Math 108, Fall 2013.
Oct. 25, 2013.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

1. Assume that propositions P and Q are true, while R and S are false. For each of the following logical expressions, determine whether it is true or false.

(a) $\sim P \vee R \vee S$

False (as $\sim P$, R, S are all False).

(b) $\sim (R \vee S)$

True (as $R \vee S$ is False).

(c) $P \vee R \Longrightarrow S$

False (as $P \vee R$ is True, but S is False)

(d) $P \wedge R \Longleftrightarrow (P \wedge S \Longrightarrow S)$

False ($P \wedge R$ is False, and
$P \wedge S$ is False, so $P \wedge S \Rightarrow S$ is True.
The two sides of $\Leftrightarrow$ have different values.)

2. For each statement below, determine whether it is true or false for the universe $\mathbb{N} = \{1, 2, 3, \ldots\}$. Justify your answers.

(a) $(\forall x)(\exists y)(y > 2x)$

True. For a given x, take $y = 2x+1$.

(b) $(\exists x)(\exists y)((y \leq 5) \wedge (y > 2x))$

True. Take $x = 1, y = 3$.

(c) $(\forall x)(\forall y)(x + y \geq 2)$

True. As $x \geq 1, y \geq 1$, $x+y \geq 2$.

(d) $(\exists x)(\forall y)(y - x \leq 20)$

False. Negation $(\forall x)(\exists y)\ (y - x \geq 20)$ is true as we can, for a given x, take $y = x + 21$.

(e) $(\exists x)(\forall y)((x + y \leq 20) \vee (y \geq 15))$

True. Take $x = 1$.

Choose a y.

Then we need to check that either $1+y \leq 20$ or $y \geq 15$, that is, either $y \leq 19$ or $y \geq 15$. This is clearly true.

3. Suppose a and b are elements of $\mathbb{N}$. Prove that

$$2a \leq b, \text{ and } b \leq a+3, \text{ and } b \text{ divides } 3a^2 - 1$$

if and only if

$$a = 1 \text{ and } b = 2.$$

(*Hint*. How do the two inequalities restrict a?)

Proof. ($\Leftarrow$) Check : when $a=1$ and $b=2$,
$2\cdot 1 \leq 2$, $2 \leq 1+3$, $2 \mid 3\cdot 1^2 - 1$.

($\Rightarrow$) As $2a \leq b$ and $b \leq a+3$,
$2a \leq a+3$, $a \leq 3$. So we have 3 cases:

Case 1: $a=1$. Then $2 \leq b \leq 4$, and $b \mid 2$.
The only possibility is $b=2$.

Case 2: $a=2$. Then $4 \leq b \leq 5$ and $b \mid 11$.
Impossible as neither 4 nor 5 divide 11.

Case 3: $a=3$. Then $6 \leq b \leq 6$ and $b \mid 26$.
Impossible as 6 does not divide 26.

Case one gives the only possibility: $a=1$, $b=2$. $\square$

4. Assume that A, B, C are arbitrary subsets of $\mathbb{N}$. For each statement below, prove it or provide a counterexample.

(a) Prove: If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

$$x \in A \cup C \Rightarrow (x \in A) \vee (x \in C)$$

$$(\text{as } A \subseteq B) \Rightarrow (x \in B) \vee (x \in C)$$

$$\Rightarrow x \in B \cup C$$

(b) Give a counterexample: Converse of (a). That is, $A \cup C \subseteq B \cup C \Rightarrow A \subseteq B$

Take $A = \mathbb{N}$, $B = \emptyset$, $C = \mathbb{N}$,

(c) Give a counterexample: If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Take $A = \{1\}$, $B = \{2\}$, $C = A$.

(d) Prove: If $1 \in A$, then $\{1\} \in \mathcal{P}(A)$.

If $1 \in A$, then $\{1\} \subseteq A$, and then $\{1\} \in \mathcal{P}(A)$.

5. Prove (by induction or any other correct method you know) that for every $n \in \mathbb{N}$, $10^n - 3^n$ is divisible by 7.

$(n=1)$ $10^1 - 3^1 = 7$, clearly divisible by 7.

$(n \to n+1)$ We need to show that $10^{n+1} - 3^{n+1}$ is divisible by 7. By I.H., $10^n - 3^n = 7k$, for an integer k. Therefore,

$$\begin{aligned}
10^{n+1} - 3^{n+1} &= 10 \cdot 10^n - 3^{n+1} \\
&= 10 \cdot (7k + 3^n) - 3^{n+1} \\
&= 7 \cdot 10k + 10 \cdot 3^n - 3 \cdot 3^n \\
&= 7 \cdot 10k + 7 \cdot 3^n \\
&= 7(10k + 3^n),
\end{aligned}$$

a multiple of 7. $\square$