A General Theory of Anomalous Shock Refraction

E.G. Puckett*, **L.F. Henderson**† **and P. Colella**‡
*Department of Mathematics and Institute of Theoretical Dynamics, U.C. Davis, Davis, CA 95616, USA
†Department of Applied Mathematics and Statistics, SUNY Stony Brook, Stony Brook, NY 11794, USA
‡Department of Mechanical Engineering, U.C. Berkeley, Berkeley, CA 94720, USA

Abstract. Anomalous refraction in gases was discovered by Jahn (1956) during experiments with shocks refracting at an Air/CO_2 gas interface. Jahn's experiments were confined to the case when the wave impedance decreases during the refraction, $|Z_t| < |Z_i|$, in which case the incident shock is overrun by upstream-moving expansions that locally weaken it and cause it to curve backward. We show that Jahn's model for this phenomenon (classical anomalous refraction) must be modified in one important respect, namely that there is still a centered supersonic expansion at the node R. We also show that anomalous refraction can occur when the impedance increases, $|Z_t| > |Z_i|$, in which case the incident shock is overrun by upstream moving compressions that locally strengthen it and cause it to move forward.

Key words: Anomalous shock refraction, Wave impedance

1. The refraction law

Suppose that a shock wave i is propagating with a velocity $\boldsymbol{U}_i$ in a material whose properties and state are known, and suppose also that it enters another material with different properties, or state, that are also known, and that causes its velocity to change to $\boldsymbol{U}_t$. The shock will be said to refract whenever $\boldsymbol{U}_i$ differs in either magnitude or direction from $\boldsymbol{U}_t$. For simplicity it will be assumed the boundary m between the two materials is a plane surface, and that i encounters it at some angle of incidence α_i (Fig.1). If the incident and transmitted shocks meet at a single point on the interface, and travel at the same speed along it, then the angle of transmission α_t is related to α_i by the refraction law,

$$\frac{U_i}{\sin \alpha_i} = \frac{U_t}{\sin \alpha_t} \tag{1}$$

where $U_{i,t} = |\boldsymbol{U}_{i,t}|$ are the wave speeds measured in the laboratory frame (Fig.1). The relative refractive index η is defined as,

$$\eta \equiv \frac{U_i}{U_t} = \frac{\sin \alpha_i}{\sin \alpha_t} \tag{2}$$

from which it is concluded that the shock will be refracted by the materials whenever $\eta \neq 1$.

2. Wave impedance

The wave impedance for an arbitrary one-dimensional shock i is defined as

$$\mathcal{Z}_i \equiv \frac{P_1 - P_0}{U_{pi}} \tag{3}$$

where P is the pressure, the subscripts $0, 1$ refer to the conditions upstream and downstream of the shock, and U_{pi} is the (signed) speed of the piston associated with the i shock (see Henderson 1989 or Henderson et al. 1991). In other words, $U_{pi} \equiv \boldsymbol{u}_1 - \boldsymbol{u}_0$, where $\boldsymbol{u}_{0,1}$ is the velocity of the material upstream (*resp.* downstream) of the shock. If $\mathcal{Z}_i$ is interpreted as the mass flux, then Eq.3 is simply the momentum equation for the flow perpendicular to i.

The impedance $\mathcal{Z}_e$ of a one-dimensional expansion wave e may also be defined as in Eq.3. Thus,

$$\mathcal{Z}_e \equiv \frac{P_2 - P_1}{U_{pe}} \tag{4}$$

Shock Waves @ Marseille IV
Editors: R. Brun, L. Z. Dumitrescu

where $P_{2,1}$ are the pressure upstream and downstream of the wave, U_{pe} is the magnitude of the piston velocity $\boldsymbol{U}_{pe} \equiv \boldsymbol{u}_2 - \boldsymbol{u}_1$ associated with the expansion, and $\boldsymbol{u}_{1,2}$ are the velocity of the medium upstream and downstream of the wave. Note that $\boldsymbol{U}_{pe}$ is the velocity of a piston which is *withdrawing* in the direction opposite that of the wave propagation.

When $\alpha_i = 0$, there will in general be a reflected wave propagated back into the initial material as i passes into the receiving material. The reflection may either be a compression r, or an expansion e. More precisely it will be a compression if the magnitude of the wave impedance increases $|\mathcal{Z}_t| > |\mathcal{Z}_i|$ and an expansion if it decreases $|\mathcal{Z}_t| < |\mathcal{Z}_i|$, but there will be no reflection if there is no change in the impedance $\mathcal{Z}_t = \mathcal{Z}_i$. In this last case the wave system consists only of two shocks and we call it a *shock pair* (Henderson and Puckett 1993).

For an oblique shock, that is one for which $\alpha_i \neq 0$, the theory is simplified if we define an *effective impedance* Z_i by

$$Z_i \equiv \frac{P_1 - P_0}{U_{pi}\cos\beta_i}, \tag{5}$$

where β_i is the angle that i makes with the *disturbed* (downstream) interface labelled d in Fig.1. For the r and t waves Z_r and Z_t are defined in a manner analogous to Eq.5 with $\beta_{r,t}$ and $\boldsymbol{U}_{pr,t}$ defined as shown in Fig.1. The effective impedance is the natural generalization of the *effective acoustic impedance* (Kinsler et al. 1982) to shock waves.

The definition of the effective impedance Z_e for a centered expansion wave in two dimensions is a bit more difficult, since an expansion of the Prandtl-Meyer type is not a single plane surface, but a fan of such surfaces. We have,

$$Z_e \equiv \frac{P_2 - P_1}{\int d(U_{pj}\cos\beta_j)} = \frac{P_2 - P_1}{(\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\boldsymbol{n}}, \tag{6}$$

where U_{pj} denotes the magnitude of the piston velocity $\boldsymbol{U}_{pj}$ associated with the "jth" wave in the expansion fan, β_j denotes the angle this wave makes with the disturbed gas and $\boldsymbol{n}$ is a unit vector normal to the disturbed gas interface.

3. The classical model of anomalous refraction

Anomalous refraction in gases was discovered by Jahn (1956) during his experiments with the Air/CO_2 combination of materials. When α_i was sufficiently small the refraction was regular, and its reflected wave was a centered (Prandtl-Meyer) expansion (Fig.3a), this is a regular-refraction-with-a-reflected-expansion and is denoted by RRE. With α_i increasing continuously, a critical condition was attained where the flow Mach number M_1 downstream of, and relative to i, became sonic: $M_1 = 1$. The critical angle of incidence α_i^* is:

$$\tan^2\alpha_i^* = \frac{U_i^2}{a_1^2 - (U_i - U_{pi})^2} = \frac{q_{n0}^2}{a_1^2 - q_{n1}^2}. \tag{7}$$

Here U_i and U_{pi} are measured in laboratory coordinates as shown in Fig.1, while q_{n0} and q_{n1} are the normal components of the particle velocity upstream and downstream of the incident shock i, measured in coordinates at rest with respect to the shock. It will be noticed that Eq.7 is valid for any material and that it does not depend on the properties of the receiving material, nor on any wave or boundary parameter of the transmitted shock t.

Jahn observed that when $\alpha_i > \alpha_i^*$ the expansion waves were able to overrun part of i and to cause some attenuation of it, and consequently that part i' was swept in the downstream direction, and curved backward. At the same time the reflected wave spreads out into a distributed band of expansions (Fig.2 and Fig.3e). The result was an anomalous-refraction-with-reflected-expansions (ARE). (The upper case "E", indicates sonic or supersonic flow downstream of i.) The Jahn

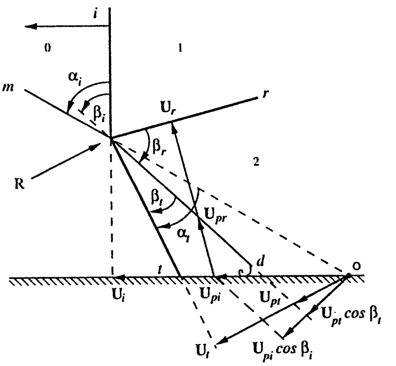

Fig. 1. Regular refraction

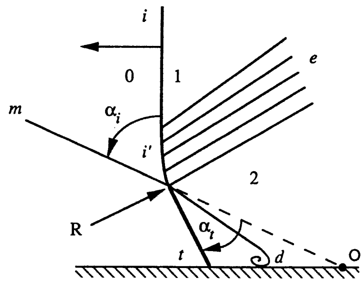

Fig. 2. Classical anomalous refraction

model assumes that the disturbed part of the shock i' has a sonic surface everywhere at its rear. Consequently Eq.7 is satisfied at every point on i'. This is only possible if i' weakens monotonically, and also the speed of sound a_1 decreases monotonically, as i' approaches the refraction node R.

4. A general theory of anomalous refraction

Let $\xi_i \equiv P_0/P_1$ denote the (inverse) incident shock strength. For a given ξ_i let α_p denote the angle at which equality of the (effective) impedance $Z_t = Z_i$ occurs. Now define ξ_{ip}^* to be the shock strength for which the condition $Z_t = Z_i$ coincides with the onset of anomalous refraction $\alpha_i = \alpha_i^*$. In other words, when $\xi_i = \xi_{ip}^*$ we have $\alpha_i = \alpha_i^* = \alpha_p$. By examining the i and t shock polars one can show that for the Air/CO_2 gas combination $\alpha_p < \alpha_i^* \iff \xi_i > \xi_{ip}^*$, and also that $\alpha_i < \alpha_p \iff |Z_i| > |Z_t|$ (Puckett et al. 1993). We call $\xi_i > \xi_{ip}^*$ *weak* refractions and $\xi_i < \xi_{ip}^*$ *strong* refractions. By chance rather than design Jahn only studied weak refractions, and hence (the magnitude of) the wave impedance was always decreasing $|Z_t| < |Z_i|$ in the anomalous regime $\alpha_i > \alpha_i^*$. In the following we examine numerically one of his experimental sequences, and then also examine a strong refraction sequence in Air/CO_2.

4.1 Air/CO_2 with $\xi_i = 0.85$ (weak refractions)

A sequence with $\xi_i = 0.85$ held constant was studied experimentally by Jahn (1956). It is a weak refraction sequence since $\xi_i = 0.85 > \xi_{ip}^* = 0.29064$. By the theory developed in Puckett et al. (1993) the only possible refractions are RRE and ARE. Some numerical results for this sequence are presented in Fig.3. We plot contours of $\log P$ in Figs.3a,c,e,g and contours of the Mach number $\tilde{M}$ in *self-similar* coordinates in Figs.3b,d,f,h. In these latter figures solid lines indicate the sonic and supersonic contours $\tilde{M} \geq 1$. The sonic contour $\tilde{M} = 1$ is the last solid contour.

Figs.3a-b are for $\alpha_i = 69^\circ < \alpha_i^* = 71.116^\circ$ which is an RRE. Transition to the anomalous system, RRE $\rightleftharpoons$ ARE, takes place at $\alpha_i = \alpha_i^* = 71.116^\circ$; the results are presented in Figs.3c-d. For $\alpha_i = 75.0^\circ > \alpha_i^*$, there is an ARE (Figs.3e-f). The centered expansion wave has now partly spread out and over-run a portion of the incident shock i and thus produces the partly attenuated shock i'. In Fig.3f, the sonic contour coincides with the rear of i', which is as it should do if the Jahn model of ARE (Fig.2) is correct. However the results do differ from his model in one important respect, namely that *there is still a centered supersonic expansion at the node.* This has also been mentioned by Grove and Menikoff (1990).

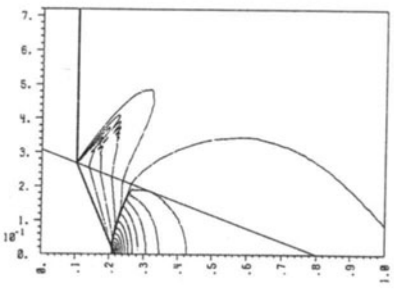

Fig. 3a. RRE at $\alpha_i = 69°$ ($\log P$)

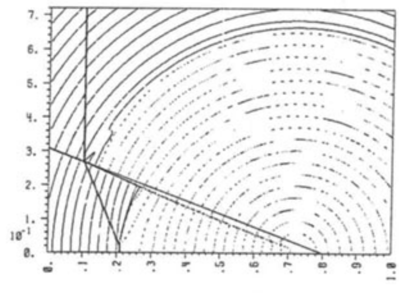

Fig. 3b. RRE at $\alpha_i = 69°$ ($\tilde{M}$)

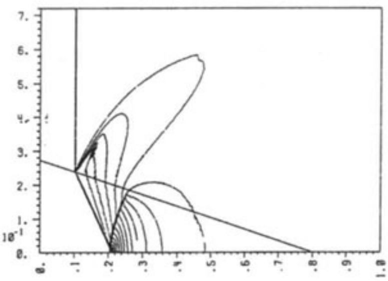

Fig. 3c. RRE $\rightleftharpoons$ ARE at $\alpha_i = 71.116°$ ($\log P$)

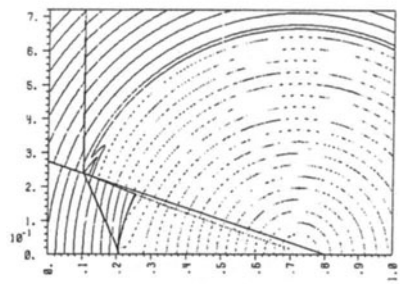

Fig. 3d. RRE $\rightleftharpoons$ ARE at $\alpha_i = 71.116°$ ($\tilde{M}$)

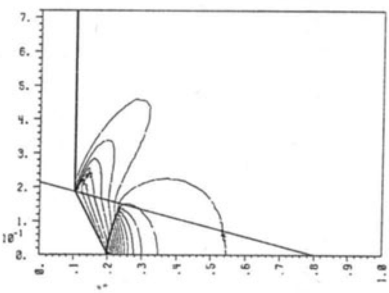

Fig. 3e. ARE at $\alpha_i = 75°$ ($\log P$)

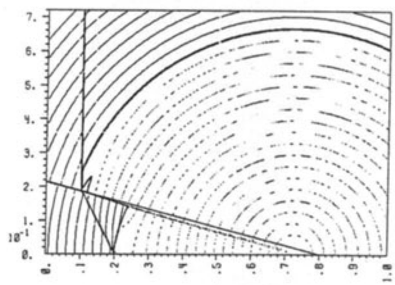

Fig. 3f. ARE at $\alpha_i = 75°$ ($\tilde{M}$)

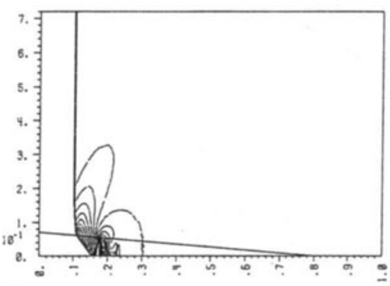

Fig. 3g. ARE at $\alpha_i = 85°$ ($\log P$)

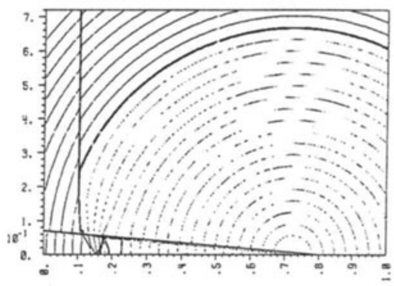

Fig. 3h. ARE at $\alpha_i = 85°$ ($\tilde{M}$)

Fig. 3. Shock refraction sequence in Air/CO_2 with $\xi_i = 0.85$

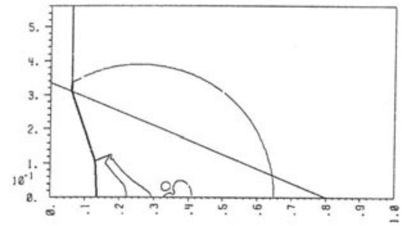

Fig. 4a. ARc at $\alpha_i = 67.2428°$ ($\log P$)

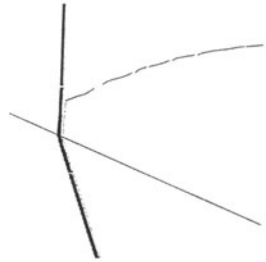

Fig. 4b. Enlargement of Fig. 4a

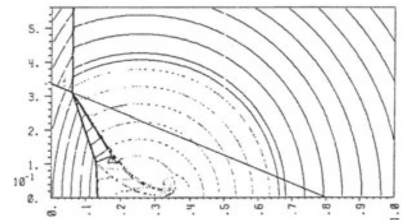

Fig. 4c. ARc at $\alpha_i = 67.2428°$ ($\tilde{M}$)

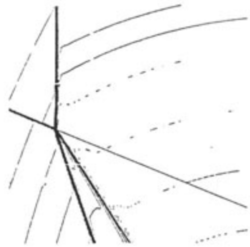

Fig. 4d. Enlargement of Fig. 4c

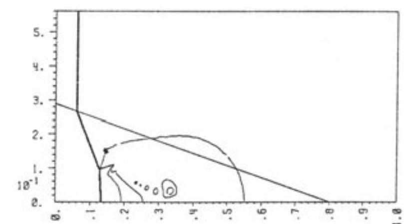

Fig. 4e. RSP at $\alpha_i = 70.21°$ ($\log P$)

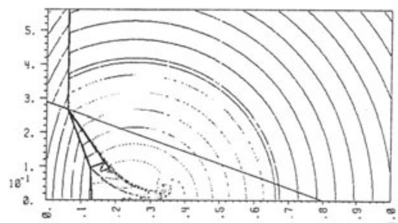

Fig. 4f. RSP at $\alpha_i = 70.21°$ ($\tilde{M}$)

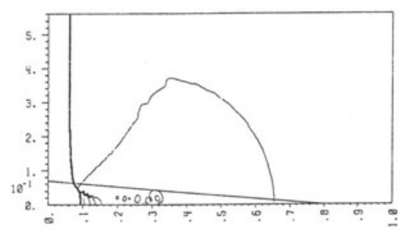

Fig. 4g. ARe at $\alpha_i = 85°$ ($\log P$)

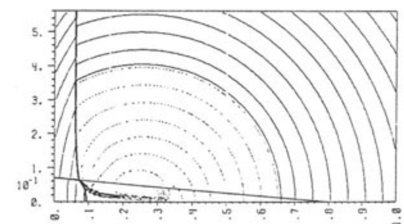

Fig. 4h. ARe at $\alpha_i = 85°$ ($\tilde{M}$)

Fig. 4. Shock refraction sequence in Air/CO_2 with $\xi_i = 0.1$

With further development to $\alpha_i = 85°$ (Figs.3i-j) the centered fan has all but vanished, although the sonic condition remains everywhere at the rear of i' (Fig.3j). This last computation corresponds to the experiment shown in Fig.14h, Plate 11 of Jahn (1956). According to our results his interpretation of the structure of the refraction is essentially correct because the remains of the centered expansion wave has all but vanished in this case. It is concluded that the structure of an ARE is as described by Jahn, except for the addition of a centered (Prandtl-Meyer) expansion wave which emanates from the refraction node.

4.2 Air/CO_2 with $\xi_i = 0.10$ (strong refractions and variable impedances)

When the refraction is weak $\xi_i > \xi_{ip}^*$, the equality of impedance condition $Z_i = Z_t$ occurs in the regular range $\alpha_p < \alpha_i^*$ for Air/CO_2. Therefore in the anomalous range $\alpha_i > \alpha_i^* > \alpha_p$, and the only possible system is one with sonic expansions at the rear of i', that is an ARE. This was well supported by the numerical results presented in §4.1. The same conclusions apply to the other sequence ($\xi_i = 0.3$) studied experimentally by Jahn, since here also $\xi_i = 0.3 > \xi_{ip}^* = 0.29064$. If a reflected compression is to occur for the Air/CO_2 combination, then the $Z_i = Z_t$ condition should lie in the anomalous range; that is, it must be a strong shock refraction, $\alpha_p > \alpha_i^*$.

In order to make numerical tests of this conclusion, an Air/CO_2 sequence was chosen with $\xi_i = 0.1 < \xi_{ip}^* = 0.29064$, with three values of α_i. One value was $\alpha_i = 67.2428°$ which is midway between $\alpha_i^* = 64.272°$ and $\alpha_p = 70.2135°$, so $|Z_i| < |Z_t|$. According to the theory this should be an anomalous-refraction-with-reflected-compressions (ARc), and indeed it is, as one can see in Figs.4a-d. (The lower case "c"indicates subsonic flow downstream of i.) Fig.4b is an enlargement of Fig.4a about the node R, and it will be noticed that i' now moves forward of i as it is overtaken by subsonic compressions arising downstream. The fact that the flow downstream of i' is subsonic is demonstrated in Figs.4c-d. The second value was $\alpha_i = \alpha_p = 70.2135°$ where $Z_i = Z_{i'} = Z_t$, and it is shown in Figs.4e-f. Notice that i and i' are now indistinguishable and that $i = i'$ and t are all planar. Consequently the flows are all uniform about the node and are subsonic downstream of $i = i'$ and supersonic downstream of t (Fig.4f). We call it a *regular shock pair* (RSP). The third value was $\alpha_i = 85° > \alpha_p$, where $|Z_i| > |Z_t|$, and as expected i' now leans backward of i (Fig.4g-h), and the flow downstream of i' is subsonic, in other words it is an ARe. These results are also consistent with the predictions of the theory. The theory is also supported by experiment (Abd-el-Fattah and Henderson 1978) and numerical work (Puckett et al. 1993) with the Air/SF_6 gas combination.

References

Abd-el-Fattah AM, Henderson LF (1978) Shock waves at a fast-slow gas interface. J. Fluid Mech. 86: 15-32.

Grove JW, Menikoff R (1990) Anomalous reflection of a shock wave at a fluid interface. J. Fluid Mech. 219: 313-336.

Henderson LF (1989) On the refraction of shock waves. J. Fluid Mech. 198: 365-386.

Henderson LF, Colella P, Puckett EG (1991) On the refraction of shock waves at a slow-fast gas interface. J. Fluid Mech. 224: 1-27.

Henderson LF, Puckett EG (1993) Anomalous refraction of shock waves in materials with general equations of state. Part I: The shock pair system. Trans. Roy. Soc. London A (in print)

Jahn RG (1956) The refraction of shock waves at a gaseous interface. J. Fluid Mech. 1: 457-489.

Kinsler LE, Frey AR, Coppens AB, Sanders JV (1982) Fundamentals of Acoustics. John Wiley and Sons, New York.

Puckett EG, Henderson LF, Colella P (1993) The anomalous refraction of shock waves in materials with general equations of state. Part II: Anomalous refraction wave systems. Trans.Roy.Soc.London A (in print)