits in $\mathcal{W}_1(\mathbb{F}_p)$ for $i = 1, 2, 3$

12. The curious case of $\mathcal{W}_4(\mathbb{F}_p)$ with $p \equiv 1 \pmod 8$

We close with the curious case of $\mathcal{W}_4(\mathbb{F}_p)$, which seems to consistently have two large orbits when $p \equiv 1 \pmod 8$. We remark that the classical affine surface $\mathcal{M}_{1,4}$, which is known as the Cayley surface, also has an unusual $\mathbb{F}_p$-orbit structure due to the fact that it admits a double cover by $(\mathbb{G}_m)^2$ in which the involutes $\sigma_1, \sigma_2, \sigma_3$ become monomial maps; see for example [19]. There are analogous MK3 surfaces in which $(\mathbb{G}_m)^2$ is replaced by E^2, but the fibers of such surfaces are all isomorphic curves, while the j-invariants of the fibers of $\mathcal{W}_4$ vary, so $\mathcal{W}_4$ does not appear to be an MK3 analogue of the Cayley surface. In any case, we list in Table 6 the sizes of the components of $\mathcal{W}(\mathbb{F}_p)$ for all primes $p \le 113$ satisfying $p \equiv 1 \pmod 8$.

Remark 12.1 (**Addendum**)**.** After submission of this paper, one of the authors spoke about the material in this section at a conference. Evan O'Dorney, who was in the audience, then came up with an explanation [29]. His proof uses an ingenious construction of a $\mathcal{G}$-invariant map

$$\mathcal{W}_4(\mathbb{F}_p) \longrightarrow \mathbb{F}_p^*/(\mathbb{F}_p^*)^2,$$

which he uses to show that $\mathcal{W}_4(\mathbb{F}_p)$ has $\mathcal{G}$-invariant sets, each of size roughly $\frac{1}{2}p^2$.

p	small orbits	two largest orbits
17	$4, 16, 24, 48^2$	$64, 288$
41	$4, 24, 40, 48, 72, 120, 160, 192^3, 216$	$288, 576$
73	$4, 24, 40, 48, 120, 160, 192, 288^2$	$1920, 2976$
89	$4, 24, 48, 160^2, 192^2, 288^2$	$3264, 4512$
97	$4, 24, 48, 192, 960$	$3840, 5408$
113	$4, 24, 48$	$6656, 7488$

Table 6. Orbit sizes in $\mathcal{W}_4(\mathbb{F}_p)$ for $p \equiv 1 \pmod 8$. We omit the trivial orbit $\{(0,0,0)\}$ and that of $(\infty, \infty, 0)$ of size 3. The notation N^d indicates d orbits of size N.

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Appendix A. Geometry of $(2,2)$-curves in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we briefly discuss the well-known geometry of smooth $(2,2)$-curves $C \subset \mathbb{P}^1 \times \mathbb{P}^1$. Such a curve C has genus 1, so its Jacobian

$$E = \operatorname{Jac}(C) = \operatorname{Pic}^0(C)$$

is an elliptic curve. There is a natural action of E on C. If we identify C with $\operatorname{Pic}^1(C)$, then the action is simply

$$\underbrace{\operatorname{Pic}^0(C) \times \operatorname{Pic}^1(C)}_{\text{this is } E \times C} \longrightarrow \operatorname{Pic}^1(C), \quad \bigl(\mathcal{L}, \mathcal{O}_C(D)\bigr) \longmapsto \mathcal{L} \otimes \mathcal{O}_C(D).$$

The two double-cover projections $\pi_i : C \to \mathbb{P}^1$ have associated involutions s_i characterized by $\pi_i \circ s_i = \pi_i$. The degree 2 line bundles $\mathcal{L}_i = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ are independent elements of $\operatorname{Pic}(C)$, so their difference $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ gives a non-trivial element of $\operatorname{Pic}^0(C) = E$. The involutions and line bundles are related as follows. To ease notation, we write

$$s_1(P_1, P_2) = (P_1, \bar{P}_2) \quad \text{and} \quad s_2(P_1, P_2) = (\bar{P}_1, P_2).$$

Then

$$\begin{aligned} \mathcal{L}_1 = \mathcal{O}_C\bigl(\pi_1^*(P_1)\bigr) &= \mathcal{O}_C\bigl((P_1, P_2) + (P_1, \bar{P}_2)\bigr) \\ &= \mathcal{O}_C\bigl((P_1, P_2) + s_1(P_1, P_2)\bigr), \end{aligned}$$

and similarly

$$\mathcal{L}_2 = \mathcal{O}_C\bigl((P_1, P_2) + s_2(P_1, P_2)\bigr).$$

Hence

$$\begin{aligned} \mathcal{O}_C\bigl(s_1(P_1, P_2)\bigr) &= \mathcal{L}_1 \otimes \mathcal{O}_C\bigl((P_1, P_2)\bigr)^{-1}, \\ \mathcal{O}_C\bigl(s_2(P_1, P_2)\bigr) &= \mathcal{L}_2 \otimes \mathcal{O}_C\bigl((P_1, P_2)\bigr)^{-1}, \end{aligned}$$

so the composition $s_2 \circ s_1$ is given by

$$\begin{aligned} \mathcal{O}_C\bigl(s_2 \circ s_1(P_1, P_2)\bigr) &= \mathcal{L}_2 \otimes \mathcal{O}_C\bigl(s_1(P_1, P_2)\bigr)^{-1} \\ &= \mathcal{L}_2 \otimes \Bigl(\mathcal{L}_1 \otimes \mathcal{O}_C\bigl((P_1, P_2)\bigr)^{-1}\Bigr)^{-1} \\ &= \mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \otimes \mathcal{O}_C\bigl((P_1, P_2)\bigr). \end{aligned}$$

Thus $s_2 \circ s_1 : C \to C$ is translation by the point $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \in E$ that is constructed using the two projection maps.

One can use invariant theory to give explicit formulas for E and its translation point explicitly in terms of the equation for C. We thank Wei Ho for providing us with these formulas, which appear in her joint paper [6, Section 6.1].

We write the equation of

$$C \subset \mathbb{P}^1 \times \mathbb{P}^1 = \operatorname{Proj} K[X_1, X_2] \times \operatorname{Proj} K[Y_1, Y_2]$$

as

$$C : \begin{pmatrix} X_1^2 & X_1X_2 & X_2^2 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} Y_1^2 \\ Y_1Y_2 \\ Y_2^2 \end{pmatrix} = 0. \tag{50}$$

N.B. The $a, b, c, \ldots$ we use in (50) and Table 7 are not the same as the $a, b, c, \ldots$ used for the coefficients of MK3 surfaces.

Table 7 gives explicit formulas for four invariants $X_0, Y_0, A, B \in \mathbb{Z}[a, b, \ldots, h, i]$ of C having respective degrees 2, 3, 4, and 6. Then $E = \operatorname{Jac}(C)$ and the point $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$ on E are given by

$$E : Y^2 = X^3 + AX + B \quad \text{and} \quad (X_0, Y_0) \in E.$$

Appendix B. Generic TIK3 and MK3 surfaces are K3, and all $\mathcal{W}_k$ surfaces are K3

At the suggestion of a referee, we sketch a proof that smooth minimal models of generic TIK3 and MK3 are K3 surfaces, and that smooth minimal models of all $\mathcal{W}_k$ surfaces are K3 surfaces. We start with the case of TIK3 surfaces.

Proposition B.1. *A minimal regular model of a generic TIK3 surface is a K3 surface.*

Proof. A generic TIK3 surface $\mathcal{W}$ is smooth, so we may assume that $\mathcal{W}$ is a smooth surface of type $(2, 2, 2)$ in $(\mathbb{P}^1)^3$. The fact that such surfaces are K3 is well-known, but for the convenience of the reader, we sketch a proof. To ease notation, we momentarily write $\mathsf{P} = (\mathbb{P}^1)^3$.

We need to show that $H^1(\mathcal{O}_\mathcal{W}) = 0$ and $\mathcal{K}_\mathcal{W} \cong \mathcal{O}_\mathcal{W}$. Taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_\mathsf{P}(-2, -2, -2) \longrightarrow \mathcal{O}_\mathsf{P} \longrightarrow \mathcal{O}_\mathcal{W} \longrightarrow 0$$

gives a long exact sequence containing the fragment

$$\longrightarrow H^1(\mathsf{P}, \mathcal{O}_\mathsf{P}) \longrightarrow H^1(\mathsf{P}, \mathcal{O}_\mathcal{W}) \longrightarrow H^2\big(\mathsf{P}, \mathcal{O}_\mathsf{P}(-2, -2, -2)\big) \longrightarrow$$

The left-hand group is $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^3 = 0$. For the right-hand group, we compute

$$\begin{aligned} H^2\big(\mathsf{P}, \mathcal{O}_\mathsf{P}(-2, -2, -2)\big) &\cong H^0\big(\mathsf{P}, \mathcal{K}_\mathsf{P} \otimes \mathcal{O}_\mathsf{P}(-2, -2, -2)^\vee\big)' \\ &\qquad \text{Serre duality [23, III.7.7],} \\ &\cong H^0\big(\mathsf{P}, \mathcal{O}_\mathsf{P}(-2, -2, -2) \otimes \mathcal{O}_\mathsf{P}(2, 2, 2)\big)' \\ &\qquad \text{since } \mathcal{K}_\mathsf{P} \cong \mathcal{O}_\mathsf{P}(-2, -2, -2) \end{aligned}$$

$$
\begin{aligned}
X_0 &= 3(e^2 - 4df + 8cg - 4bh + 8ai),\\
Y_0 &= 108(ceg - bfg - cdh + afh + bdi - aei),\\
A &= -27(e^4 - 8de^2f + 16d^2f^2 - 8ce^2g - 16cdfg + 24befg - 48af^2g\\
&\quad + 16c^2g^2 + 24cdeh - 8be^2h - 16bdfh + 24aefh - 16bcgh\\
&\quad + 16b^2h^2 - 48ach^2 - 48cd^2i + 24bdei - 8ae^2i - 16adfi\\
&\quad - 48b^2gi + 224acgi - 16abhi + 16a^2i^2),\\
B &= -54(e^6 - 12de^4f + 48d^2e^2f^2 - 64d^3f^3 - 12ce^4g + 24cde^2fg\\
&\quad + 36be^3fg + 96cd^2f^2g - 144bdef^2g - 72ae^2f^2g + 288adf^3g\\
&\quad + 48c^2e^2g^2 + 96c^2dfg^2 - 144bcefg^2 + 216b^2f^2g^2 - 576acf^2g^2\\
&\quad - 64c^3g^3 + 36cde^3h - 12be^4h - 144cd^2efh + 24bde^2fh + 36ae^3fh\\
&\quad + 96bd^2f^2h - 144adef^2h - 144c^2degh + 24bce^2gh + 48bcdfgh\\
&\quad - 144b^2efgh + 720acefgh - 144abf^2gh + 96bc^2g^2h + 216c^2d^2h^2\\
&\quad - 144bcdeh^2 + 48b^2e^2h^2 - 72ace^2h^2 + 96b^2dfh^2 - 144acdfh^2\\
&\quad - 144abefh^2 + 216a^2f^2h^2 + 96b^2cgh^2 - 576ac^2gh^2 - 64b^3h^3 + 288abch^3\\
&\quad - 72cd^2e^2i + 36bde^3i - 12ae^4i + 288cd^3fi - 144bd^2efi + 24ade^2fi\\
&\quad + 96ad^2f^2i - 576c^2d^2gi + 720bcdegi - 72b^2e^2gi - 480ace^2gi - 144b^2dfgi\\
&\quad - 960acdfgi + 720abefgi - 576a^2f^2gi - 576b^2cg^2i + 2112ac^2g^2i - 144bcd^2hi\\
&\quad - 144b^2dehi + 720acdehi + 24abe^2hi + 48abdfhi - 144a^2efhi + 288b^3ghi\\
&\quad - 960abcghi + 96ab^2h^2i - 576a^2ch^2i + 216b^2d^2i^2 - 576acd^2i^2 - 144abdei^2\\
&\quad + 48a^2e^2i^2 + 96a^2dfi^2 - 576ab^2gi^2 + 2112a^2cgi^2 + 96a^2bhi^2 - 64a^3i^3).
\end{aligned}
$$

TABLE 7. Invariants of the biquadratic form (50)

$$\cong H^0(\mathsf{P}, \mathcal{O}_{\mathsf{P}})' = 0.$$

Hence $H^1(\mathcal{O}_{\mathcal{W}}) = 0$.

The general formula for the canonical bundle on a smooth subvariety [23, II.8.20] gives

$$\mathcal{K}_{\mathcal{W}} \cong \mathcal{K}_{\mathsf{P}} \otimes \mathcal{O}_{\mathsf{P}}(\mathcal{W}) \otimes \mathcal{O}_{\mathcal{W}} \cong \mathcal{O}_{\mathsf{P}}(-2,-2,-2) \otimes \mathcal{O}_{\mathsf{P}}(2,2,2) \otimes \mathcal{O}_{\mathcal{W}} \cong \mathcal{O}_{\mathcal{W}}.$$

This completes the proof that $\mathcal{W}$ is a K3 surface. $\square$

We next recall the classical geometric classification of elliptic surfaces over $\mathbb{P}^1$ via their minimal Weierstrass models.

Lemma B.2. *Let K be a field of characteristic not equal to 2 or 3, let $A(T), B(T) \in K[T]$ be polynomials such that*

$$\Delta(T) := 4A(T)^3 + 27B(T)^2 \neq 0$$

and

$$\gcd(A(T)^3, B(T)^2) \text{ is } 12\text{th-power-free in } K[T],$$

let

$$r = \max\left\{ \left\lceil \tfrac{1}{4}\deg(A) \right\rceil, \left\lceil \tfrac{1}{6}\deg(B) \right\rceil \right\}, \tag{51}$$

i.e., r is the smallest integer satisfying $r \geq \frac{1}{4}\deg(A)$ and $r \geq \frac{1}{6}\deg(B)$, and let

$$\delta(A,B) := \frac{1}{12}\Big(\deg\Delta(T) + \operatorname{ord}_{T=0} T^{12r}\Delta(T^{-1})\Big).$$

Let $E_{A,B}/K(T)$ be the elliptic curve defined by the Weierstrass equation

$$E_{A,B} : Y^2 = X^3 + AX + B,$$

and let $\mathcal{E}_{A,B} \to \mathbb{P}^1$ be the minimal regular elliptic surface with generic fiber $E_{A,B}$. Then the quantity $\delta(A,B)$ is a non-negative integer, and we have the following classification:

$\delta(A,B)$	*Geometry of $\mathcal{E}_{A,B}$*
0	*$\mathcal{E}_{A,B}$ is a product*
1	*$\mathcal{E}_{A,B}$ is a rational surface*
2	*$\mathcal{E}_{A,B}$ is a K3 surface*
≥ 3	*$\mathcal{E}_{A,B}$ is an elliptic surface of Kodaira dimension 1*

Proof. Let $\mathbb{L}$ be the fundamental line bundle attached to $E_{A,B}$ as defined in [28, II §4], so in particular the minimal discriminant $\Delta(T)$ defines a global section $\overline{\Delta}$ of $\mathbb{L}^{12}$.

We first compute the order of $\overline{\Delta}$ at ∞ by setting $T = S^{-1}$ and changing coordinates of the Weierstrass equation to obtain a minimal model at (S). Thus

$$E_{A,B} : Y^2 = X^3 + S^{4r}A(S^{-1})X + S^{6r}B(S^{-1}), \tag{52}$$

where defining r by (51) makes (52) into a minimal Weierstrass equation at (S). Then

$$\begin{aligned} \operatorname{ord}_\infty(\overline{\Delta}) &= \operatorname{ord}_{S=0}\Big(4\big(S^{4r}A(S^{-1})\big)^3 + 27\big(S^{6r}B(S^{-1})\big)^2\Big) \\ &= \operatorname{ord}_{S=0} S^{12r}\Delta(S^{-1}). \end{aligned} \tag{53}$$

We use this to compute

$$12\deg(\mathbb{L}) = \deg(\overline{\Delta}) \quad \text{since } \overline{\Delta} \text{ is a section of } \mathbb{L}^{12},$$

$$
\begin{aligned}
&= \left(\sum_{\gamma\in\mathbb{A}^1} \operatorname{ord}_\gamma(\overline{\Delta})\right) + \operatorname{ord}_\infty(\overline{\Delta}) \\
&= \left(\sum_{\gamma\in\mathbb{A}^1} \operatorname{ord}_{T=\gamma}\bigl(\Delta(T)\bigr)\right) + \operatorname{ord}_\infty(\overline{\Delta}) \\
&= \deg\bigl(\Delta(T)\bigr) + \operatorname{ord}_{T=0} T^{12r}\Delta(T^{-1}) \quad \text{from (53)}.
\end{aligned}
$$

This shows that $\deg(\mathbb{L}) = \delta(A,B)$, and the classification given in the table is then an immediate consequence of [28, Lemma III.4.6(a)]. □

We now have the tools required to show that generic MK3 surfaces are K3 surfaces, and that all $\mathcal{W}_k$ surfaces (always with the restriction that $k \neq 0$) are K3 surfaces.

Proposition B.3. *Let $\mathcal{W}/K$ be an MK3 surface defined over a field of characteristic not equal to 2 or 3, so from Proposition 6.5 there exist $a,b,c,d,e \in K$ so that $\mathcal{W} = \mathcal{W}_{a,b,c,d,e}$ is defined by a $(2,2,2)$-form of the following shape*:

$$
\begin{aligned}
F_{a,b,c,d,e}(x,y,z) = ax^2y^2z^2 &+ b(x^2y^2 + x^2z^2 + y^2z^2) \\
&+ cxyz + d(x^2+y^2+z^2) + e = 0. \qquad (54)
\end{aligned}
$$

Let $\widehat{\mathcal{W}}_{a,b,c,d,e}$ be a minimal regular model for $\mathcal{W}_{a,b,c,d,e}$.

(a) *For generic $(a,b,c,d,e) \in \mathbb{A}^5$, the surface $\widehat{\mathcal{W}}_{a,b,c,d,e}$ is a K3 surface.*

(b) *The surface $\widehat{\mathcal{W}}_k := \widehat{\mathcal{W}}_{1,0,k,1,0}$ is a K3 surface for all values of $k \neq 0$.*

Proof. Viewing $\mathcal{W}_{a,b,c,d,e}$ as being fibered via $\pi_3 : \mathcal{W}_{a,b,c,d,e} \to \mathbb{P}^1$, we can write the affine equation (54) for $\mathcal{W}_{a,b,c,d,e}$ as a matrix product

$$
\begin{pmatrix} x^2 & x & 1 \end{pmatrix}
\begin{pmatrix} az^2+b & 0 & bz^2+d \\ 0 & cz & 0 \\ bz^2+d & 0 & dz^2+e \end{pmatrix}
\begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix} = 0
$$

The material in Section A, and in particular the formulas in Table 7, say that the Jacobian of $\mathcal{W}_{a,b,c,d,e}$ has Weierstrass equation

$$
\mathcal{E}_{a,b,c,d,e} = \operatorname{Jac}(\mathcal{W}_{a,b,c,d,e}) : Y^2 = X^3 + AX + B,
$$

where A B, and Δ have the form

$$
\begin{aligned}
A(z) &= A(a,b,c,d,e;z) \\
&= -2^4\cdot 3^3\cdot(a^2d^2 + 14ab^2d + b^4)z^8 + (\text{l.o.t.}), \\
B(z) &= B(a,b,c,d,e;z) \\
&= 2^7\cdot 3^3\cdot(ad+b^2)\cdot(a^2d^2 - 34ab^2d + b^4)z^{12} + (\text{l.o.t.}), \\
\Delta &= \Delta(a,b,c,d,e;z)
\end{aligned}
$$

$$= 2^8 \cdot 3^{12} \cdot f_1 \cdot f_2 \cdot f_3^2 \cdot f_4^2 \cdot f_5^2, \tag{55}$$

with $f_1, \dots, f_5$ given by the formulas

$$f_1 = az^2 + b, \qquad f_2 = dz^2 + e, \qquad f_3 = bz^2 + d, \tag{56}$$

$$f_4 = 4(ad - b^2)z^4 - 4bcz^3 + (4ae - 4bd - c^2)z^2 \\ - 4cdz + 4(be - d^2), \tag{57}$$

$$f_5 = 4(ad - b^2)z^4 + 4bcz^3 + (4ae - 4bd - c^2)z^2 \\ + 4cdz + 4(be - d^2). \tag{58}$$

(a) Since we are interested in generic values of a, b, c, d, e, we may assume that

$$\begin{aligned} d^2a^2 + 14db^2a + b^4 &\neq 0, \\ (ad + b^2) \cdot (a^2d^2 - 34ab^2d + b^4) &\neq 0, \\ abd(ad - b^2) &\neq 0. \end{aligned}$$

These conditions ensure that

$$\deg(A) = 8, \quad \deg(B) = 12, \quad \deg(\Delta) = 24,$$

so in the notation of Lemma B.2, we have

$$r = 2 \quad \text{and} \quad \delta(A, B) = \frac{1}{12}\bigl(24 + (24 - 24)\bigr) = 2.$$

It follows from Lemma B.2 that $\widehat{\mathcal{W}}_{a,b,c,d,e}$ is a K3 surface provided that we check the minimal Weierstrass equation condition that generically, the quantity $\gcd\bigl(A(z)^3, B(z)^2\bigr)$ is 12th-power-free. The definition $\Delta = 4A^3 + 27B^2$ implies that

$$\Delta \text{ is 12th-power-free} \quad \Longrightarrow \quad \gcd(A^3, B^2) \text{ is 12th-power-free.}$$

so it suffices to show that Δ is generically 12th-power-free. Using the factorization (55) of Δ, it suffices to restrict to values of (a, b, c, d, e) such that the polynomials $f_1, \dots, f_5$ described by (56)–(58) are square-free and pairwise relatively prime as polynomials in z. It thus suffices to take (a, b, c, d, e) satisfying

$$\left(\prod_{i=1}^{5} \mathrm{Disc}_z(f_i)\right) \cdot \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{5} \mathrm{Res}_z(f_i, f_j)\right) \neq 0. \tag{59}$$

The non-equality (59) is a Zariski open condition in $\mathbb{A}^5$. This completes the proof that there is a non-empty Zariski open subset U of $\mathbb{A}^5$ such that $\widetilde{\mathcal{W}}_{a,b,c,d,e}$ is a K3 surface for all $(a, b, c, d, e) \in U$.

(b) For the $\mathcal{W}_k$ surfaces, the formulas are much simplified, and we find that

$$\begin{aligned} A &= -432z^8 + 216k^2z^6 + \cdots, \\ B &= -3456z^{12} + 2592k^2z^{10} + \cdots, \\ \Delta &= 2^8 \cdot 3^{12} \cdot z^4 \cdot (2z^2 - kz - 2)^2 \cdot (2z^2 - kz + 2)^2 \\ &\qquad\qquad \cdot (2z^2 + kz - 2)^2 \cdot (2z^2 + kz + 2)^2. \end{aligned} \tag{60}$$

As in the proof of (a), it suffices to show that Δ is 12th-power-free. The explicit factorization (60) of Δ tells us that

$$\Delta \text{ is } \underline{\text{not}} \text{ 12th-power-free} \quad \Longrightarrow \quad \begin{pmatrix} \text{at least two of the} \\ \text{polynomials } 2z^2 \pm kz \pm 2 \\ \text{have a common root} \end{pmatrix}.$$

There are six pairs of polynomials, and their pairwise resultants can be computed from

$$\begin{aligned} \operatorname{Res}(2z^2 + \epsilon kz + 2\epsilon, 2z^2 + \delta kz - 2\epsilon) &= 64\epsilon^2 + 4k^2\epsilon(\delta^2 - \epsilon^2), \\ \operatorname{Res}(2z^2 + \epsilon kz + \epsilon 2, 2z^2 - \epsilon kz + \epsilon 2) &= 16k^2\epsilon^3, \end{aligned}$$

by taking $\delta, \epsilon \in \{\pm 1\}$. In particular, we see that

$$k \ne 0 \quad \Longrightarrow \quad \Delta \text{ is 12th-power-free},$$

and indeed $k \ne 0$ implies that Δ is 5th-power-free.

This allows us to compute, using the notation from Lemma B.2,

$$\begin{aligned} r &= \max\left\{ \left\lceil \tfrac{1}{4}\deg(A) \right\rceil, \left\lceil \tfrac{1}{6}\deg(B) \right\rceil \right\} = \max\left\{ \left\lceil \tfrac{8}{4} \right\rceil, \left\lceil \tfrac{12}{6} \right\rceil \right\} = 2, \\ \delta(A,B) &= \frac{1}{12}\Bigl(\deg \Delta(z) + \operatorname{ord}_{z=0} z^{12r}\Delta(z^{-1}) \Bigr) \\ &= \frac{1}{12}\Bigl(20 + \operatorname{ord}_{z=0} z^{24}(2^{16}3^{12}z^{-20} + \cdots) \Bigr) \\ &= 2. \end{aligned}$$

It follows from Lemma B.2 that $\widehat{\mathcal{W}}_k$ is a K3 surface for all $k \ne 0$. $\square$

Appendix C. Orbits of $\mathcal{W}_k$ over finite fields

This appendix contains tables listing the orbit sizes for $\mathcal{W}_k(\mathbb{F}_p)$.

p	k	orbit sizes
3	1	4
5	1	$4, 48$
7	1	64
7	2	24
7	3	4
11	1	144
11	2	64
11	3	24
11	4	$4, 128$
11	5	$24, 64$
13	1	$24, 48, 192$
13	2	$24, 40, 48, 64, 120$
13	4	$4, 48, 192$
17	1	$4, 16, 24, 48^2, 64, 288$
17	2	$48, 96, 192$
17	3	$24, 48, 384$
17	6	$24, 48, 160, 192$
19	1	$24, 160$
19	2	$24, 160$
19	3	320
19	4	$4, 320$
19	5	$24, 288$
19	6	$24, 288$
19	7	432
19	8	288
19	9	$48, 64, 144^2$
23	1	$24, 448$
23	2	$256, 352$
23	3	$24, 336$
23	4	$4, 96, 288$
23	5	$24, 112, 160$
23	6	448
23	7	576
23	8	$24, 448$
23	9	608
23	10	448
23	11	$24, 384$

p	k	orbit sizes
29	1	$40, 48, 120, 144, 192, 352$
29	2	$24, 48, 352, 672$
29	3	$24^2, 48, 1152$
29	4	$4, 48, 192^2, 288^2$
29	6	$24^2, 48, 1184$
29	8	$24, 48, 64, 96, 288, 576$
29	11	$48, 144, 192^2, 384$
31	1	$24, 800$
31	2	$24, 144, 544$
31	3	896
31	4	$4, 768$
31	5	$24, 688$
31	6	$24, 160, 256, 384$
31	7	$24, 864$
31	8	864
31	9	864
31	10	1024
31	11	1056
31	12	$24, 624$
31	13	1120
31	14	$24, 800$
31	15	1024
37	1	$36^2, 48, 72^2, 160, 192,$ $216, 288, 384$
37	2	$24, 48, 72, 216, 576, 672$
37	3	$24^2, 48, 768, 1056$
37	4	$4, 48, 192, 384, 960$
37	5	$24^2, 48, 1792$
37	8	$24, 48, 480, 1152$
37	9	$24, 48, 160, 192, 1312$
37	10	$24, 48, 1664$
37	15	$48, 160, 192^2, 288, 624$

TABLE 8. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
41	1	$48, 64, 160, 1632$
41	2	$24, 40, 48, 96, 120, 192, 1536$
41	3	$24, 48, 192, 1824$
41	4	$4, 24, 40, 48, 72, 120, 160, 192^3, 216, 288, 576$
41	6	$16, 24, 48^2, 192, 1632$
41	7	$24, 48, 192, 1792$
41	8	$24, 48, 192, 1792$
41	11	$24, 48, 384, 1600$
41	12	$24^2, 48, 2160$
41	16	$48, 96, 192, 1440$
43	1	1728
43	2	$24, 48, 144, 1536$
43	3	$24, 1536$
43	4	$4, 1856$
43	5	$24, 1408$
43	6	1632
43	7	1936
43	8	1968
43	9	1760
43	10	$24, 64, 1600$
43	11	1936
43	12	$256, 1504$
43	13	$24, 1408$
43	14	1728
43	15	2032
43	16	$24, 1408$
43	17	$24, 384, 1024$
43	18	1968
43	19	$24, 1664$
43	20	$24, 256, 1408$
43	21	$24, 1728$

p	k	orbit sizes
47	1	$24, 1712$
47	2	2304
47	3	2112
47	4	$4, 1920$
47	5	$24, 2080$
47	6	2336
47	7	$64, 2016$
47	8	$24, 2080$
47	9	$24, 1776$
47	10	$24, 2080$
47	11	$64, 96, 160, 288, 1728$
47	12	$24, 64, 2016$
47	13	$24, 2080$
47	14	1984
47	15	$24, 1776$
47	16	$864, 1216$
47	17	2304
47	18	2336
47	19	$24, 1712$
47	20	$24, 2016$
47	21	$24, 1776$
47	22	2400
47	23	1984
53	1	$24^2, 48, 3456$
53	2	$48, 192, 2736$
53	3	$24^2, 48, 192, 3360$
53	4	$4, 48, 3072$
53	5	$24, 48, 64, 3168$
53	6	$24, 48, 192, 3040$
53	8	$48, 64, 192, 256, 336, 2016$
53	10	$24, 48, 192, 3072$
53	11	$24, 48, 64, 192, 288, 2688$
53	13	$24, 48, 192, 288, 2752$
53	15	$24, 48, 192, 2944$
53	17	$24, 48, 192, 3040$
53	22	$24, 48, 192^2, 2752$

TABLE 9. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
59	1	3232
59	2	3328
59	3	3360
59	4	$4, 3392$
59	5	$24, 2880$
59	6	$24, 3264$
59	7	3696
59	8	$24, 160, 2848$
59	9	$144, 160, 3328$
59	10	$24, 3008$
59	11	$24, 2880$
59	12	3792
59	13	$24, 3328$
59	14	$24, 2880$
59	15	$160, 3072$
59	16	$24, 3008$
59	17	3600
59	18	3232
59	19	3632
59	20	3328
59	21	$24, 3264$
59	22	3232
59	23	$24, 96, 288, 2944$
59	24	$24, 3328$
59	25	$24, 2880$
59	26	3632
59	27	$24, 3328$
59	28	$24, 3136$
59	29	3696
61	1	$24, 48, 4224$
61	2	$24^2, 48, 4512$
61	3	$24, 48, 192, 256, 384, 3424$
61	4	$4, 48, 192, 384, 3456$
61	5	$24^2, 48, 4480$
61	7	$24, 48, 192, 4032$
61	8	$24^2, 48, 192, 4288$
61	9	$24^2, 48, 192^2, 4192$
61	10	$36^2, 48, 72, 192, 288, 3168$
61	13	$48, 64, 544, 3248$
61	14	$24, 48, 352, 3904$
61	15	$24, 48, 96, 288^3, 3264$
61	19	$48, 192^2, 288, 3184$
61	20	$48, 288, 3568$
61	25	$24, 48, 192, 3936$
67	1	4320
67	2	$24, 4256$
67	3	$24, 3808$
67	4	$4, 4544$
67	5	$24, 4256$
67	6	4656
67	7	$24, 3936$
67	8	4624
67	9	$24, 4320$
67	10	$24, 3808$
67	11	4720
67	12	4352
67	13	$24, 4128$
67	14	4624
67	15	4352
67	16	$24, 3936$
67	17	4224
67	18	$24, 4256$
67	19	$24, 4256$
67	20	$24, 3936$
67	21	$24, 3808$
67	22	4720
67	23	4320
67	24	$24, 3808$
67	25	$24, 4128$
67	26	$480, 3840$
67	27	$96, 160, 288, 4080$
67	28	$288, 4528$
67	29	$24, 4320$
67	30	4624
67	31	$48, 144, 4032$
67	32	4352
67	33	$24, 3808$

TABLE 10. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
71	1	5280
71	2	4768
71	3	$24, 4560$
71	4	$4, 4608$
71	5	$24, 4800$
71	6	$24, 4864$
71	7	5376
71	8	$24, 4368$
71	9	5184
71	10	4864
71	11	5280
71	12	$24, 4304$
71	13	$96, 288, 384, 4096$
71	14	$24, 4864$
71	15	5216
71	16	$24, 4800$
71	17	$24, 4864$
71	18	$24, 4672$
71	19	5184
71	20	$24, 4864$
71	21	5216
71	22	4864
71	23	$24, 4368$
71	24	4864
71	25	4768
71	26	5216
71	27	$24, 4672$
71	28	$24, 4304$
71	29	4864
71	30	$24, 4304$
71	31	4864
71	32	5216
71	33	$24, 4368$
71	34	$24, 144, 4224$
71	35	$24, 4800$
73	1	$48, 192, 5248$
73	2	$24, 48, 96, 5760$
73	3	$24, 48, 64, 5920$
73	4	$4, 24, 40, 48, 120, 160, 192, 288^2, 1920, 2976$
73	5	$24^2, 48, 6448$
73	6	$48, 192, 5376$
73	7	$24, 48, 5952$
73	9	$24^2, 48, 6288$
73	10	$48, 192, 5248$
73	12	$24, 48, 192, 5792$
73	13	$48, 192, 672, 4576$
73	15	$48, 192, 544, 4704$
73	17	$24, 48, 192, 5760$
73	18	$24^2, 48, 160, 192, 6000$
73	20	$16, 24, 48^2, 192, 5728$
73	23	$24, 48, 5856$
73	26	$24^2, 48, 6256$
73	31	$24, 48, 192, 5792$
79	1	$24, 5856$
79	2	$24, 5424$
79	3	$24, 5488$
79	4	$4, 5760$
79	5	$24, 6048$
79	6	$24, 144, 5344$
79	7	5952
79	8	5792
79	9	$24, 5488$
79	10	$24, 5984$
79	11	$24, 5984$
79	12	$24, 5424$
79	13	6432
79	14	$24, 6048$
79	15	$24, 5488$
79	16	6400
79	17	$24, 5984$
79	18	6592
79	19	6400
79	20	6048
79	21	5952
79	22	$24, 5488$
79	23	6496
79	24	6496
79	25	6048
79	26	6432
79	27	$24, 5984$
79	28	6080
79	29	5792
79	30	6496
79	31	$24, 6048$
79	32	5952
79	33	$24, 5984$
79	34	6592
79	35	$96, 288, 6112$
79	36	$24, 96, 288, 5664$
79	37	$24, 5680$
79	38	5952
79	39	$24, 64, 5616$

TABLE 11. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
83	1	$24, 96, 288, 5664$
83	2	7248
83	3	6720
83	4	$4, 7040$
83	5	$24, 6176$
83	6	7088
83	7	$24, 6048$
83	8	$24, 6496$
83	9	$24, 6496$
83	10	$24, 6176$
83	11	7248
83	12	6720
83	13	$24, 6624$
83	14	7056
83	15	6688
83	16	6432
83	17	7088
83	18	$24, 6688$
83	19	7152
83	20	6688
83	21	$24, 6688$
83	22	7088
83	23	7088
83	24	6592
83	25	$24, 6496$
83	26	6592
83	27	$24, 6048$
83	28	$24, 96, 288, 6304$
83	29	$24, 6048$
83	30	6688
83	31	6688
83	32	$24, 6176$
83	33	$24, 6176$
83	34	$24, 6176$
83	35	7056
83	36	7088
83	37	$24, 6624$
83	38	$24, 6048$
83	39	$24, 64, 6624$
83	40	$24, 6496$
83	41	6688

p	k	orbit sizes
89	1	$24, 48, 192^2, 8320$
89	2	$24, 48, 96, 192, 8320$
89	3	$24, 48, 96, 192, 288^2, 7872$
89	4	$4, 24, 48, 160^2, 192^2, 288^2, 3264, 4512$
89	5	$24, 48, 8608$
89	6	$24, 48, 192, 8416$
89	7	$48, 192, 288, 7584$
89	9	$24, 48, 8448$
89	10	$24, 48, 8448$
89	11	$24, 48, 192, 8512$
89	12	$24^2, 48, 9264$
89	14	$24^2, 48, 9072$
89	15	$16, 48^2, 8128$
89	17	$48, 8192$
89	19	$24^2, 48, 144, 192^2, 8640$
89	20	$48, 192, 7872$
89	22	$24, 48, 8608$
89	25	$24, 48, 8736$
89	27	$24, 48, 8704$
89	30	$40, 48, 120, 8032$
89	33	$24, 48, 8704$
89	38	$24^2, 48, 144, 192, 8768$

TABLE 12. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
97	1	$48, 192, 9504$
97	2	$24, 48, 96, 672, 9408$
97	3	$16, 24, 48^2, 160, 10080$
97	4	$4, 24, 48, 192, 960, 3840, 5408$
97	5	$24, 48, 10304$
97	6	$48, 192, 9376$
97	7	$24^2, 48, 10672$
97	8	$24, 48, 10304$
97	10	$48, 192, 9376$
97	11	$24, 40, 48, 120, 9856$
97	12	$24, 48, 10304$
97	14	$24, 48, 192, 10080$
97	15	$24, 48, 10304$
97	16	$48, 9696$
97	19	$24^2, 48, 10864$
97	20	$24, 48, 192^2, 9792$
97	21	$48, 9696$
97	24	$24, 48, 192, 10080$
97	25	$24, 48, 192, 10080$
97	28	$24^2, 48, 192, 10576$
97	29	$24^2, 48, 192, 10512$
97	33	$24, 48, 192, 10080$
97	37	$24, 48, 96, 288^2, 9344$
97	42	$24, 48, 192, 9824$

p	k	orbit sizes
101	1	$24, 48, 192, 10912$
101	2	$24, 48, 11104$
101	3	$24, 48, 192, 10944$
101	4	$4, 48, 192^2, 288^2, 9792$
101	5	$24, 48, 192, 10912$
101	6	$24^2, 48, 11552$
101	7	$24^2, 48, 11712$
101	8	$24, 48, 192^2, 10464$
101	9	$24^2, 48, 192, 11360$
101	12	$48, 60^2, 120, 192^2, 9728$
101	13	$24^2, 48, 192, 11328$
101	14	$24, 48, 352, 10656$
101	15	$48, 10608$
101	16	$24, 48, 160, 192, 10656$
101	17	$24, 48, 11104$
101	18	$24^2, 48, 11552$
101	23	$48, 10352$
101	24	$24, 48, 11008$
101	25	$48, 64, 96^2, 144, 192, 288^2, 9184$
101	26	$24, 48, 11104$
101	27	$24, 40, 48, 120, 192, 480, 10272$
101	34	$48, 144, 192, 10080$
101	35	$48, 10416$
101	36	$24^2, 40, 48, 120, 192^2, 11296$
101	45	$24, 48, 192, 10816$

TABLE 13. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
103	1	10112
103	2	24, 10304
103	3	10400
103	4	4, 9984
103	5	24, 9616
103	6	10368
103	7	24, 10176
103	8	11136
103	9	10400
103	10	10272
103	11	24, 9616
103	12	24, 9984
103	13	24, 9552
103	14	10848
103	15	96, 288, 10464
103	16	24, 9552
103	17	11008
103	18	10816
103	19	24, 9808
103	20	64, 10048
103	21	24, 10368
103	22	24, 10368
103	23	10368
103	24	10848
103	25	10400

p	k	orbit sizes
103	26	10912
103	27	11008
103	28	10400
103	29	24, 9616
103	30	24, 9616
103	31	24, 9616
103	32	24, 10176
103	33	11008
103	34	24, 10176
103	35	24, 64, 10240
103	36	10112
103	37	24, 9616
103	38	10112
103	39	24, 10304
103	40	64, 10944
103	41	24, 9808
103	42	24, 9808
103	43	24, 10368
103	44	24, 9808
103	45	24, 10304
103	46	10272
103	47	24, 10304
103	48	10848
103	49	24, 10304
103	50	10816
103	51	10912

TABLE 14. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
107	1	24, 11136
107	2	11696
107	3	24, 10752
107	4	4, 11264
107	5	24, 10368
107	6	11104
107	7	24, 11008
107	8	24, 96, 288, 10624
107	9	96, 288, 11280
107	10	11104
107	11	24, 10496
107	12	11232
107	13	11696
107	14	11200
107	15	24, 10368
107	16	11696
107	17	11696
107	18	10944
107	19	11104
107	20	11760
107	21	11104
107	22	24, 11136
107	23	24, 10368
107	24	24, 64, 10432
107	25	11664
107	26	11664

p	k	orbit sizes
107	27	11760
107	28	24, 10816
107	29	24, 10368
107	30	11984
107	31	24, 10496
107	32	11856
107	33	24, 10496
107	34	11200
107	35	11104
107	36	11984
107	37	24, 10368
107	38	24, 10496
107	39	11200
107	40	24, 10496
107	41	11200
107	42	24, 11136
107	43	24, 11008
107	44	24, 11008
107	45	24, 11136
107	46	10944
107	47	24, 10496
107	48	24, 288, 10912
107	49	11984
107	50	24, 96, 288, 10816
107	51	11200
107	52	24, 11200
107	53	24, 11200

TABLE 15. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

p	k	orbit sizes
109	1	$24, 48, 12864$
109	2	$24^2, 48, 13408$
109	3	$24^2, 48, 13632$
109	4	$4, 48, 192, 12288$
109	5	$24, 48, 192^2, 12224$
109	6	$24, 48, 12768$
109	7	$48, 12112$
109	8	$24^2, 48, 192, 13312$
109	9	$24, 48, 12864$
109	11	$24^2, 48, 13504$
109	12	$48, 192, 288, 11568$
109	14	$24, 48, 12768$
109	15	$24, 48, 192, 12576$
109	16	$24, 48, 192, 12416$
109	18	$48, 192^2, 11920$
109	19	$48, 12304$
109	21	$24, 48, 192^3, 12032$
109	22	$24, 48, 160, 12736$
109	24	$24, 48, 12864$
109	25	$24, 48, 64, 96, 192, 288, 11968$
109	28	$24, 48, 192, 12704$
109	31	$24^2, 48, 192, 480, 12672$
109	32	$24, 48, 192, 12416$
109	35	$48, 192, 11920$
109	38	$24, 48, 192, 12576$
109	41	$48, 12304$
109	48	$24^2, 48, 13408$

p	k	orbit sizes
113	1	$24, 48, 13792$
113	2	$48, 96, 192, 12672$
113	3	$24^2, 48, 14256$
113	4	$4, 24, 48, 6656, 7488$
113	5	$24^2, 40, 48, 120, 192, 480, 13456$
113	6	$24, 48, 192, 13344$
113	7	$48, 13088$
113	9	$24, 48, 192, 13504$
113	10	$48, 288, 12800$
113	11	$24, 48, 160, 192^2, 13152$
113	12	$24, 48, 13824$
113	13	$24, 48, 192, 13344$
113	14	$24, 48, 192, 256, 13344$
113	17	$48, 12960$
113	18	$24^2, 48, 14288$
113	19	$48, 192, 12768$
113	20	$24, 48, 192, 288, 13312$
113	21	$40, 48, 120, 192^2, 480, 12064$
113	25	$16, 24, 48^2, 13728$
113	26	$24^2, 48, 192, 14160$
113	27	$48, 13088$
113	28	$24, 48, 96, 192, 13408$
113	33	$24^2, 48, 14256$
113	34	$48, 192^3, 12768$
113	35	$24, 48, 96, 192, 288^2, 12832$
113	41	$24^2, 48, 14448$
113	42	$24^2, 48, 14288$
113	49	$24, 48, 13824$

TABLE 16. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 9.3. The notation N^d indicates d orbits of size N.

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