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The top-weight cohomology of A_g

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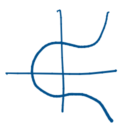
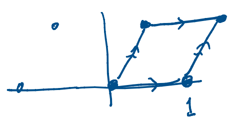
elliptic curve



$\mathbb{C}/\Lambda$ $\quad \Lambda \cong \mathbb{Z}^2$



$\Lambda = \langle 1, \tau \rangle$ $\quad \operatorname{Im} \tau > 0.$

$M_{1,1} \cong \mathbb{H} / SL_2\mathbb{Z}.$

moduli of elliptic curves

M_g moduli space of curves of genus $g \geq 2$ $\xrightarrow{t_g}$ A_g moduli space of principally polarized abelian varieties of dimension g.

Tropical moduli: Chan–Galatius–Payne

$\mathbb{C}^g/\Lambda$ $\quad \Lambda \cong \mathbb{Z}^{2g}$

is an abelian variety iff

Λ generated by the columns of $g\big[\underbrace{\,Id \mid \tau\,}_{2g}\big]$

where τ symmetric with $\operatorname{Im} \tau > 0.$

$M_g \cong \text{Teich}/\text{Mod}_g.$

Let X complex variety. Deligne defined a weight filtration on $H^*(X;\mathbb{Q})$

$$0 = W_0 \subset \cdots \subset W_{2i} = W_{2i+1} = \cdots = H^i(X;\mathbb{Q}).$$

Let $\operatorname{Gr}_j^W H^i(X;\mathbb{Q}) = W_j H^i(X;\mathbb{Q}) / W_{j-1} H^i(X;\mathbb{Q}).$

Let $d = \dim X$. We'll call $\operatorname{Gr}_{2d}^W H^*(X;\mathbb{Q})$ the top-weight cohomology of X.

$$\operatorname{Gr}_{2d}^W H^*(X;\mathbb{Q}) \twoheadleftarrow H^*(X;\mathbb{Q})$$

Λ generated by the columns of $[\underbrace{I \quad \tau}_{2g}]$,

where τ symmetric with $\operatorname{Im}\tau > 0$.

thus $\underbrace{\mathcal{A}_g \cong \mathbb{H}_g}_{\text{Siegel upper half space}} / Sp(2g,\mathbb{Z})$

$H^*(X;\mathbb{Q})$

$$M_g \cong \mathrm{Teich}/\mathrm{Mod}_g.$$

$$H^*(M_g;\mathbb{Q}) \cong H^*(\mathrm{Mod}_g;\mathbb{Q})$$

$$H^*(A_g;\mathbb{Q}) \cong H^*(Sp(2g,\mathbb{Z});\mathbb{Q})$$

$$\dim_{\mathbb{C}} A_g = \frac{g(g+1)}{2}$$

Note: $H^*(A_2;\mathbb{Q})$ known. $Gr_6^W H^*(A_2;\mathbb{Q}) = 0$.

$H^*(A_3;\mathbb{Q})$ computed by Hain (2002), $Gr_{12}^W H^k(A_3;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=6 \\ 0 & \text{else.} \end{cases}$

$H^*(A_4;\mathbb{Q})$ not fully understood, but $Gr_{20}^W H^*(A_4;\mathbb{Q}) = 0$ from Hulek-Tommasi (2012).

Thm (BBCMMW)

$$Gr_{30}^W H^k(A_5;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=15,20 \\ 0 & \text{else.} \end{cases}$$

$$Gr_{42}^W H^k(A_6;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=30 \\ 0 & \text{else} \end{cases}$$

$$Gr_{56}^W H^k(A_7;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=28,33,37,42 \\ 0 & \text{else.} \end{cases}$$

Problem (Grushevsky *The geometry of A_g and its compactifications* '09): Does A_g or any of its compactifications have cohomology in odd degree? Rem: For M_g, such classes are known to abound, Tomassi '05 $H^5(M_4;\mathbb{Q}) \neq 0$, $\boxed{\text{Harer-Zagier}}$

Thm (BBCMMW)

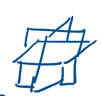
$$Gr_{72}^W H^i(A_8;\mathbb{Q}) = 0 \qquad i \geq 60,$$

$$Gr_{90}^W H^i(A_9;\mathbb{Q}) = 0 \qquad i \geq 79,$$

$$Gr_{110}^W H^i(A_{10};\mathbb{Q}) = 0 \qquad i \geq 99.$$

$/\mathbb{C}$

① X variety, smooth, $\overline{X} \supset X$ compactification with $D = \overline{X} \setminus X = D_1 \cup \cdots \cup D_s$ D_i smooth, intersect transversely. (snc).



Fix $j \geq 0$.

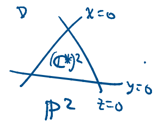
$$0 \to H^j(\overline{X};\mathbb{Q}) \xrightarrow{\delta_0} \bigoplus_{i_0} H^j(D_{i_0};\mathbb{Q}) \xrightarrow{\delta_1} \bigoplus_{i_0<i_1} H^j(D_{i_0} \cap D_{i_1};\mathbb{Q}) \xrightarrow{\delta_2} \cdots \to 0.$$

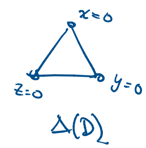
Then $\mathrm{Gr}^W_j H_c^{i+j}(X;\mathbb{Q}) \cong \ker \delta_i / \operatorname{im} \delta_{i-1}$.

$\quad \cong$

$(\mathrm{Gr}^W_{2d-j} H^{2d-(i+j)}(X;\mathbb{Q}))^\vee$

When $j=0$, this complex is $\cong$ the cochain complex associated to the dual complex of D $\Delta(D)$





② Toroidal compactifications of $\mathcal{A}_g$. A toroidal compactification is $(\mathcal{U} \subset X)$ complete that is étale/analytically locally $\underbrace{\ }_{\text{isomorphic to}}$ $T \subset X_\Sigma$

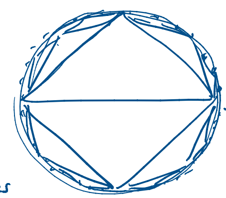
Let Ω_g positive define cone $\subset \mathrm{Sym}^2 \mathbb{R}^g$

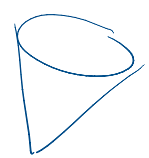
$\cap$

$\Omega_g^{rat} = \{$ positive semidefinite $g \times g$ matrices with rational nullspace $\}$.

$GL_g\mathbb{Z} \curvearrowright \Omega_g, \Omega_g^{rat}$

$A \cdot X = AXA^T$.





Def. An admissible decomposition of Ω_g^{rat} is a face-to-face tiling by polyhedral rational cones which are permuted under $GL_g\mathbb{Z}$ into finitely many orbits.

$\Omega_2 \subset \Omega_2^{rat}$ $\quad [\cdot\ \cdot]$ $\quad g=2$.

Voronoi 1908

Given an adm decomp Σ, one obtains a toroidal compactification

$\mathcal{A}_g \subset \overline{\mathcal{A}_g}^\Sigma$.

We study the perfect cone ~~~~ admissible decomposition of $\mathcal{A}_g$ $\quad \subset \overline{\mathcal{A}_g}^{perf}$.

$\Delta(\mathcal{A}_g \subset \mathcal{A}_g^\Sigma) = \underbrace{\Sigma / GL_g\mathbb{Z}}$. $\quad$ Ex $g=2$

studied using Elbaz-Vincent-Gangl-Soulé

interpretable as a moduli space of tropical p.p.a.v.s (Brannetti-Melo-Viviani)

$0 \to P^{(g-1)} \to P^{(g)} \to \underset{\sim}{V}^{(g)} \to 0$

$$0 \longrightarrow P \longrightarrow P \longrightarrow \underbrace{V}_{\text{Voronoi complex}} \longrightarrow 0$$

q

21	0	0	0	0	0	0	0	$\mathbb{Q}$			
20	0	0	0	0	0	0	0	0			
19	0	0	0	0	0	0	0	0			
18	0	0	0	0	0	0	0	0			
17	0	0	0	0	0	0	0	0			
16	0	0	0	0	0	0	0	$\mathbb{Q}$			
15	0	0	0	0	0	0	0	0			
14	0	0	0	0	0	0	0	0			
13	0	0	0	0	0	0	0	0			
12	0	0	0	0	0	0	0	$\mathbb{Q}$			
11	0	0	0	0	0	0	0	0			
10	0	0	0	0	0	$\mathbb{Q}$	0	0			
9	0	0	0	0	0	0	0	0			
8	0	0	0	0	0	0	0	0			
7	0	0	0	0	0	0	0	$\mathbb{Q}$			
6	0	0	0	0	0	0	$\mathbb{Q}$	0			
5	0	0	0	0	0	$\mathbb{Q}$	0	0			
4	0	0	0	0	0	0	0	0	0		
3	0	0	0	$\mathbb{Q}$	0	0	0	0	0		
2	0	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9	10

$\to p$

$\mathbb{Q}$

TABLE 4. The page $E^1_{p,q} = \mathrm{Gr}^W_0 H^{p+q}_c(\mathcal{A}_p;\mathbb{Q}) \Rightarrow \mathrm{Gr}^W_0 H^{p+q}(\mathcal{A}^{\mathrm{Sat}}_g;\mathbb{Q})$ of the Gysin spectral sequence, for g sufficiently large. The blank entries for $p \geq 8$ are currently unknown.

Chenevier-Lee, Chen-Looijenga

$\exists$ cohomology classes in wt 0 of A_g^{Sat}

$y_6, y_{10}, y_{14}, \ldots$

$$\mathrm{Gr}^W_0 H^{p+q}_c(A_p;\mathbb{Q}) \cong \mathrm{Gr}^W_{2d} H^{2d-p-q}(A_p;\mathbb{Q}).$$

$$A_g^{Sat} = A_g \sqcup A_{g-1} \sqcup \cdots \sqcup A_0$$

$$\mathrm{Gr}^W_0 H^j_c(A_g;\mathbb{Q}) \times \mathrm{Gr}^W_{(g+1)g} H^{(g+1)g-j}(A_g;\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}.$$