2-dim COHASs of curves and surfaces, and their categorification

The present talk is based on arXiv:1903.07253, arXiv:2004.13685, and ongoing project with Diaconescu, Schiffmann, and Vasserot

1. Heuristics about COHASs

- $A = \text{(nice) abelian category}$
- $M_A = \text{moduli stack of objects of } A$
- $M_A^{\text{ext}} = \text{moduli stack of extensions of objects of } A$

We have a “convolution diagram”:

\[
\begin{align*}
\text{ev}_2 \times \text{ev}_2 &= p \\
M_A \times M_A &\xrightarrow{p} M_A^{\text{ext}} \\
&\xrightarrow{q = \text{ev}_2} M_A
\end{align*}
\]

where:

- $p: (0 \to E_2 \to E \to E_2 \to 0) \mapsto (E_2, E_2)$
- $q: (0 \to E_2 \to E \to E_2 \to 0) \mapsto E$

Fix a homology theory with “nice” functorial properties: $A^*(-)$
Examples:

- $A_*(-) = H_*^{BM}(-) = \text{Borel-Moore homology}$
- $A_*(-) = G_0(-) = \text{Grothendieck group of coherent sheaves}$
- $A_*(-) = \text{Chow group}$

We would like to define the **Cohomological Hall algebra** of $A$:

\[
H_A := \text{associative algebra } (A_*(M_A), m = \text{product}) : \\
m: A_*(M_A) \otimes A_*(M_A) \xrightarrow{\otimes} A_*(M_A \times M_A) \xrightarrow{q_* \circ p^*} A_*(M_A)
\]

⚠️ This definition works only if $\text{gl.dim.}(A) \leq 2$, indeed:

- $q$ is proper representable $\Rightarrow \exists q_* = \text{proper push-forward, but:}$
- If $\text{gl.dim.}(A) = 1 \Rightarrow p$ is smooth $\Rightarrow \exists p^* = \text{pull back}$
- If $\text{gl.dim.}(A) = 2 \Rightarrow p^*$ has to be defined carefully (as we will see later)

Remark: if $\text{gl.dim.}(A) = 3$, one has to use Kontsevich-Soibelman's theory of COHAs which I am not going to introduce today.
2. Motivating example

Let

- \( \mathcal{A} = \text{category of 0-dim coherent sheaves on } \mathbb{C}^2 \)

- \( \mathcal{M}_\mathcal{A} = \text{Coh}_0(\mathbb{C}^2) \) = moduli stack of 0-dim coherent sheaves on \( \mathbb{C}^2 \)

\( \sim \bigcup_{d \geq 0} \text{Coh}_{0,d}(\mathbb{C}^2) \) -- these stacks have a "simple" explicit description

stratified w.r.t. the number of pts ("counted with multiplicities") of the support

- \( \text{Coh}_{0,d}(\mathbb{C}^2) \cong \left[ \left\{ (A, B) \in \text{Mat}(d, \mathbb{C}) : [A, B] = 0 \right\} \bigg/ \text{GL}(d, \mathbb{C}) \right] \)

\( C_d = \text{commuting variety} \)

- \( A^*_\mathcal{A}(-) = H^*_{\text{equiv}}(-) = \text{equivariant Borel-Moore homology} \)

**Theorem** (Schiffmann-Vasserot)

1. \( \exists \) associative algebra structure on

\[ H_A = \bigvee_{\mathbb{C}^2} = H^*_{\text{equiv}}(\text{Coh}_0(\mathbb{C}^2)) \cong \bigoplus_{d \geq 0} H^*_{\text{equiv}}(\text{GL}(d) \times (\mathbb{C}^*)^2)(C_d) \]
2. \((\mathcal{Y}_{C^2})^+_\text{loc} = \text{positive nilpotent part of Maulik-Okounkov Yangian} \ (\mathcal{Y}_{\text{1-loop}}^+)_\text{quiver loc} \) (deformation of \(U(gl(1)[u])\))

**Important Slogan:**

\(\mathcal{Y}_{\text{1-loop}}^+\) is the "largest" algebra acting on \(\mathcal{H}_{x}^{C^* \times C^*} (\text{Hilbert schemes of points on } C^2)\)

**Consider:**

\[ \text{Hilb}^n := \text{Hilb}^n(C^2) = \text{Hilbert scheme of } n \text{ pts in } C^2 \]

\[ = \text{moduli space of } 0\text{-dim. subschemes } Z \subset C^2 \text{ with } \dim H^0(O_Z) = n \]

**Fact:** \(\text{Hilb}^n = \text{smooth quasi-projective variety of dim. } 2n\)

**Set:** \(\text{Hilb} = \bigsqcup_{n \geq 0} \text{Hilb}^n\)

**Theorem (Schiffmann-Vasserot):**

1. \(\mathcal{Y}_{C^2}\) acts on \(\mathcal{H}_{x}^{C^* \times C^*}(\text{Hilb})\)
2. This induces an action of \((\mathcal{Y}_{\text{1-loop}}^+)_\text{quiver loc}\) on \(\mathcal{H}_{x}^{C^* \times C^*}(\text{Hilb})\)_loc
1. This result generalizes Grojnowski-Nakajima's construction of an action:

\[ \text{Heisenberg algebra} = \text{Heis} \otimes H^*_c(C^* \times C^*(\text{Hilb}))_{\text{loc}} \]

2. Negut described explicitly the action (2) in terms of explicit "geometric" operators.

**Important:** Negut's operators have been generalized in two different contexts:

- Negut: study of \( K_0 \) (moduli space of stable sheaves on K3 surface)
- Maulik-Negut: study of \( A_*(\text{Hilb}(K3)) \) and Beauville-Voisin's conjectures

3. 2-dim. COHAs of quivers

Note that in the previous example:

\[ \text{preprojective algebra of } 1\text{-loop quiver} \]

\[ \text{Coh}_{p,d}(C^2) \cong \left[ C_d / GL(d) \right] \cong \text{Rep}(\mathbb{W}_{1\text{-loop}}, d) \]

\( \rightarrow \mathfrak{g}_{\text{e}^2} \) is an example of 2-dim. COHAs of quivers.
The general framework is as follows:

1. $\mathcal{Q} = \text{quiver} = (\mathcal{Q}_0 = \{\text{vertices}\}, \mathcal{Q}_1 = \{\text{edges}\})$

2. $\mathcal{Q}^{db} = \text{double quiver} = (\mathcal{Q}_0, \mathcal{Q}_1^{db} = \mathcal{Q}_1 \cup \mathcal{Q}_1^{opp})$

3. $\prod_{\mathcal{Q}} \equiv \mathbb{C} \mathcal{Q}^{db} / \left< \sum_{e \in \mathcal{Q}_1^{db}} x_e x_e^* - x_e^* x_e \right> = \text{quotient of the path algebra } \mathbb{C} \mathcal{Q}^{db} \text{ by the preprojective relations}$

Then

1. $\mathcal{A} = \text{category of f.d. representations of } \prod_{\mathcal{Q}}$

2. $\mathcal{M}_\mathcal{A} = \text{Rep}(\prod_{\mathcal{Q}}) \cong \text{T}^* \text{Rep}(\mathcal{Q}) \xrightarrow{\text{categorical stack}} \text{Rep}(\mathcal{Q})$

3. $A_* = \text{BM homology, } G_0 = \text{Grothendieck group of coh. sheaves, etc}$

**Theorem** (Schiffmann-Vasserot, Yang-Zhao)

An associative algebra structure on

$$\mathcal{H}_{\mathcal{Q}}^{(c^*)} = A_* \left( \text{Rep}(\prod_{\mathcal{Q}}) \right)$$
4. COHA's of Curves and Surfaces, and their categorification

$A =$ (nice) abelian category of gl.dim $(A) = 2$

⚠️: a problem in defining $\mathcal{H}_A$ is the existence of $p^*$!

Previous solutions:

- Schiffmann-Vasserot for $A = \{ f.d. representations of T_{\mathbb{Q}} \}:
  \operatorname{Rep}(T_{\mathbb{Q}}) =$ quotient stack and $\sim T^*\operatorname{Rep}(\mathbb{Q}) \Rightarrow p^* =$ refined Gysin pullback

- S.-Schiffmann, Minets in $rK=0$: construction of $\mathcal{H}_A$ for $A = \{ $ Higgs sheaves $(F, F \to F \otimes \omega_X)$ on a smooth projective curve $X/\mathbb{C} \}:
  M_A$ is locally of the form $T^*[Z/G] \Rightarrow p^* =$ refined Gysin pullback

- Kapranov-Vasserot, Yu Zhao in $rK=0$: construction of $\mathcal{H}_A$ for $A = \{ $ (properly supported) coherent sheaves on smooth (quasi-)proj. surface $\}:
  p^* =$ virtual pull back (à la Behrend-Fantechi)

Problems: These approaches could NOT be useful for an arbitrary heart of a t-structure and they are NOT suitable for categorification

Pocket-S.'s solution: use of derived algebraic geometry
$S =$ smooth (quasi-)projective surface / $\mathbb{C}$.

**Proposition 1** (Patak - S.)

$\exists$ derived enhancements $\mathsf{IRGl}(S)$ and $\mathsf{IRCoh}^{\text{ext}}(S)$ of $\mathsf{Gl}(S)$ and $\mathsf{Coh}^{\text{ext}}(S)$ s.t.

The derived map

$$\mathsf{IRp} : \mathsf{IRCoh}^{\text{ext}}(S) \longrightarrow \mathsf{IRGl}(S) \times \mathsf{IRGl}(S)$$

is derived l.c.i.

**Consequence:**

$\exists$ $\mathsf{IRp}^{\ast} : \mathsf{Coh}^{b}(\mathsf{IRCoh}^{\text{ext}}(S)) \longrightarrow \mathsf{Coh}^{b}(\mathsf{IRGl}(S) \times \mathsf{IRGl}(S))$

Here, $\mathsf{Coh}^{b}(-) =$ dg enhancement of $\mathsf{D}^{b}(\mathsf{Coh}(-))$.

⚠️ $\mathsf{IRCoh}(S)$ has an hidden "algebraic structure" which encodes the "higher associative conditions" associated to the "iterated" convolution diagrams.

**Proposition 2** (Patak - S.)

$\exists$ a simplicial derived stack $\mathsf{S} \cdot \mathsf{IRCoh}(S) \in \mathsf{Fun}(\Delta^{\text{op}}, \mathsf{dGeom})$ s.t.

$$\mathsf{S}_{0} \mathsf{IRCoh}(S) \simeq \mathbb{C}, \quad \mathsf{S}_{1} \mathsf{IRCoh}(S) \simeq \mathsf{IRCoh}(S), \quad \mathsf{S}_{2} \mathsf{IRCoh}(S) \simeq \mathsf{IRCoh}^{\text{ext}}(S)$$

which is a 2-Segal space in the sense of Dwyer-Hoff-Kapranov.
Theory of 2-Segal spaces + Proposition 2 imply:

\[ \text{IRCoh}(S) \text{ has the structure of an IE,-algebra in } \text{Corr}(d\text{Geom})_{rps,all} \]

represented by proper schemes

Here:

\[ \text{Corr}(d\text{Geom})_{rps,all} = \left\{ \begin{array}{l}
\text{objects: geometric derived stacks} \\
\text{morphisms } X \rightarrow Y: X \leftarrow P \rightarrow Z \\
 q \downarrow \\
 Y \end{array} \right. \]

- rps morphism

**Example:**

- The convolution diagram is a morphism in \( \text{Corr}(d\text{Geom})_{rps,all} \):

\[
\begin{array}{c}
\text{IRCoh}(S) \times \text{IRCoh}(S) \\
\downarrow \text{IRq} \\
\text{IRCoh}(S)
\end{array}
\begin{array}{c}
\text{IRcoh}(S) \times \text{IRcoh}(S) \\
\downarrow \text{IRq} \\
\text{IRcoh}(S)
\end{array}
\]

\[
\begin{array}{c}
\text{IRcoh}(S) \times \text{IRcoh}(S) \\
\downarrow \text{IRq} \\
\text{IRcoh}(S)
\end{array}
\begin{array}{c}
\text{IRcoh}(S) \times \text{IRcoh}(S) \\
\downarrow \text{IRq} \\
\text{IRcoh}(S)
\end{array}
\]

- Similarly, all the iterated convolution diagrams are morphisms in \( \text{Corr}(d\text{Geom})_{rps,all} \)
Proposition 1 implies:

\[ \text{IRcoh}(S) \text{ has the structure of an } \mathcal{E}_1 \text{-algebra in } \text{Corr} \text{(dGeom)} \]

\[ \text{rps, lc.i.} \]

\[ \text{l.c.i. morphisms} \]

\[ \text{represented by proper schemes} \]

Gaitsgory-Rozenblyum: \( \exists \text{ lax monoidal functor } \text{Coh}^b: \text{Corr} \text{(dGeom)} \text{ rps, lc.i.} \rightarrow \text{Cat}_{\text{stable}} \)

**Theorem (Porta-S.)**

\( \text{Coh}^b(\text{IRcoh}(S)) \) is endowed with the structure of an \( \mathcal{E}_1 \) -algebra in \( \text{Cat}_{\text{stable}} \).

**Corollary** \( \text{D}(\text{Coh}(\text{IRcoh}(S))) \) is endowed with a monoidal structure \( \otimes_{\text{Hall}} \).

\[ \text{⚠️: } \text{D}(\text{Coh}(\text{IRcoh}(S))) \nless \text{D}(\text{Coh}(\text{Coh}(S))) \]

\[ \Rightarrow \text{available machinery to construct a monoidal structure on } \text{D}(\text{Coh}(\text{Coh}(S))) \]

\[ \text{⚠️: by passing to K-theory, we obtain } \mathcal{H}_{\text{Coh}(S)} \text{ for } \Lambda(-) = G_0(-) \]

**Important Example:** \( \text{Coh}^b_{(\mathbb{C}^*)^2}(\text{IRcoh}_0(\mathbb{C}^2)) \) categorifies \( \mathcal{H}_{\text{Coh}_0(\mathbb{C}^2)} \) for \( \Lambda(-) = G_0(-) \)

\[ \sim \text{ positive part of the Elliptic Hall algebra} \]
Curve case

\[ X = \text{smooth proj. curve/} \mathbb{C} \]

Consider

\[ A_{\text{dol}}(X) = \{ \text{Higgs sheaves on } X \} \]

\[ A_{\text{dR}}(X) = \{ \text{flat vector bundles on } X \} \quad \text{gl. dim. = } 2 \]

\[ A_B(X) = \{ \text{f.d. representations of } \pi_1(X) \} \]

**Theorem** (Pantev–Toën)

1. \( \exists \) \( \text{lie}_1 \)-algebra structures on

\[ \text{Coh}^b(\text{IRGcoh}_{\text{dol}}(X)), \text{Coh}^b(\text{IRGcoh}_{\text{dR}}(X)), \text{Coh}^b(\text{IRGcoh}_B(X)) \]

2. \( \exists \) categorified Hall type versions of the Riemann–Hilbert and non-abelian Hodge correspondences.

**Expectation**: \( \mathcal{C} = \text{CY2 category} \quad \exists \text{lie}_1 \)-algebra structure on \( \text{Coh}^b(\text{IRM}_\mathbb{C}) \)

**Remark**: Toda used the categorified Hall product to study his categorified DT invariants.
5. COHAS of quivers and Yangians

Recall again:

\[(Y_{\mathfrak{c}^2})_{\text{loc}} \simeq \text{positive nilpotent part of Maulik-Okounkov Yangian}\]

\[\left( Y_{\text{loop}} \right)_{\text{loc}} \quad \text{(deformation of } U(\hat{\mathfrak{gl}}(1)[n]) \text{)}\]

If one wants to remove \((-)_{\text{loc}}\) above, we need to replace:

\[\text{Rep}(\Pi_Q) \sim \Lambda_Q = \text{Lusztig(-Bozec) nilpotent stack} = \text{moduli stack of nilpotent representations of } \Pi_Q\]

\[\mathcal{H}_Q \sim \exists \text{ nilpotent COHA } Y_{\Lambda_Q}^{\text{nil}} := (A^*_s(\Lambda_Q), m)\]

\[Y_{\Lambda_Q}^{\text{nil}} := (H^{C^\ast}_s(\Lambda_Q), m)\]

**Theorem** (Schiffmann-Vasserot, Yeng-Zheo for $Q$ without loops)

\[\exists \text{ a surjective morphism } \Phi_Q : Y_Q \twoheadrightarrow Y_{\Lambda_Q}^{\text{nil}}\]

**Remark** $Y_Q$ = deformation of $U(g_{Q}[n])$:

\[\begin{cases} g_{Q}^{\text{H}} = \text{certain graded Borokhov-Kac-Moody algebra} \\ g_{Q}^{\text{H}}[0] \simeq g_{Q} \text{ BKM Lie algebra of } Q \end{cases}\]
Conjecture: $\psi_Q$ is an isomorphism.

Remark: conjecture true for $Q =$ one-loop, finite/affine Dynkin quivers.

6. COHA of the minimal resolution of ADE singularity (Diaconescu-S.-Schiffmann - Vasserot)

- $G \subset SL(2, \mathbb{C})$ finite group
- $Q^\text{fin} = \text{Dynkin diagram associated to } G$, $Q = \text{affine diagram corresponding to } Q^\text{fin}$
- $\pi : S \longrightarrow \mathbb{C}^2 / G$ minimal resolution of singularities
- $C := \pi^{-1}(0) = \bigcup_{i=1}^{N} C_i$, $C_i \cong \mathbb{P}^2$, $(C_i : C_j) = - (\text{Cartan matrix of } Q^\text{fin})$

Let

$$(\text{IR})\text{Coh}_C(S) = (\text{derived}) \text{ moduli stack of properly supported sheaves on } S$$

set-theoretically supported on $C$

$s\cong \mathrm{colim}_n (\text{IR})\text{Coh}(C^{(n)})$ w.r.t. closed embeddings $J_{n,m}$

Here

- $V_{n \geq 1}$, $C^{(n)} := \text{infinitesimal neighborhood of order } n \text{ of } C \text{ inside } Y$
- $\forall m \geq n$, $\exists$ closed embedding $J_{n,m} : (\text{IR})\text{Coh}(C^{(n)}) \longrightarrow (\text{IR})\text{Coh}(C^{(m)})$
Theorem
1. There exists an associative algebra structure \( Y_{S,C} \) on \( \mathbb{H}_x^{C^*}(\text{IR Coh}_C(S)) \)
2. There exists a surjective morphism of algebras

\[
Y_{x} \otimes \cdots \otimes Y_{x} \rightarrow Y_{x}
\]

where \( Y_{x} \) is the COHA structure on \( \mathbb{H}_x^{C^*}(\text{IR Coh}_C(T)) \), such that the kernel is generated by the relations "depending on two intersecting \( C_i \) and \( C_j \).

Remark: (2) follows from the "canonical filtration" of a sheaf \( \mathcal{E} \) on \( Y \) set-theoretically supported at \( C \). For example, \( N=2 \):

\[
0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0
\]

\( \text{max. subsheaf} \quad \text{quotient} \)

set-theoretically supported on \( C_1 \) \quad set-theoretically supported on \( C_2 \)

Expectations: \( \exists \ Y_{S,C} \rightarrow (Y_\mathbb{Q})^* \)

This is justified by the "derived McKay correspondence": \( D^b(Coh(S)) \xrightarrow{\Phi} D^b(\text{Mod}(\Pi_\mathbb{Q})) \)

\( \Phi \) should arise from a COHA associated to the whole \( D^b(Coh(S)) \)

\( \Phi \) does NOT preserve the t-structure \( \Rightarrow \) different t-structures give rise to different "halves" of \( Y_\mathbb{Q} \)
Let us restrict ourselves: \( S = T^* \mathbb{P}^1 \supset C = \mathbb{P}^1 \)

**Slogan:** the "building blocks" of \( \mathcal{Y}_{T^* \mathbb{P}^1, \mathbb{P}^1} \) are the semistable COHAs

\[
(\mathcal{I}R) \mathcal{C}oh_C(S) \overset{\text{ss}}{\longrightarrow} (\text{derived}) \text{ moduli stack of } D\text{-semistable}
\]
\[
\int_{\text{open}} \text{ properly supported sheaves on } S, \text{ of slope } \mu \in \mathbb{Q} \cup \{0\}
\]
\[
(\mathcal{I}R) \mathcal{C}oh_C(S) \overset{\text{set-theoretically supported on } C}{\longrightarrow}
\]

**Facts:**

\( \mu \in \mathbb{Q}: (\mathcal{I}R) \mathcal{C}oh_C(S) \overset{\text{ss}}{\longrightarrow} \wedge_{A_1} \Rightarrow H^*_c((\mathcal{I}R) \mathcal{C}oh_C(S) \overset{\text{ss}}{\longrightarrow}) \overset{\text{Y}}{\longrightarrow} \overset{\text{Y}^+}{\longrightarrow} (\mathfrak{sl}(2)) \)

\( \mu = \infty: (\mathcal{I}R) \mathcal{C}oh_C(S) \overset{\text{ss}}{\longrightarrow} (\text{derived}) \text{ moduli stack of } 0\text{-dim. sheaves on } S = T^* \mathbb{P}^1, \text{ set-theoretically supported on } C = \mathbb{P}^1 \)

\[\Rightarrow H^*_c((\mathcal{I}R) \mathcal{C}oh_C(S) \overset{\text{ss}}{\longrightarrow}) \overset{\text{Y}}{\longrightarrow} \overset{\otimes}{\longrightarrow} \overset{\text{Y}^+}{\longrightarrow} \overset{\text{Y}^+((\hat{\mathfrak{g}}/\mathfrak{a}))}{\longrightarrow} \left( \mathfrak{g}^{1/2} \right) \otimes \left( \mathfrak{g}^{1/2} \right)
\]

"Corresponding to the two fixed pts of \( \mathbb{P}^1 \)"
Theorem
1. $Y(\gamma_1^\ast, \gamma_1^-) \cong \gamma^\ast(\hat{g}(1))^\otimes 2 \times \gamma^\ast(Lsl(2))$

2. PBW decomposition: $Y(\gamma_1^\ast, \gamma_1^-) \cong \gamma^\ast(\hat{g}(1))^\otimes 2 \otimes_{\mu \in \mathbb{Z}} \gamma^\ast(s(\mathfrak{s}))$

Idea behind the Theorem:

\[(\text{IR Coh}\ C(S)) \cong \text{colim}_n (\text{IR Coh}\ C^{(n)})\]

Each $(\text{IR Coh}\ C^{(n)})$ can be exhausted by open substacks given by Hélder-Narasimhan stratification:

\[(\text{IR Coh}\ C^{(n)}) \cong \text{colim}_\ell (\text{IR Coh}\ C^{(n)}) \rightarrow \mu_{\text{min}} \rightarrow \mu_{\text{max}} \rightarrow \ell\]

Now, we "twist by $-\otimes_S(-D) =: T$":

\[(\text{IR Coh}\ C^{(n)}) \rightarrow \ell \leftarrow \text{T}^{\ell}((\text{IR Coh}\ C^{\otimes})_{\leq 0})\]
Summarizing:

\[ Y_{\mathcal{P}^{p}}, \mathcal{P}^{p} = \text{colim} \lim_{n} H_{x}^{C} (\bigwedge_{Q, n}^{\leq 0}) \]

Important Facts:

1. Under the derived McKay correspondence: \( T \hookrightarrow T_{s}^{\mathfrak{b}} \in \mathcal{B}_{Q}^{\mathfrak{e}} = \text{extended affine braid group} \)

2. \( H_{x}^{C} (\bigwedge_{Q}^{\leq 0}) \) is a quotient of \( H_{x}^{C} (\bigwedge_{Q}) = Y_{Q}^{\text{nil}} \approx Y_{Q}^{+} \)

\[ \Rightarrow \text{One needs to understand the compatibility between } \mathcal{B}_{Q}^{\mathfrak{e}} \text{ and } Y_{Q}^{\text{nil}}. \]