A Discontinuous Galerkin (DG) Method for Solving Temperature Advection-Diffusion Equation in the Mantle Convection Code ASPECT

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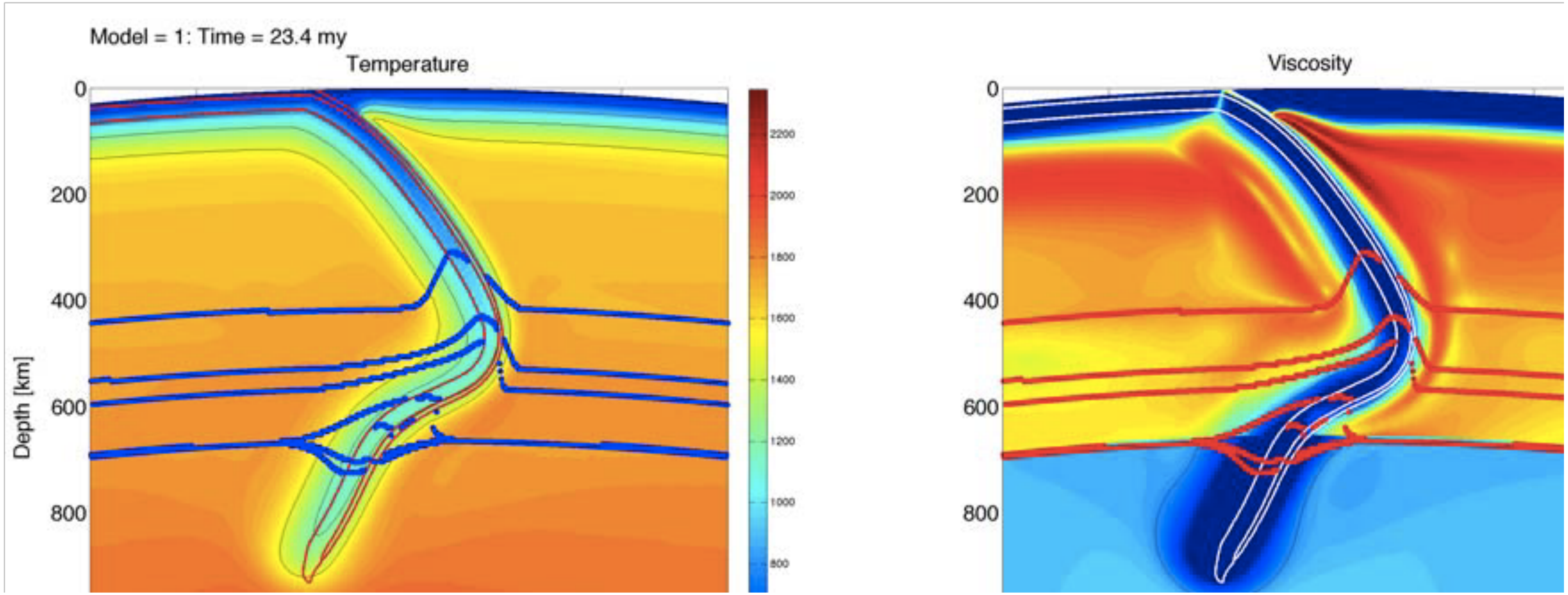
Geodynamics Computations involve Temperature

Crustal layers for slab dynamics



Arredondo & Billen, J. Geodyn. submitted

Incompressible Stokes equations with an advection-diffusion equation for the temperature and a Bousinesq approximation for the density

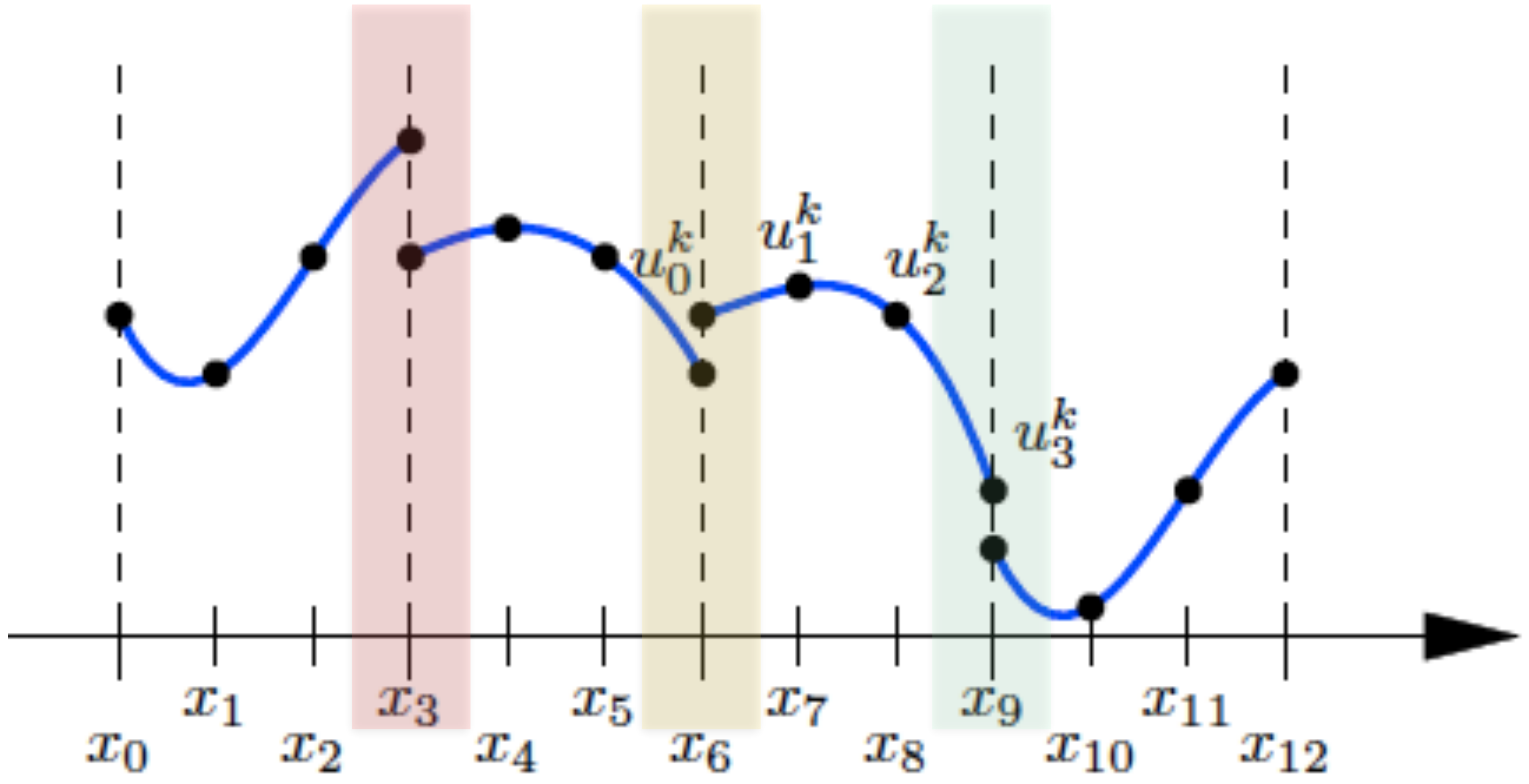
$$
\begin{aligned}
-\nabla \cdot [2\eta\epsilon(\mathbf{u})] + \nabla p &= \rho \mathbf{g}, \\
\nabla \cdot \mathbf{u} &= 0, \\
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \nabla \cdot (\kappa \nabla T), \\
\rho &= \rho_0(1 - \alpha(T - T_0)).
\end{aligned}
$$

Convection-dominated if the Péclet Number $Pe = \dfrac{L\|u\|_\infty}{\kappa} >> 1$

Note, if $\kappa = 0$, there is no diffusion, only advection and $Pe = +\infty$.

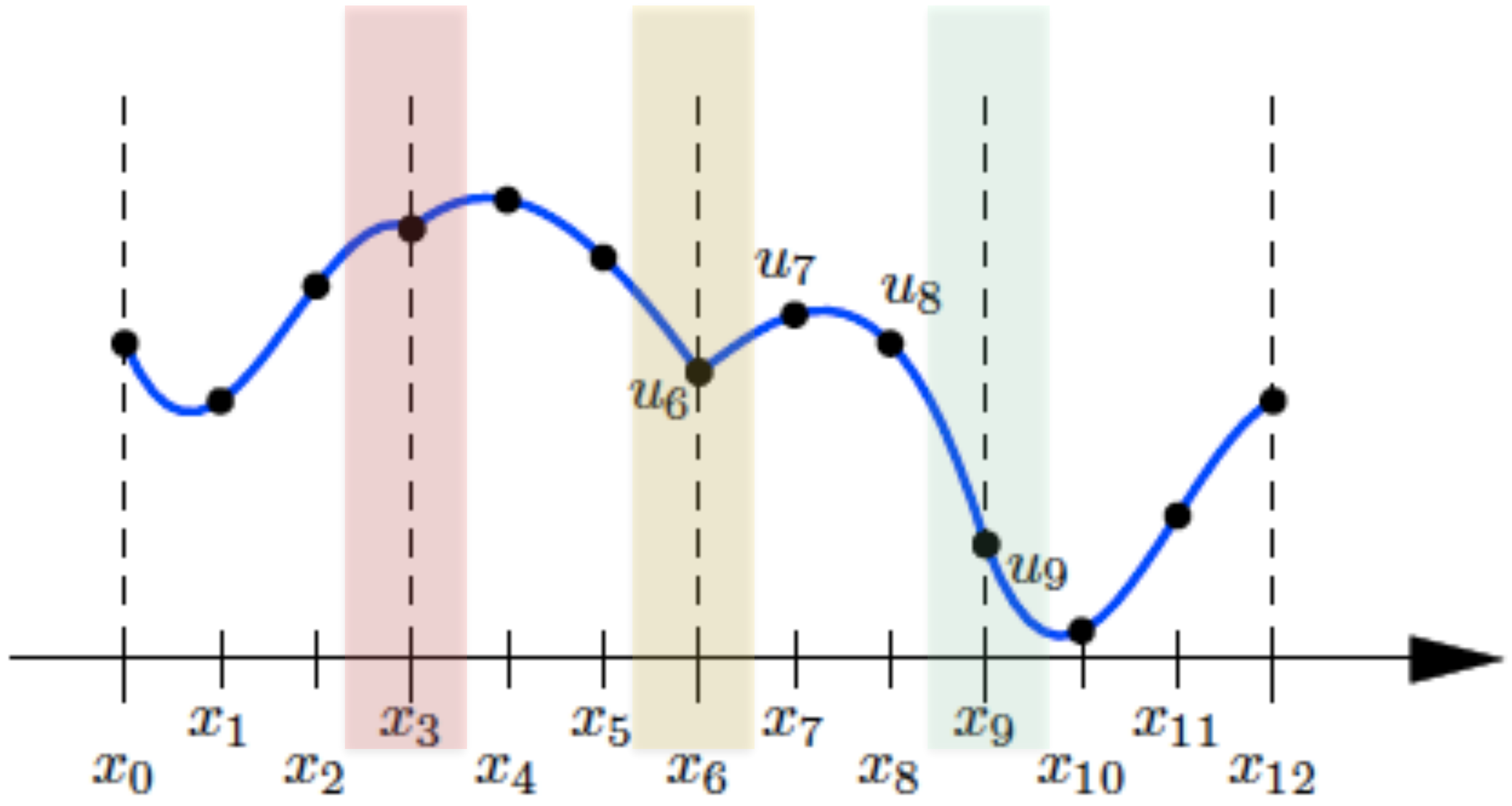
In ASPECT we have two options

Discontinuous Galerkin Method
(DG)

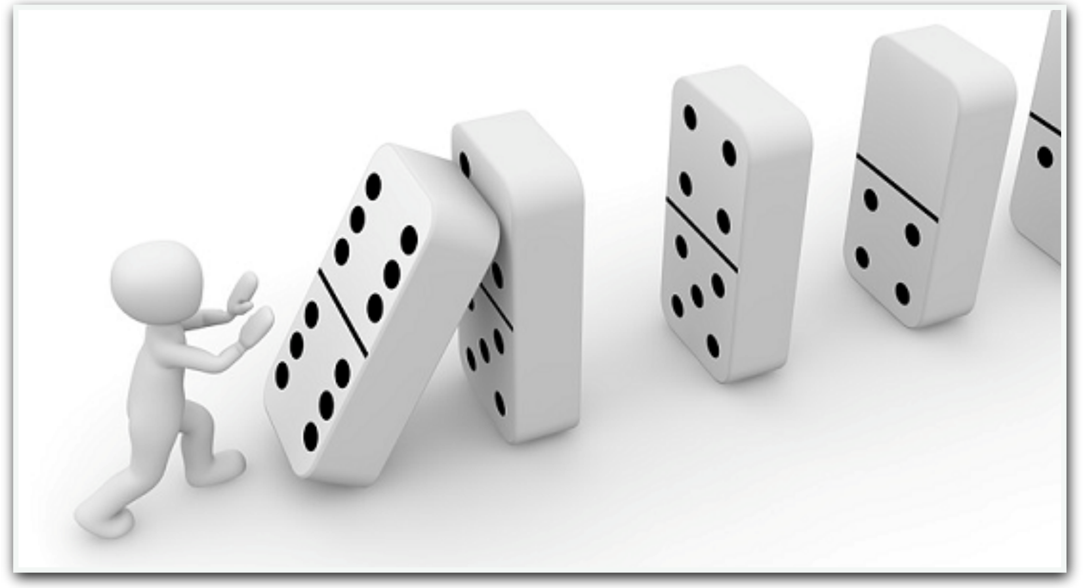


CAN easily make LOCAL cell changes!

Finite Element Method
(FEM)



CANNOT make local changes WITHOUT affecting neighboring cell values!



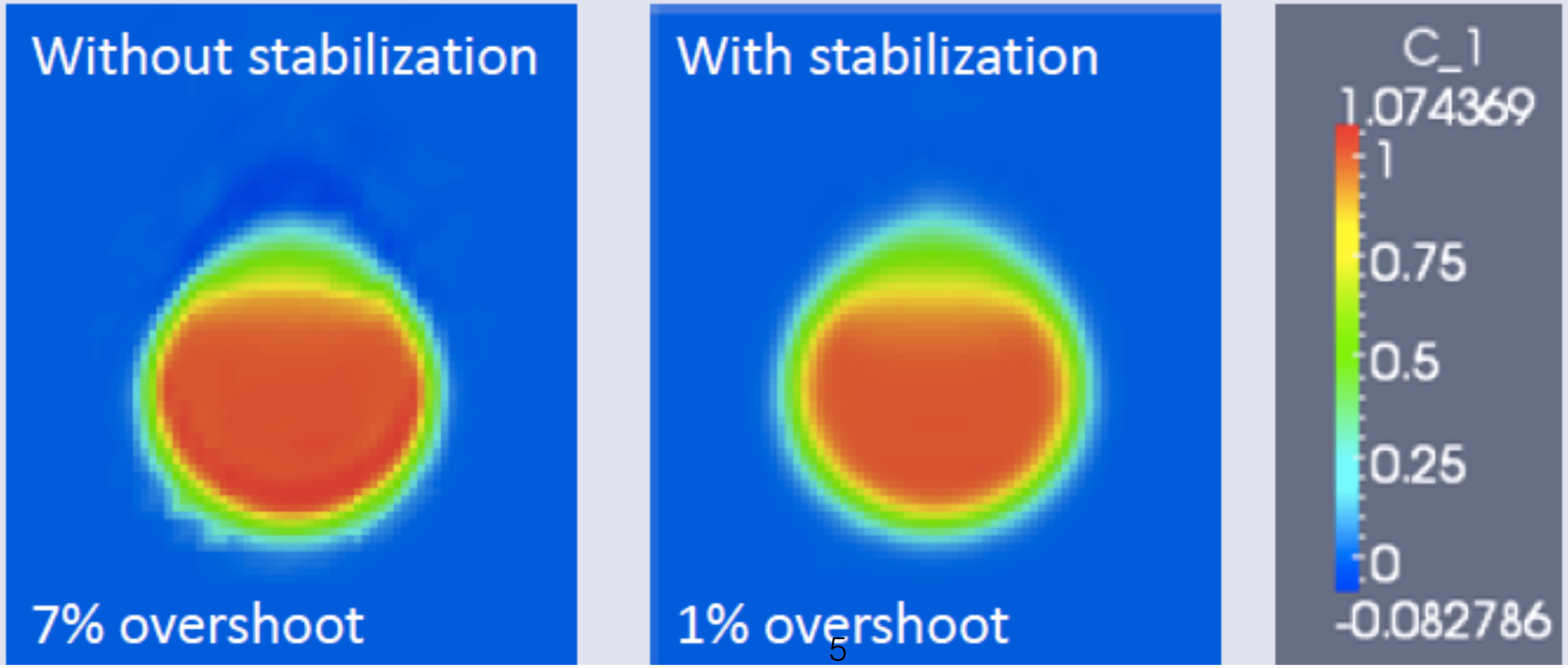
The original advection-diffusion algorithm in ASPECT

The equation for temperature is modified as follows

FEM stabilization scheme: ***entropy viscosity***

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \nabla \cdot (\kappa + \nu_h(T))\nabla T = \mathbf{0}$$

- Result: *positive valued entropy viscosity adds artificial diffusion*



The Discontinuous Galerkin (DG) approach

Post processing of DG solutions for Overshoot/Undershoot

1. Find the trouble cell using local Min/Max, global Min/Max

2. Apply the Bound Preserving limiter

Zhang, X., Shu, C.-W., 2010a

Y. He, G. Puckett, and B. Magali, to appear, PEPI 2016

For Local oscillation with high order discretization

1. Use trouble cell indicator: for example, the minmod limiter, the shock detection technique. Need local and neighboring values.

2. Apply the WENO limiter, reconstruction depends on adjacent cells.

Qiu, J., Shu, C.-W., 2005

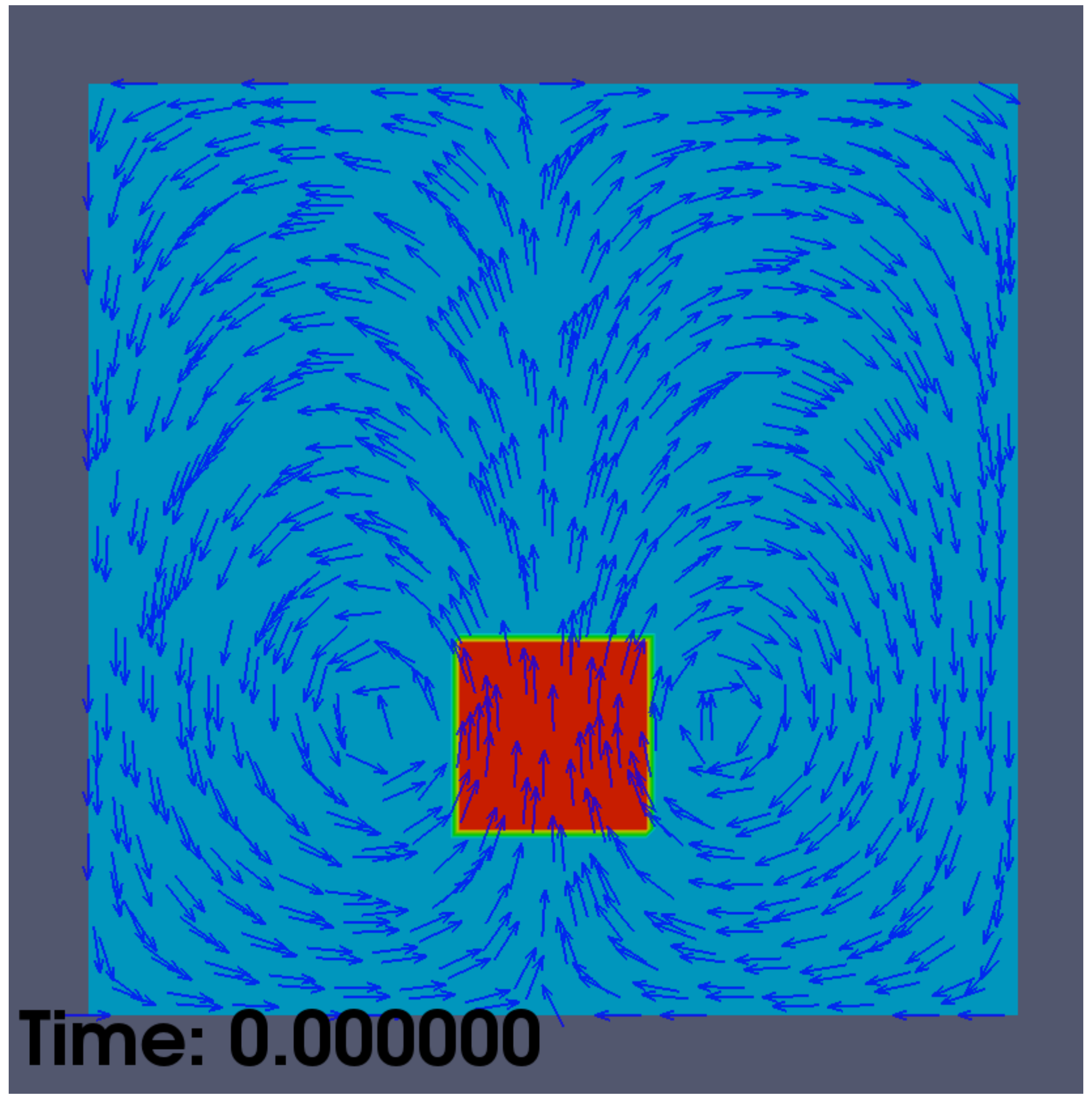
Zhu, J., Qiu, J., Shu, C.-W., Dumbser, M., 2008

Numerical Results

Solve the nondimensional problem

$$Pe = \frac{L\|u\|_\infty}{\kappa} = \frac{1}{\kappa}, \text{ if } L = \|u\|_\infty = 1$$

Initial Temperature
and velocity field
on a unit box



Typical layout in the movie.
Play the movie!

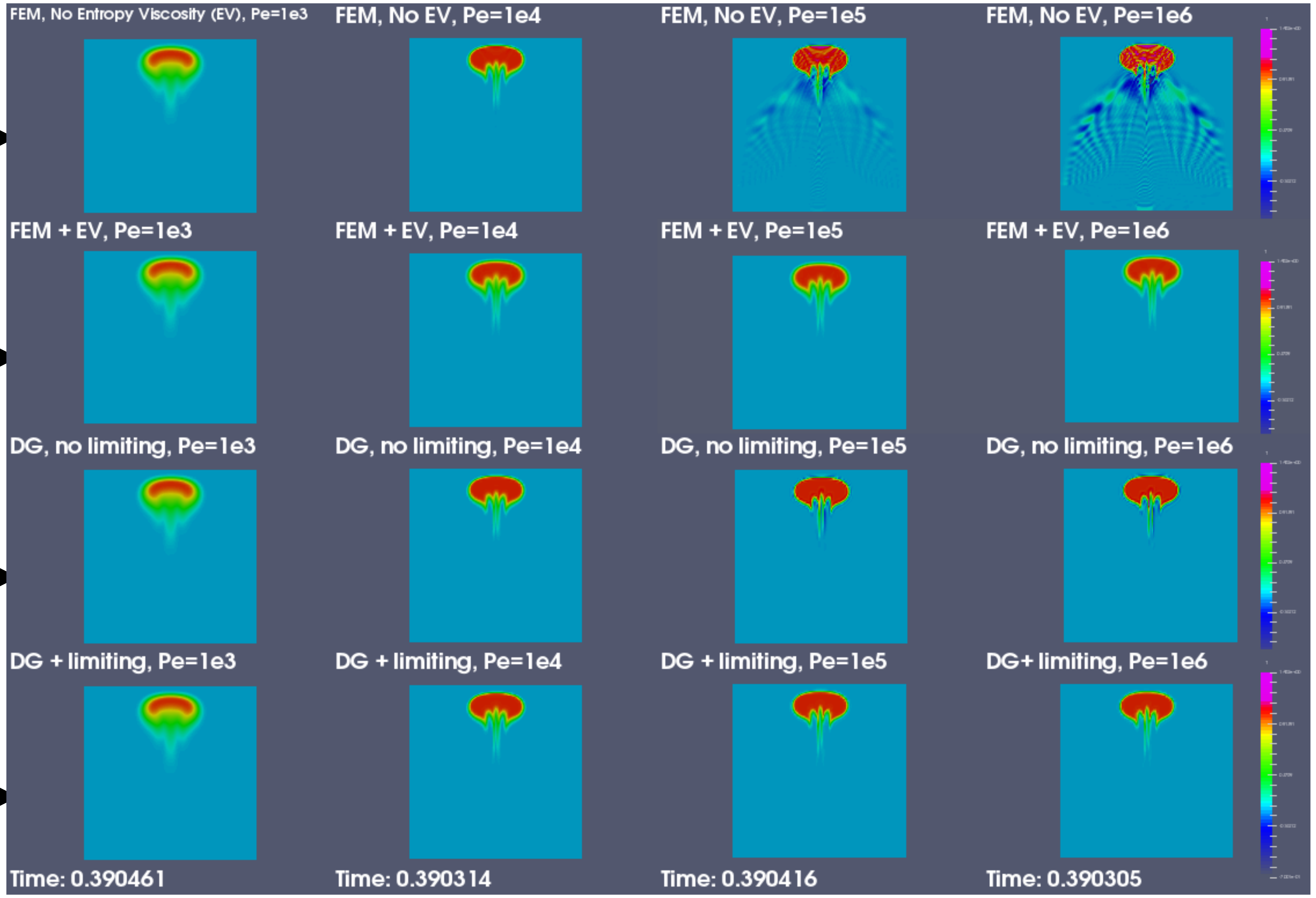
Pe=1e3 **Pe=1e4** **Pe=1e5** **Pe=1e6**

FEM only →

FEM + EV →

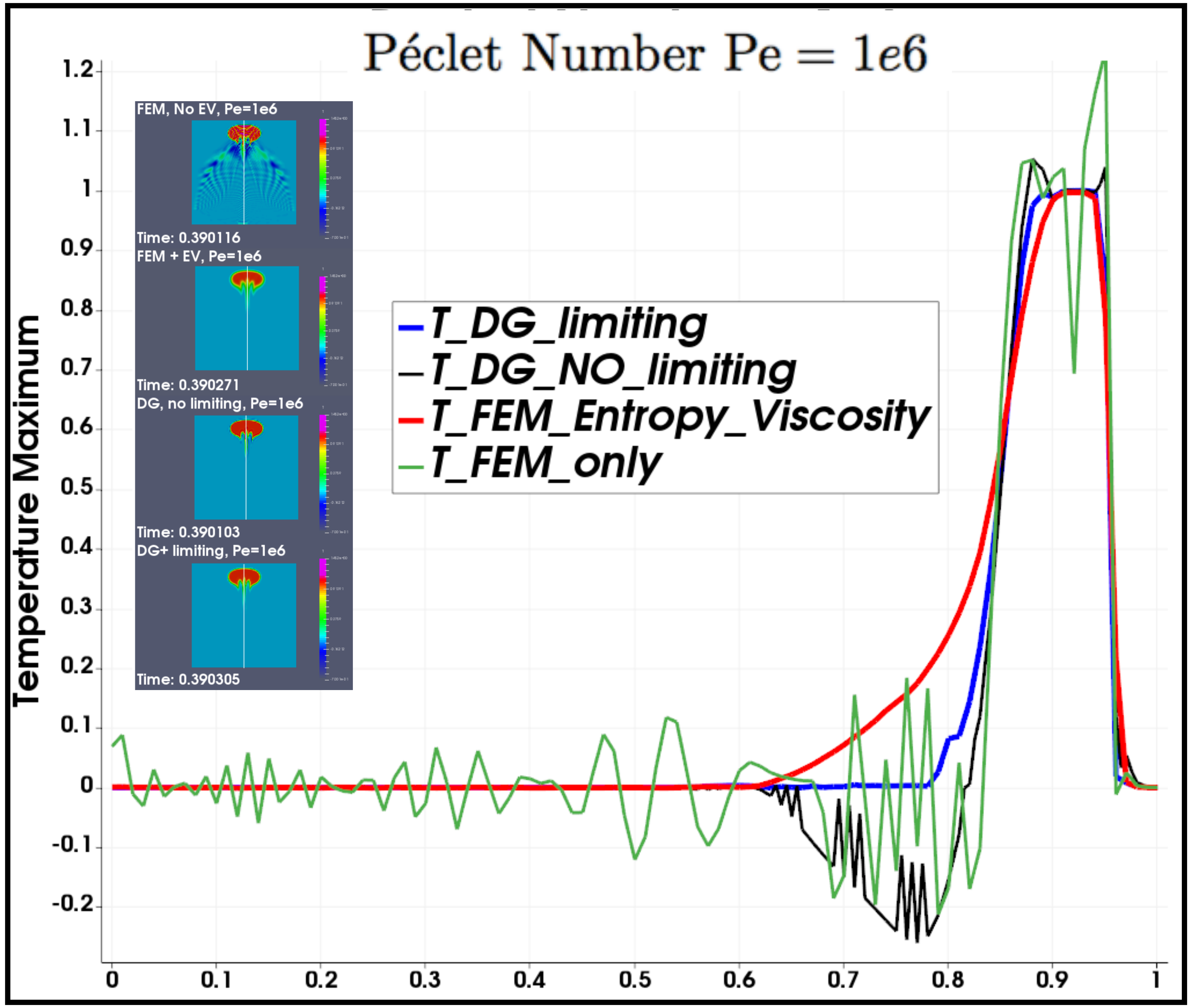
DG only →

DG + limiting →

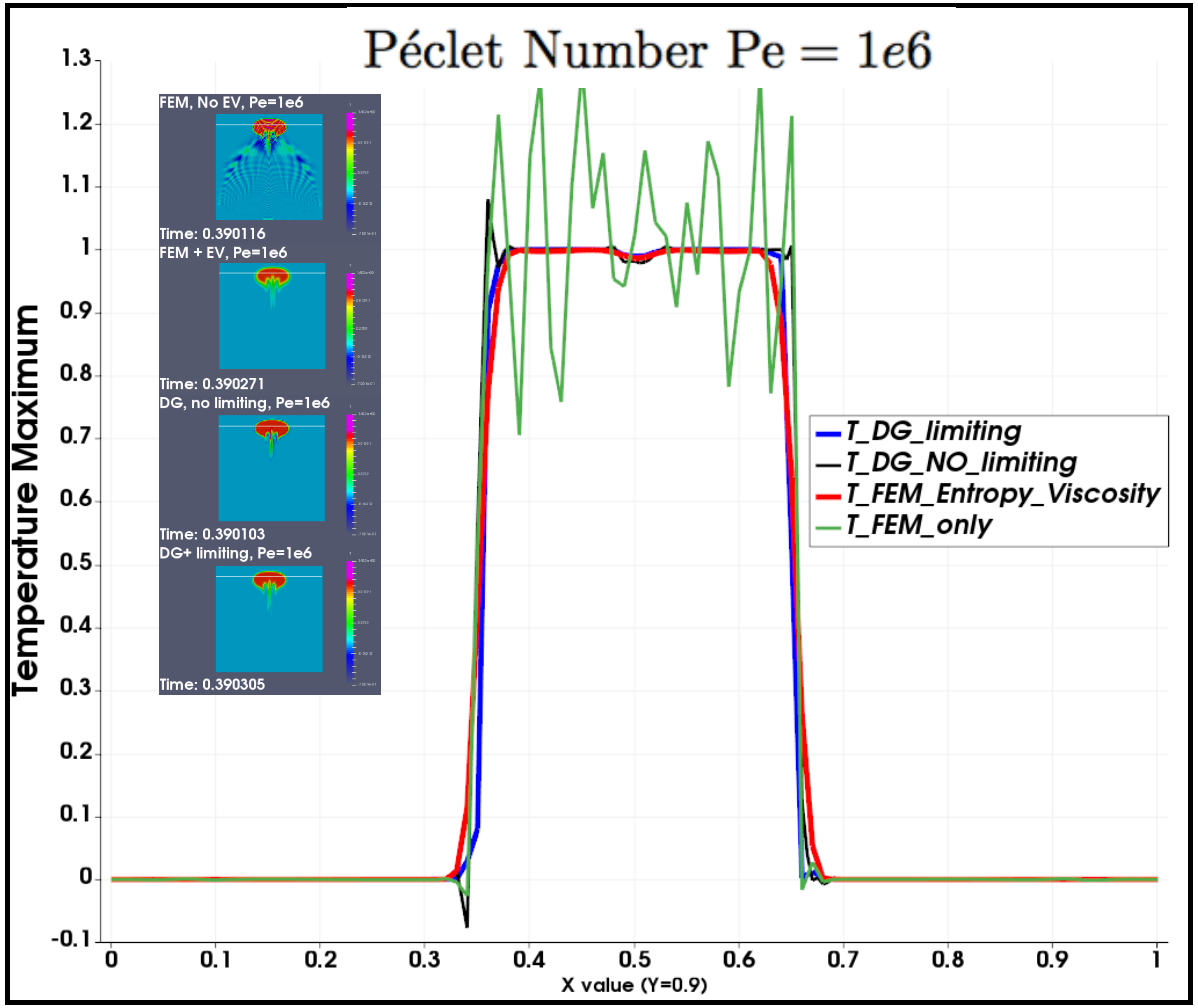


Numerical Results

Vertical Profile at x=0.5



Horizontal Profile at y=0.9



Conclusions and Future Work

Conclusions

- We have implemented a stable, accurate, and efficient method for the temperature advection-diffusion equation within ASPECT*:* DG+bound preserving limiter+WENO limiter.
- We studied the iso-viscous test cases of a rising square with Peclet numbers from 1e3 to 1e6.
- Our numerical results have demonstrated that*, compared to FEM with entropy viscosity,* DG with limiting is more suitable for convection dominated flows; i.e., higher Peclet numbers.

Future Work

- Adaptive mesh refinement (AMR)
- Apply this to subduction type models, where temperature overshoots occur ahead of the slab tip when there is a sharp viscosity contrast. The sharp viscosity contrast produces thin thermal boundary layers that "act" like iso-viscous flows with large Peclet numbers.

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Thank you!

1, Y. He, G. Puckett, and B. Magali, A Discontinuous Galerkin Method with a Bound Preserving Limiter for the Advection of non-Diffusive Fields in Solid Earth Geodynamics, to appear, Physics of the Earth and Planetary Interiors 2016 *DOI 10.1016/j.pepi.2016.12.001*

2, Kronbichler, M., Heister, T., Bangerth, W., 2012. High accuracy mantle convection simulation through modern numerical methods. Geophysical Journal International 191 (1), 12-29

3, Zhang, X., Shu, C.-W., 2010a. On maximum-principle-satisfying high order schemes for scalar conservation laws. Journal of Computational Physics 229 (9)

4, Qiu, J., Shu, C.-W., 2005. Runge Kutta discontinuous Galerkin method using WENO limiters. SIAM Journal on Scientic Computing 26 (3), 907-929.

5, Zhu, J., Qiu, J., Shu, C.-W., Dumbser, M., 2008. Range Kutta discontinuous Galerkin method using WENO limiters ii: unstructured meshes. Journal of Computational Physics 227 (9), 4330-4353.

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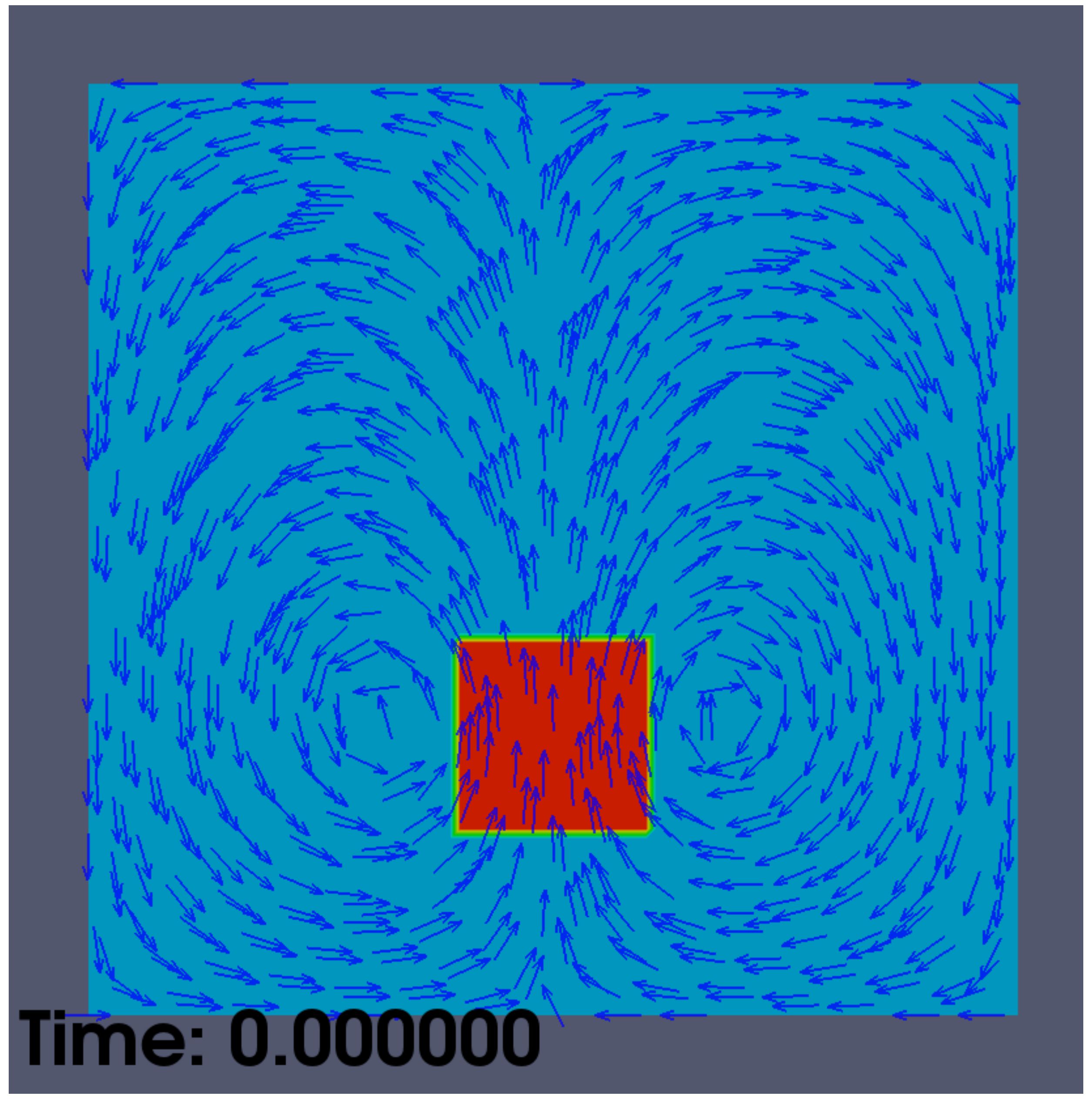

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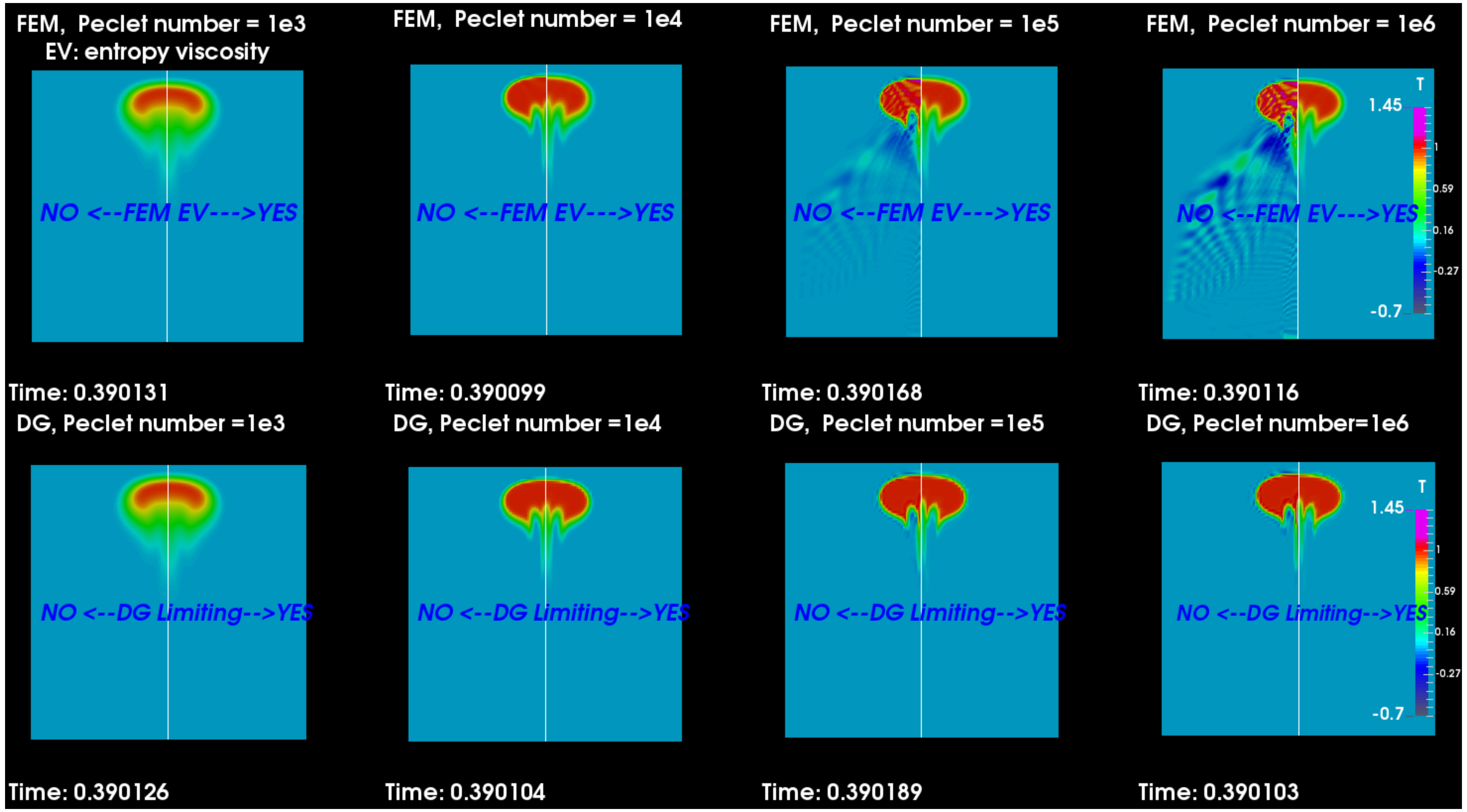
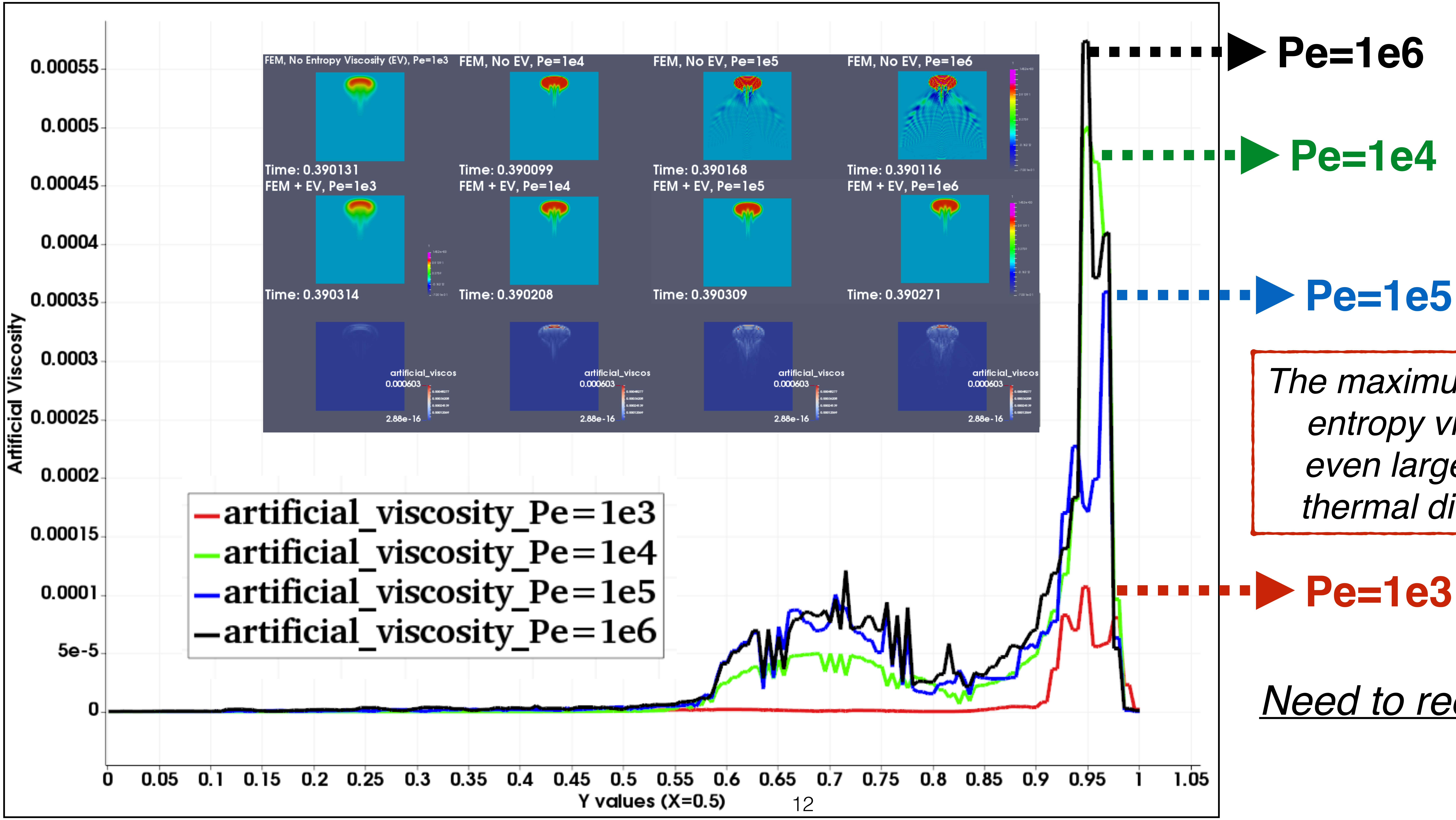
DG →



Numerical Results: entropy viscosity



The maximum added entropy viscosity even larger than thermal diffusivity

<u>Need to reduce "h"</u>

Numerical Results

Peclet Number
= 4.16E3

$$\rho = \rho_0(1 - \beta(T - T_0))$$
$$\eta = \eta_0 e^{-\frac{T - T_0}{T_0}}$$

T= 0 year