Bott vanishing using GIT and quantization

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Outline

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1. Introduction
2. GIT
3. Quantization
4. How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$

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for every $i > 0$, $j \geq 0$ and L ample.

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- Very restrictive property.
- Not clear geometric meaning.

Example

Suppose Y is Fano and satisfies Bott vanishing. Then

$$H^1(Y, T_Y) = H^1(Y, \Omega_Y^{n-1} \otimes K_Y^*) = 0.$$

In particular, Y must be rigid.

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- Quintic del Pezzo surface (Totaro 2019).
- Bott vanishing holds for K3 surfaces of degree $= 20$ or ≥ 24, fails for K3 surfaces of degree < 20 (Totaro 2019).

Introduction

The quintic del Pezzo is isomorphic to $\bar{M}_{0,5}$ can be obtained as a GIT quotient $(\mathbb{P}^1)^5 /\!\!/_{\mathcal{O}(2,2,2,2,2)} PGL_2$. It parametrizes 5-tuples of points on $\mathbb{P}^1$ where no three of them coincide.

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From now on, we will work over $\mathbb{C}$.

Introduction

Theorem (T)

Let Y be a GIT quotient $(\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$ given by a linearization with no strictly semi-stable locus. Then Y satisfies Bott vanishing.

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- Quantization. This allows us to compute cohomology on Y as cohomology on X of a suitable object $\mathcal{F}$.
- Geometric syzygies. The cohomologies of $\mathcal{F}$ correspond to the Koszul resolution of certain locus in $X \times \mathbb{P}(\mathfrak{g})$.
- Gelfand-MacPherson correspondence. This allows us to see global invariant sections as polynomials in the Plücker minors, and we characterize these as directed graphs.

GIT

Definition (GIT quotient)

Let $X = \operatorname{Proj} R$ be a variety with an action by a group G. Extend the action of G to R. Then the *GIT* quotient $X /\!\!/ G$ is defined as $\operatorname{Proj} R^G$.

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There are two choices involved:

- The coordinate ring R. This amounts to specifying an ample line bundle $\mathcal{L}$, so that $R = \bigoplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})$.
- The action of G on R. This amounts to extending the action of G on X to the total space of $\mathcal{L}$.

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Let $\mathbb{C}^*$ act on $\mathbb{A}^{n+1}$ by multiplication. Extend this action to $\mathcal{O}$ and twist it by the character $t \mapsto t$, that is: $t \cdot p(x_0, \ldots, x_n) = tp(t^{-1}x_0, \ldots, t^{-1}x_n)$. Then one obtains the GIT quotient $\mathbb{A}^{n+1} /\!\!/ \mathbb{C}^* = \mathbb{P}^n$.

GIT

Definition

Given an action of G on X and a G-linearized ample line bundle $\mathcal{L}$, the *semi-stable* locus is defined as

$$X^{ss} = \{x \in X \mid \exists \sigma \in H^0(X, \mathcal{L}^{\otimes k})^G, \sigma(x) \neq 0\}$$

and then we have a quotient map

$$\pi : X^{ss} \to X /\!\!/_{\mathcal{L}} G.$$

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We are interested in the cases when $X^{ss} = X^s$.

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Let $\mathcal{L} = \mathcal{O}(d_1, \ldots, d_n)$ be a PGL_2-linearized ample line bundle in $X = (\mathbb{P}^1)^n$. The semi-stable (resp. stable) locus consists of tuples $(z_1, \ldots, z_n) \in (\mathbb{P}^1)^n$ such that whenever $\sum_{i \in I} d_i > \sum_{i \in I^c} d_i$ (resp. $\geq$) for some $I \subset \{1, \ldots, n\}$, the coordinates $\{z_i, i \in I\}$ do not all coincide.

The GIT quotient $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$ parametrizes such configurations of n points up to projective equivalence.

Quantization

We are interested in computing cohomologies $H^i(Y, F)$, for certain vector bundles F. Under certain circumstances, this can be computed as $H^i(X, \mathcal{F})^G$ for some suitable object $\mathcal{F}$.

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Quantization Theorem (Teleman, Halpern-Leistner)

Suppose $\mathcal{F}$ descends to F. Take a Kempf-Ness stratification of the unstable locus $X \backslash X^{ss} = \sqcup S_\alpha$. If all the weights of $\mathcal{F}$ on S_α are $< \eta_\alpha$, then

$$H^i(Y, F) = H^i(X, \mathcal{F})^G.$$

Quantization

How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$

In our case, we have $X = (\mathbb{P}^1)^n$, $G = PGL_2$ and let $\mathfrak{g} = \mathfrak{sl}_2$ be the Lie algebra. The action of G induces a map of sheaves $\Omega_X \to \mathfrak{g}^\vee$.

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Definition

We denote by $L_{\mathfrak{X}}$ the PGL_2-linearized two-step chain complex $[\Omega_X \to \mathfrak{g}^\vee]$.

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Remark

$L_{\mathfrak{X}}$ descends to Ω_Y. This is because of the following short exact sequence

$$0 \to \pi^*\Omega_Y \to \Omega_{X^{ss}} \to \mathfrak{g}^\vee \to 0.$$

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Similarly, $\Lambda^j L_{\mathfrak{X}}$ descends to Ω_Y^j, where $\Lambda^j L_{\mathfrak{X}}$ is the chain complex

$$0 \to \Omega_X^j \to \Omega_X^{j-1} \otimes \mathfrak{g}^\vee \to \cdots \to S^j\mathfrak{g}^\vee \to 0.$$

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Lemma

Let $X = (\mathbb{P}^1)^n$, $G = PGL_2$ and $\mathcal{L} = \mathcal{O}(d_1, \ldots, d_n)$ a PGL_2-linearized ample line bundle such that $X^{ss} = X^s$. Then $\wedge^j L_{\mathfrak{X}} \otimes \mathcal{L}$ satisfies the hypotheses of the Quantization theorem, and so

$$H^i(Y, \Omega_Y^j \otimes L) = H^i(X, \wedge^j L_{\mathfrak{X}} \otimes \mathcal{L})^{PGL_2}$$

where $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$ and L is the descent of $\mathcal{L}$.

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$$H^i(Y, \Omega^j_Y \otimes L) = H^i(X, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})^{PGL_2}$$

where $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$ and L is the descent of $\mathcal{L}$.

In fact, to show Bott vanishing on Y, it suffices to check that

$$H^i(X, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})^{PGL_2} = 0$$

for $i > 0$.

How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$

Recall $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$ is the complex

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Lemma

$H^i(X, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})$ can be computed as the cohomology of the complex of global sections

$$0 \to H^0(\Omega^j_X \otimes \mathcal{L}) \to H^0(\Omega^{j-1}_X \otimes \mathcal{L}) \otimes \mathfrak{g}^\vee \to \cdots \to H^0(\mathcal{L}) \otimes S^j \mathfrak{g}^\vee \to 0.$$

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For example, for $j = 1$, showing that $H^1(Y, \Omega_Y \otimes L) = 0$ is equivalent to showing that the map of invariant global sections

$$H^0(X, \Omega_X \otimes \mathcal{L})^{PGL_2} \to (H^0(X, \mathcal{L}) \otimes \mathfrak{g}^\vee)^{PGL_2}$$

is surjective.

How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$

How to think of PGL_2-invariant global sections of a line bundle $\mathcal{O}(d_1, \ldots, d_n)$ on $(\mathbb{P}^1)^n$?

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Gelfand-MacPherson correspondence

Consider the action of the torus $(\mathbb{C}^*)^n$ on the Grassmannian $\mathrm{Gr}(2, n)$. Let $\mathcal{O}(1)$ be the ample line bundle on $\mathrm{Gr}(2, 5)$ given by the Plücker embedding. We endow it with a $(\mathbb{C}^*)^n$-linearization by choosing the character $(d_1, \ldots, d_n)$. Then

$$\bigoplus_{k \geq 0} H^0((\mathbb{P}^1)^n, \mathcal{O}(kd_1, \ldots, kd_n))^{PGL_2} = \bigoplus_{k \geq 0} H^0(\mathrm{Gr}(2, n), \mathcal{O}(k))^{(\mathbb{C}^*)^n}$$

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In particular, PGL_2-invariant global sections of $\mathcal{O}(d_1, \dots, d_n)$ can be found in the coordinate ring of the Grassmannian.

How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$

In fact, $H^0((\mathbb{P}^1)^n, \mathcal{O}(d_1, \ldots, d_n))^{PGL_2}$ consists of polynomials in $x_i y_j - x_j y_i$ having homogeneous degree $d_1, \ldots, d_n$ in the variables $x_1, y_1; \ldots; x_n, y_n$, subject to the Plücker equivalence relations.

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Each such section can be described as a linear combination of (directed) graphs having n vertices, $v_1, \ldots, v_n$, each of them of degree $\deg v_i = d_i$.

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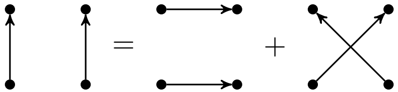
The Plücker relations can be depicted as follows:



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This way one can show that $H^0(X, \Omega_X \otimes \mathcal{L})^{PGL_2} \twoheadrightarrow (H^0(X, \Omega_X) \otimes \mathfrak{g}^\vee)$ and so

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Using similar techniques (plus an argument with Koszul complexes), one can also show that

$$H^i(Y, \Omega^j \otimes L) = H^i(X, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})^{PGL_2} = 0, \quad i > 0, j \geq 0$$

and so Y satisfies Bott vanishing, as long as the linearization does not admit strictly semi-stable locus.

The toric case

Interestingly, quantization can also be applied succesfully towards toric varieties. In fact, a smooth projective toric variety Y can be written as a GIT quotient $Y = \mathbb{A}^d /\!\!/ (\mathbb{C}^*)^{d-n}$, where $(\mathbb{C}^*)^{d-n} = \mathrm{Hom}(\mathrm{Pic}\, Y, \mathbb{C}^*)$, and the action is free on the semi-stable locus.

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Using quantization and similar techniques, we recover yet another proof of the following well-known result.

Theorem

A smooth projective toric variety satisfies Bott vanishing.

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Thanks!