2-dim COHAs of curves and surfaces, and their categorification

The present talk is based on arXiv: 1903.07253, arXiv: 2004.13685, and ongoing project with Diaconescu, Schiffmann, and Vasserot

1. Heuristics about COHAs

- $\mathcal{A}$ = (nice) abelian category
- $\mathfrak{M}_{\mathcal{A}}$ = moduli stack of objects of $\mathcal{A}$
- $\mathfrak{M}_{\mathcal{A}}^{ext}$ = moduli stack of extensions of objects of $\mathcal{A}$

We have a "convolution diagram":

$$\begin{tikzcd}
 & \mathfrak{M}_{\mathcal{A}}^{ext} \arrow[dl, "ev_3 \times ev_1 = p"'] \arrow[dr, "q = ev_2"] & \\
\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} & & \mathfrak{M}_{\mathcal{A}}
\end{tikzcd}$$

where:

- $p : (0 \to \mathcal{E}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0) \longmapsto (\mathcal{E}_1, \mathcal{E}_2)$
- $q : (0 \to \mathcal{E}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0) \longmapsto \mathcal{E}$

Fix a homology theory with "nice" functorial properties: $A_*(-)$

Examples:

- $A_*(-) = H_*^{BM}(-)$ = Borel-Moore homology
- $A_*(-) = G_0(-)$ = Grothendieck group of coherent sheaves
- $A_*(-)$ = Chow group

We would like to define the Cohomological Hall algebra of $\mathcal{A}$:

$$\mathcal{H}_{\mathcal{A}} := \text{associative algebra } \left(A_*(\mathcal{M}_{\mathcal{A}}),\ m = \text{product}\right) :$$

$$m:\ A_*(\mathcal{M}_{\mathcal{A}}) \otimes A_*(\mathcal{M}_{\mathcal{A}}) \xrightarrow{\boxtimes} A_*(\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}}) \xrightarrow{q_* \circ p^*} A_*(\mathcal{M}_{\mathcal{A}})$$

⚠: This definition works only if $\mathrm{gl.dim.}(\mathcal{A}) \leq 2$, indeed:

- q is proper representable $\Rightarrow \exists\ q_*$ = proper pushforward,

but:

- If $\mathrm{gl.dim.}(\mathcal{A}) = 1 \Rightarrow p$ is smooth $\Rightarrow \exists\ p^*$ = pull back
- If $\mathrm{gl.dim.}(\mathcal{A}) = 2 \Rightarrow p^*$ has to be defined carefully (as we will see later)

Remark: if $\mathrm{gl.dim.}(\mathcal{A}) = 3$, one has to use Kontsevich-Soibelman's theory of COHAs which I am not going to introduce today.

2. Motivating example

Let

- $\mathcal{A}$ = category of 0-dim coherent sheaves on $\mathbb{C}^2$
- $M_{\mathcal{A}} = \underline{Coh}_0(\mathbb{C}^2)$ = moduli stack of 0-dim coherent sheaves on $\mathbb{C}^2$

$\simeq \coprod_{d \geq 0} \underline{Coh}_{0,d}(\mathbb{C}^2)$ – these stacks have a "simple" explicit description

stratified w.r.t. the number of pts ("counted with multiplicities") of the support

- $\underline{Coh}_{0,d}(\mathbb{C}^2) \simeq \Big[\underbrace{\{ (A,B) \in Mat(d,\mathbb{C}) : [A,B] = 0 \}}_{\substack{\parallel \\ C_d \text{ = commuting variety}}} / GL(d,\mathbb{C}) \Big]$

- $A_*(-) = H_*^{equiv}(-)$ = equivariant Borel-Moore homology

Theorem (Schiffmann-Vasserot)

1. $\exists$ associative algebra structure on

$$\boxed{\mathcal{H}_{\mathcal{A}} = \mathcal{Y}_{\mathbb{C}^2} = H_*^{\mathbb{C}^* \times \mathbb{C}^*}(\underline{Coh}_0(\mathbb{C}^2)) \simeq \bigoplus_{d \geq 0} H_*^{GL(d) \times (\mathbb{C}^*)^2}(C_d)}$$

2. $\boxed{(\mathcal{Y}_{\mathbb{C}^2})_{loc} \simeq \text{positive nilpotent part of Maulik-Okounkov Yangian } (\mathbb{Y}_{\substack{1\text{-loop}\\ \text{quiver}}})_{loc}}$

(deformation of $U(\widehat{\mathfrak{gl}}(1)[u])$)

Important Slogan:

$\boxed{\mathbb{Y}_{\substack{1\text{-loop}\\ \text{quiver}}} \text{ is the "largest" algebra acting on } H_*^{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilbert schemes of points on } \mathbb{C}^2)}$

Consider:

$$\mathrm{Hilb}^n := \mathrm{Hilb}^n(\mathbb{C}^2) = \text{Hilbert scheme of } n \text{ pts in } \mathbb{C}^2$$
$$= \text{moduli space of } 0\text{-dim. subschemes } Z \subset \mathbb{C}^2 \text{ with } \dim H^0(\mathcal{O}_Z) = n$$

Fact: Hilb^n = smooth quasi-projective variety of dim. $2n$

Set: $\mathrm{Hilb} = \bigsqcup_{n \geq 0} \mathrm{Hilb}^n$

Theorem (Schiffmann-Vasserot)

1. $\mathcal{Y}_{\mathbb{C}^2}$ acts on $H_*^{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb})$
2. This induces an action of $(\mathbb{Y}_{\substack{1\text{-loop}\\ \text{quiver}}})_{loc}$ on $H_*^{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb})_{loc}$

Remark

1. This result generalizes Grojnowski-Nakajima's construction of an action:

$$\text{Heisenberg algebra} = \mathcal{H}eis \curvearrowright H_*^{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb})_{loc}$$

2. Negut described explictly the action (2) in terms of explicit "geometric" operators

Important: Negut's operators have been generalized in two different contexts:

- Negut: study of K_0(moduli space of stable sheaves on K3 surface)
- Maulik-Negut: study of $A_*(\mathrm{Hilb}(K3))$ and Beauville-Voisin's conjectures

3. 2-dim. COHAs of quivers

Note that in the previous example:

$$\underline{\mathrm{Coh}}_{0,d}(\mathbb{C}^2) \simeq [C_d / GL(d)] \simeq \underline{\mathrm{Rep}}(\underbrace{\Pi_{\text{1-loop quiver}}}_{\text{preprojective algebra of 1-loop quiver}}, d)$$

$\Rightarrow \mathcal{Y}_{\mathbb{C}^2}$ is an example of 2-dim. COHAs of quivers

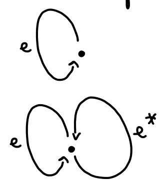
The general framework is as follows:

- Q = quiver = $\left(Q_0 = \{\text{vertices}\},\ Q_1 = \{e : i \longrightarrow j \text{ edges}\}\right)$

$$\rightsquigarrow Q^{db} = \text{double quiver} = \Big(Q_0,\ Q_1^{db} := Q_1 \sqcup \underbrace{Q_1^{opp}}_{\{e^* : j \longrightarrow i \ :\ \forall e \in Q_1\}}\Big)$$

Ex: Q = 1-loop quiver

Q^{db} :



- $\Pi_Q := \mathbb{C}Q^{db} \Big/ \langle \sum_{e \in Q_1^{db}} x_e x_{e^*} - x_{e^*} x_e \rangle$ = quotient of the path algebra $\mathbb{C}Q^{db}$ by the preprojective relations

Then

- $\mathcal{A}$ = category of f.d. representations of Π_Q

$\mathbb{C}^*$-action scaling the fibers

- $$\mathcal{M}_{\mathcal{A}} = \underline{\mathrm{Rep}}(\Pi_Q) \simeq \underbrace{T^* \underline{\mathrm{Rep}}(Q)}_{\text{cotangent stack}} \longrightarrow \underline{\mathrm{Rep}}(Q)$$

- A_* = BM homology, G_0 = Grothendieck group of coh. sheaves, etc

Theorem (Schiffmann–Vasserot, Yang–Zhao)

$\exists$ associative algebra structure on

$$\mathcal{H}_Q^{(\mathbb{C}^*)} = A_*^{(\mathbb{C}^*)}\left(\underline{\mathrm{Rep}}(\Pi_Q)\right)$$

4. COHAs of Curves and Surfaces, and their categorification

$\mathcal{A}$ = (nice) abelian category of gl. dim. $(\mathcal{A}) = 2$

⚠ : a problem in defining $\mathcal{H}_{\mathcal{A}}$ is the existence of p^*!

Previous solutions:

- Schiffmann-Vasserot for $\mathcal{A} = \{$f.d. representations of $\Pi_Q\}$:
  $\underline{Rep}(\Pi_Q)$ = quotient stack and $\simeq T^*\underline{Rep}(Q) \Rightarrow p^*$ = refined Gysin pullback

- S.-Schiffmann, Minets in rk=0: construction of $\mathcal{H}_{\mathcal{A}}$ for $\mathcal{A} = \{$Higgs sheaves $(\mathcal{F}, \mathcal{F} \longrightarrow \mathcal{F} \otimes \omega_X)$ on a smooth projective curve $X/\mathbb{C}\}$:
  $\mathcal{M}_{\mathcal{A}}$ is locally of the form $T^*[Z/G] \Rightarrow p^*$ = refined Gysin pull back

- Kapranov-Vasserot, Yu Zhao in rk=0: construction of $\mathcal{H}_{\mathcal{A}}$ for $\mathcal{A} = \{$(properly supported) coherent sheaves on smooth (quasi-) proj. surface$\}$:
  p^* = virtual pull back (à la Behrend-Fantechi)

Problems: These approaches could NOT be useful for an arbitrary heart of a t-structure and they are NOT suitable for categorification

> Porta-S.'s solution: use of derived algebraic geometry

Surface case

S = smooth (quasi-) projective surface $/\mathbb{C}$.

Proposition 1 (Porta-S.)

$\exists$ derived enhancements $\mathbb{R}\underline{\mathrm{Coh}}(S)$ and $\mathbb{R}\underline{\mathrm{Coh}}^{ext}(S)$ of $\underline{\mathrm{Coh}}(S)$ and $\underline{\mathrm{Coh}}^{ext}(S)$ s.t. the derived map

$$\mathbb{R}p : \mathbb{R}\underline{\mathrm{Coh}}^{ext}(S) \longrightarrow \mathbb{R}\underline{\mathrm{Coh}}(S) \times \mathbb{R}\underline{\mathrm{Coh}}(S)$$

is derived l.c.i.

Consequence: $\boxed{\exists\ \mathbb{R}p^* : \mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}^{ext}(S)) \longrightarrow \mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}(S) \times \mathbb{R}\underline{\mathrm{Coh}}(S))}$

Here, $\mathrm{Coh}^b(-)$ = dg enhancement of $D^b(\mathrm{Coh}(-))$.

⚠: $\mathbb{R}\underline{\mathrm{Coh}}(S)$ has an hidden "algebraic structure" which encodes the "higher associative conditions" associated to the "iterated" convolution diagrams.

Proposition 2 (Porta-S.)

$\exists$ a simplicial derived stack $S_\bullet\mathbb{R}\underline{\mathrm{Coh}}(S) \in \mathrm{Fun}(\Delta^{op}, \underset{\text{"category of geometric derived stacks"}}{\underbrace{\mathrm{dGeom}}})$ s.t.

$$\boxed{S_0\mathbb{R}\underline{\mathrm{Coh}}(S) \simeq \mathbb{C}, \quad S_1\mathbb{R}\underline{\mathrm{Coh}}(S) \simeq \mathbb{R}\underline{\mathrm{Coh}}(S), \quad S_2\mathbb{R}\underline{\mathrm{Coh}}(S) \simeq \mathbb{R}\underline{\mathrm{Coh}}^{ext}(S)}$$

which is a 2-Segal space in the sense of Dyckerhoff-Kapranov.

Theory of 2-Segal spaces + Proposition 2 imply:

$$\boxed{\mathbb{R}\,\underline{\mathrm{Coh}}(S) \text{ has the structure of an } \mathbb{E}_1\text{-algebra in } \mathrm{Corr}(\mathrm{dGeom})_{\mathrm{rps},\mathrm{all}}}$$

rps: represented by proper schemes

Here:

$$\mathrm{Corr}(\mathrm{dGeom})_{\mathrm{rps},\mathrm{all}} = \begin{cases} \text{objects: geometric derived stacks} \\[2ex] \text{morphisms } \mathfrak{X} \longrightarrow \mathcal{Y} : \quad \begin{array}{ccc} \mathfrak{X} & \xleftarrow{\ p\ } & \mathcal{Z} \\ & & \downarrow{\scriptstyle q} \ \text{– rps morphism} \\ & & \mathcal{Y} \end{array} \end{cases}$$

Example:

- The convolution diagram is a morphism in $\mathrm{Corr}(\mathrm{dGeom})_{\mathrm{rps},\mathrm{all}}$:

$$\begin{tikzcd} \mathbb{R}\,\underline{\mathrm{Coh}}(S) \times \mathbb{R}\,\underline{\mathrm{Coh}}(S) & \mathbb{R}\,\underline{\mathrm{Coh}}^{\mathrm{ext}}(S) \arrow[l, "\mathbb{R}p"'] \arrow[d, "\mathbb{R}q"] \\ & \mathbb{R}\,\underline{\mathrm{Coh}}(S) \end{tikzcd}$$

- similarly, all the iterated convolution diagrams are morphisms in $\mathrm{Corr}(\mathrm{dGeom})_{\mathrm{rps},\mathrm{all}}$

Proposition 1 implies:

l.c.i. morphisms

$$\boxed{\mathbb{R}\underline{\mathrm{Coh}}(S) \text{ has the structure of an } \mathbb{E}_1\text{-algebra in } \mathrm{Corr}(\mathrm{dGeom})_{\mathrm{rps},\,\mathrm{l.c.i.}}}$$

represented by proper schemes

$$+$$

$$\boxed{\text{Gaitsgory-Rozenblyum}: \exists \text{ lax monoidal functor } \mathrm{Coh}^b : \mathrm{Corr}(\mathrm{dGeom})_{\mathrm{rps},\,\mathrm{l.c.i.}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{stable}}}$$

Theorem (Porta-S.)

$\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}(S))$ is endowed with the structure of an $\mathbb{E}_1$-algebra in $\mathrm{Cat}_\infty^{\mathrm{stable}}$.

Corollary $D^b(\mathrm{Coh}(\mathbb{R}\underline{\mathrm{Coh}}(S)))$ is endowed with a monoidal structure $\otimes_{\mathrm{Hall}}$.

⚠ : $D^b(\mathrm{Coh}(\mathbb{R}\underline{\mathrm{Coh}}(S))) \not\simeq D^b(\mathrm{Coh}(\underbrace{\underline{\mathrm{Coh}}(S)}_{\text{classical stack}}))$

$\Rightarrow \nexists$ available machinery to construct a monoidal structure on $D^b(\mathrm{Coh}(\underline{\mathrm{Coh}}(S)))$

⚠ : by passing to K-theory, we obtain $\mathcal{H}_{\mathrm{Coh}(S)}$ for $A_*(-) = G_0(-)$

Important Example: $\mathrm{Coh}^b_{(\mathbb{C}^*)^2}(\mathbb{R}\underline{\mathrm{Coh}}_0(\mathbb{C}^2))$ categorifies $\mathcal{H}^{(\mathbb{C}^*)^2}_{\mathrm{Coh}_0(\mathbb{C}^2)}$ for $A_*(-) = G_0(-)$

$\simeq$ positive part of the Elliptic Hall algebra

Curve case

X = smooth proj. curve$/\mathbb{C}$

Consider

- $\mathcal{A}_{Dol}(X) = \{$ Higgs sheaves on $X\}$
- $\mathcal{A}_{dR}(X) = \{$ flat vector bundles on $X\}$
- $\mathcal{A}_{B}(X) = \{$ f.d. representations of $\pi_1(X)\}$

gl. dim. = 2

Theorem (Porta–S.)

1. $\exists$ $\mathbb{E}_1$-algebra structures on

$$\mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{Dol}(X)),\ \mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{dR}(X)),\ \mathrm{Coh}^b(\mathbb{R}\underline{\mathrm{Coh}}_{B}(X))$$

2. $\exists$ categorified Hall type versions of the Riemann–Hilbert and non-abelian Hodge correspondences.

Expectation: $\mathcal{C}$ = CY2 category: $\boxed{\exists\ \mathbb{E}_1\text{-algebra structure on } \mathrm{Coh}^b(\mathbb{R}\mathcal{M}_{\mathcal{C}})}$

Remark: Toda used the categorified Hall product to study his categorified DT invariants.

5. COHAs of quivers and Yangians

Recall again:

$$(\mathcal{Y}_{\mathbb{C}^2})_{loc} \simeq \text{positive nilpotent part of Maulik-Okounkov Yangian } (\mathbb{Y}_{\text{1-loop quiver}})_{loc}$$

(deformation of $U(\widehat{\mathfrak{gl}}(1)[u])$)

If one wants to remove $(-)_{loc}$ above, we need to replace:

$\mathrm{Rep}(\Pi_Q) \rightsquigarrow \Lambda_Q$ = Lusztig(-Bozec) nilpotent stack = moduli stack of nilpotent representations of Π_Q

$\mathcal{H}_Q \rightsquigarrow$ $\exists$ nilpotent COHA $\mathcal{H}_Q^{nil} := (A_*(\Lambda_Q), m)$

$\mathcal{Y}_Q^{nil} := (H_*^{\mathbb{C}^*}(\Lambda_Q), m)$

Theorem (Schiffmann-Vasserot, Yang-Zhao for Q without loops)

$\exists$ a surjective morphism $\Phi_Q : \mathbb{Y}_Q \longrightarrow\!\!\!\!\rightarrow \mathcal{Y}_Q^{nil}$

Remark $\mathbb{Y}_Q$ = deformation of $U(\mathfrak{g}_Q^{MO}[u])$:

$$\begin{cases} \mathfrak{g}_Q^{MO} = \text{certain graded Borcherds-Kac-Moody algebra} \\ \mathfrak{g}_Q^{MO}[0] \simeq \mathfrak{g}_Q \ \text{BKM Lie algebra of } Q \end{cases}$$

Conjecture: Ψ_Q is an isomorphism.

Remark: conjecture true for Q = one-loop, finite/affine Dynkin quivers

6. COHA of the minimal resolution of ADE singularity (Diaconescu-S.-Schiffmann-Vasserot)

- $G \subset SL(2,\mathbb{C})$ finite group
- Q^{fin} = Dynkin diagram associated to G, Q = affine diagram corresponding to Q^{fin}
- $\pi : S \longrightarrow \mathbb{C}^2/G$ minimal resolution of singularities
- $C := \pi^{-1}(0) = \bigcup_{i=1}^{N} C_i$, $C_i \simeq \mathbb{P}^1$, $(C_i \cdot C_j) = -(\text{Cartan matrix of } Q^{fin})$

Let

$(\mathbb{R})\underline{Coh}_C(S)$ = (derived) moduli stack of properly supported sheaves on S set-theoretically supported on C

$\simeq \operatorname{colim}_n (\mathbb{R})\underline{Coh}(C^{(n)})$ w.r.t. closed embeddings $J_{n,m}$

Here,

- $\forall\, n \geq 1$, $C^{(n)}$:= infinitesimal neighborhood of order n of C inside Y
- $\forall\, m \geq n$, $\exists$ closed embedding $J_{n,m} : (\mathbb{R})\underline{Coh}(C^{(n)}) \longrightarrow (\mathbb{R})\underline{Coh}(C^{(m)})$

Theorem

1. $\exists$ an associative algebra structure $\mathbb{Y}_{S,C}$ on $H_*^{\mathbb{C}^*}(\mathbb{R}\underline{\mathrm{Coh}}_C(S))$
2. $\exists$ a surjective morphism of algebras

$$\mathbb{Y}_{T^*C_1,C_1} \otimes \cdots \otimes \mathbb{Y}_{T^*C_N,C_N} \twoheadrightarrow \mathbb{Y}_{S,C}$$

zero section of T^*C

where $\mathbb{Y}_{T^*C_i,C_i}$ is the COHA structure on $H_*^{\mathbb{C}^*}(\mathbb{R}\underline{\mathrm{Coh}}_C(T^*C))$, such that the Kernel is generated by "local relations" depending on two intersecting C_i, C_j

Remark: (2) follows from the "canonical filtration" of a sheaf $\mathcal{E}$ on Y set-theoretically supported at C. For example, $N=2$:

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

$\mathcal{E}_1$: max. subsheaf set-theoretically supported on C_1

$\mathcal{E}_2$: quotient set-theoretically supported on C_2

Expectations: $\exists$ $\mathbb{Y}_{S,C} \hookrightarrow (\mathbb{Y}_Q^+)^c$ $\longleftarrow$ some completion

This is justified by "derived McKay correspondence": $D^b(\mathrm{Coh}(S)) \overset{\Phi}{\simeq} D^b(\mathrm{Mod}(\Pi_Q))$

$\Longrightarrow$
- $\mathbb{Y}_Q$ should arise from a COHA associated to the whole $D^b(\mathrm{Coh}(S))$
- Φ does NOT preserve the t-structure $\Rightarrow$ different t-structures give rise to different "halves" of $\mathbb{Y}_Q$

Let us restrict ourselves: $S = T^*\mathbb{P}^1 \supset C \simeq \mathbb{P}^1$

Slogan: the "building blocks" of $Y_{T^*\mathbb{P}^1, \mathbb{P}^1}$ are the semistable COHAs

$$\begin{array}{ll} (\mathbb{R})\,\underline{\mathrm{Coh}}_C(S)^{ss}_\mu & = \text{(derived) moduli stack of } D\text{-semistable} \\ \quad \Big\downarrow \text{-open} & \quad \text{properly supported sheaves on } S, \text{ of slope } \mu \in \mathbb{Q}\cup\{\infty\}, \\ (\mathbb{R})\,\underline{\mathrm{Coh}}_C(S) & \quad \text{set-theoretically supported on } C \end{array}$$

Facts:

- $\mu \in \mathbb{Q}$: $(\mathbb{R})\,\underline{\mathrm{Coh}}_C(S)^{ss}_\mu \simeq \Lambda_{A_1} \Rightarrow H^{\mathbb{C}^*}_*(\mathbb{R}\,\underline{\mathrm{Coh}}_C(S)^{ss}_\mu) \simeq Y^+_{A_1} = Y^+(\mathfrak{sl}(2))$

- $\mu = \infty$: $(\mathbb{R})\,\underline{\mathrm{Coh}}_C(S)^{ss}_\infty \simeq$ (derived) moduli stack of 0-dim. sheaves on $S = T^*\mathbb{P}^1$, set-theoretically supported on $C = \mathbb{P}^1$

$$\Rightarrow H^{\mathbb{C}^*}_*(\mathbb{R}\,\underline{\mathrm{Coh}}_C(S)^{ss}_\infty) \simeq \underbrace{Y_{\substack{\text{1-loop}\\ \text{quiver}}} \otimes Y_{\substack{\text{1-loop}\\ \text{quiver}}} = Y(\hat{\mathfrak{gl}}(1)) \otimes Y(\hat{\mathfrak{gl}}(1))}_{\text{"corresponding to the two fixed pts of } \mathbb{P}^1\text{"}}$$

Theorem

1. $Y_{T^*\mathbb{P}^1, \mathbb{P}^1} \simeq \mathbb{Y}^+(\hat{\mathfrak{gl}}(1))^{\otimes 2} \rtimes \mathbb{Y}^+(L\mathfrak{sl}(2))$

2. PBW decomposition: $Y_{T^*\mathbb{P}^1, \mathbb{P}^1} \simeq \mathbb{Y}(\hat{\mathfrak{gl}}(1))^{\otimes 2} \otimes \overrightarrow{\bigotimes_{\mu \in \mathbb{Z}}} \underbrace{\mathbb{Y}^+(\mathfrak{sl}(2))}_{\text{lift of semistable COHAs}}$

Idea behind the Theorem:

w.r.t. closed embeddings

$$\boxed{(\mathbb{R})\underline{\mathrm{Coh}}_C(S) \simeq \operatorname{colim}_n (\mathbb{R})\underline{\mathrm{Coh}}(C^{(n)})}$$

Each $(\mathbb{R})\underline{\mathrm{Coh}}(C^{(n)})$ can be exhausted by open substacks given by Harder-Narasimhan stratification:

$$\boxed{(\mathbb{R})\underline{\mathrm{Coh}}(C^{(n)}) \simeq \operatorname{colim}_\ell (\mathbb{R})\underline{\mathrm{Coh}}(C^{(n)})^{>-\ell} \longleftarrow \mu_{D\text{-}min} > -\ell}$$

w.r.t. open embeddings

Now, we "twist by $-\otimes_S(-D) =: T$":

$$\boxed{(\mathbb{R})\underline{\mathrm{Coh}}(C^{(n)})^{>-\ell} \text{ "}\simeq\text{" } T^{\ell}\left((\mathbb{R})\Lambda_{Q,n}^{\leq 0}\right)}$$

w.r.t. HN filtration: $\mu_{max} \leq 0$

Summarizing:

w.r.t. (T^ℓ)'s

$$\boxed{Y_{T^*\mathbb{P}^1,\mathbb{P}^1} \simeq \operatorname*{colim}_n \lim_{\ell} H_*^{\mathbb{C}^*}\left(\Lambda^{\leq 0}_{Q,n}\right)}$$

Important Facts:

sum of fundamental coweights

1. under the derived McKay correspondence: $T \longleftrightarrow T_{-\rho^\vee} \in B^e_Q$ = extended affine braid group

2. $H_*^{\mathbb{C}^*}\left(\Lambda_Q^{\leq 0}\right)$ is a quotient of $H_*^{\mathbb{C}^*}\left(\Lambda_Q\right) = Y_Q^{nil} \simeq \mathbb{Y}_Q^+$

$\Rightarrow$ One needs to understand the compatibility between $\hat{B}^e_Q$ and Y_Q^{nil}.